Analytic and Algebraic Geometry 4

edited by
Tadeusz Krasiński
Stanisław Spodzieja

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Łódź 2022

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Printed directly from camera-ready materials provided to the Łódź University Press
by Department of Analytic Functions and Differential Equations

Published by Łódź University Press
First edition. W.10902.22.0.K

https://doi.org/10.18778/8331-092-3

ISBN 978-83-8331-092-3
e-ISBN 978-83-8331-093-0

Printing sheets 11.5

Łódź University Press
90-237 Łódź, 34A Jana Matejki St.
www.wydawnictwo.uni.lodz.pl
e-mail: ksiegarnia@uni.lodz.pl
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Preface

Annual Conferences in Analytic and Algebraic Geometry have been organized by Faculty of Mathematics and Computer Science of the University of Łódź since 1980. Proceedings of these conferences (mainly in Polish) were published in the form of brochures containing educational materials describing current state of branches of mathematics mentioned in the conference title, new approaches to known topics, and new proofs of known results (all the materials are available on the website: http://konfrogi.math.uni.lodz.pl/). Since 2013 proceedings are published (non-regularly) in the form of monographs. Three volumes have been published so far: Analytic and Algebraic Geometry (2013), Analytic and Algebraic Geometry 2 (2017), Analytic and Algebraic Geometry 3 (2019). The content of these volumes consists of new results and survey articles concerning real and complex algebraic geometry, singularities of curves and hypersurfaces, invariants of singularities, algebraic theory of derivations and other topics.

This volume (the fourth in the series) is dedicated to two mathematicians: Wojciech Kucharz, who celebrates 70th anniversary in 2022 and Tadeusz Winiarski who celebrated the 80th anniversary in 2020. These people were closely associated with our conferences Analytic and Algebraic Geometry. The first one is an active participant of the conferences since 2009 and the second one is a leading figure of the conferences almost from the beginning (1983). Thanks to their mathematical vigor and stimulation the conferences become more interesting and fruitful. On next pages we provide short scientific biographies of each of them.

We would like to thank many people for the help in preparing the volume. In particular, Michał Jankowski for designing the cover, referees for preparing reports on the volume and all participants of the Conferences for their good humor, atmosphere and enthusiasm during the conferences.

Tadeusz Krasiński
Stanisław Spodzieja

October 2022, Łódź

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Professor Wojciech Kucharz

Analytic and Algebraic Geometry 4
Łódź University Press 2022, 13–14
DOI: https://doi.org/10.18778/8331-092-3.01

WOJCIECH KUCHARZ – SCIENTIFIC BIOGRAPHY

Wojciech Kucharz was born in Kozłów (near Miechów) on January 2, 1952. He entered the Jagiellonian University in 1969, graduated with a degree in mathematics in 1974, and earned his Ph.D. in mathematics three years later under the supervision of Professor Józef Siciak and, informally, Professor Jacek Bochnak. His doctoral dissertation was entitled "Sufficiency of jets and finite determinacy of germs". In the years 1977–1981, Kucharz was an assistant professor at the University of Silesia. Then, after completing his visiting positions at the Free University of Amsterdam, he moved to the USA in 1984. From 1984 to 2009 he held academic positions at the University of New Mexico, where he advanced to the rank of Full Professor in 1990. Concurrently, he also held a position of Full Professor at the University of Hawaii at Manoa in 1989 and 1990. Kucharz returned to the Jagiellonian University as a visiting professor in 2009. Since 2010 he has been Full Professor at the Jagiellonian University. He met the requirements of the Polish higher education system, obtaining his habilitation in 2008 and the Polish title of professor in 2010. Professor Kucharz has travelled widely and visited many research institutes and universities in Austria, Brazil, Canada, Chile, France, Germany, Italy, Japan, the Netherlands, Spain, Switzerland, and the United Kingdom. He held visiting positions in the above-mentioned countries for a total of over 7 years.

In 2019, Professor Kucharz was elected to the Polish Academy of Sciences. In addition to the above, his honors include the Polish Mathematical Society Prize for Young Mathematicians (1979), the Presidential Lectureship at the University of New Mexico (1988–1990), the Efroymson Award at the University of New Mexico (1994, 1995), the Prime Minister of Poland Award for Scientific Achievements (2018), the Stefan Banach Prize of the Polish Mathematical Society (2019), the Jagiellonian Laurel (2020), and the Nicolaus Copernicus Prize of the Polish Academy of Arts and Sciences (2020), as well as election to the Polish Academy of Arts and Sciences (2022).

Professor Kucharz is the author or coauthor of over 150 scientific papers. He contributed to the development of several areas of mathematics, including algebraic and analytic geometry, singularity theory, complex analysis, and commutative algebra. He works on questions of central interest and importance, his solutions regularly demonstrate originality of his approaches and his results appear in the most prestigious mathematical journals. He is best known for his work on the borderline between real algebraic geometry and topology. In collaboration with Jacek Bochnak, he obtained significant results on real algebraic morphisms, algebraic cycles, and algebraic vector bundles, developing along the way several important methods that have proved to be indispensable in the works of other mathematicians. Kucharz was the first researcher to draw the attention of real algebraic geometers to the study of continuous rational maps between real algebraic varieties. Since then this line of research has led to the development of regulous geometry as an independent subfield of real algebraic geometry, showing that a very slight weakening of algebraicity implies a major change in the scope of the theory. The results of Kucharz in regulous geometry, some obtained in collaboration with János Kollár and Krzysztof Kurdyka, are surprising and contain a wealth of new ideas. Professor Kucharz presented, jointly with Professor Krzysztof Kurdyka from the University of Savoie Mont Blanc, an invited lecture at the International Congress of Mathematicians 2018 in Rio de Janeiro.

Since 2009 Professor Wojciech Kucharz has been an active participant in the Analytic and Algebraic Geometry Conference organized by the Faculty of Mathematics and Computer Science of the University of Łódź.

It is no secret for Kucharz's friends that he loves opera and likes to read biographies, and that his favorite physical activities are hiking in the mountains and swimming in warm seas.

PREPARED BY KAMIL RUSEK

Professor Tadeusz Winiarski

Analytic and Algebraic Geometry 4
Łódź University Press 2022, 17–18
DOI: https://doi.org/10.18778/8331-092-3.02

TADEUSZ WINIARSKI – SCIENTIFIC BIOGRAPHY

Tadeusz Winiarski was born on September 10, 1940. He studied mathematics at the Jagiellonian University in 1961–1966. After receiving his master's degree, he worked at the Institute of Mathematics of the Jagiellonian University, going through all stages of his scientific career from assistant to full professor until 2005, when he retired. He obtained his doctoral degree in 1971, and habilitation in 1982. In 1991, he obtained the title of professor. In 1986–1991 he was the Deputy Director of the Institute, and in 1991–2005 he headed the Chair of Analytic and Algebraic Geometry at the Institute of Mathematics. In the years 2001–2005 he was the President of the Kraków section of the Polish Mathematical Society. From 1997 he also worked for 13 years as full professor at the Institute of Mathematics of the Pedagogical University in Kraków.

Tadeusz Winiarski's research and scientific activity can be broadly divided into four parts (with non-empty intersections):

• The approximation theory of complex analytic functions. This initial part of scientific activity was related to the doctoral dissertation prepared under the supervision of Professor Józef Siciak. His first publication from 1970, "Approximation and interpolation of entire function", was extremely important. This work was inspiring and allowed for research in many directions by other mathematicians.

• Complex analytic and algebraic geometry. From around 1975, he began researching broadly understood analytic and algebraic geometry, starting to establish his own school at our Institute. Then a number of new interesting theories appear at the Institute. The combination of complex analysis with the theory of Hausdorff's measure permitted to see more insightfully the differences between analytic and algebraic sets and obtained, with K. Rusek, some new criteria for the algebraicity of analytic sets and the regularity of analytic mappings. This field also inspired some directions of research concerning polynomial automorphisms and the Jacobian Conjecture.

• Intersection theory in complex analytic geometry. This branch of mathematics appeared at the Institute of Mathematics around 1980 thanks to Tadeusz Winiarski.

His work “Total number of intersection of analytic sets” from 1981 opened new wide possibilities. In particular, it contained the famous “local Bézout theorem”. Together with R. Achilles and P. Tworzewski he also developed a complete and fully recognized theory of improper intersections of isolated analytical sets in the work “On improper isolated intersection in complex analytic geometry” from 1990.

• Gröbner’s bases theory. This branch of effective methods of analytic and algebraic geometry, unique in Poland, was developed by Professor Winiarski in cooperation with the University of Leipzig. The works from 1996 and 1998 “Reduction of everywhere convergent power series with respect to Gröbner bases” and “Intersections of sequences of ideals generated by polynomials”, with J. Apel, J. Stückrad and P. Tworzewski, were very important. Professor Winiarski’s attempt to spread Gröbner bases theory among Polish mathematicians, physicists and engineers resulted in publication in 2007, with M. Dumnicki, the only Polish book on this topic: “Bazy Gröbnera – efektywne metody w układach równań wielomianowych”.

Professor Winiarski developed two completely new branches of mathematics in the Institute of Mathematics of the Jagiellonian University: “Intersection theory in complex analytic geometry” and “Gröbner bases theory”. His scientific activity is characterized by an outstanding ability to cooperate with other mathematicians. The fruit of his many years of cooperation with foreign centers in Bochum, Leipzig, Osnabrück and Marseille is a series of joint publications. His scientific contacts are of great benefit to our environment, also because of their high efficiency.

The same feature of the scientific activity of Professor Winiarski was the reason for his exceptional success in the field of education of young scientists. At his seminars, there were never enough problems to solve for everyone. In the years 1984–2006 he was the supervisor of eight doctoral dissertations and currently he has 25 descendants.

Since 1983 Professor Tadeusz Winiarski has been an active participant (and even a leading figure) in the Analytic and Algebraic Geometry Conferences organized annually by the Faculty of Mathematics and Computer Science of the University of Łódź.

Prepared by Marcin Dumnicki, Kamil Rusek and Piotr Tworzewski

SCIENTIFIC ARTICLES

Analytic and Algebraic Geometry 4
Łódź University Press 2022, 21–51
DOI: https://doi.org/10.18778/8331-092-3.03

CONVEXIFYING OF POLYNOMIALS BY CONVEX FACTOR

ABDULLJABAR NAJI ABDULLAH, KLAUDIA ROSIAK, AND STANISŁAW SPODZIEJA

Abstract. Let $X \subset \mathbb{R}^n$ be a convex closed and semialgebraic set and $b : \mathbb{R}^n \to (0, +\infty)$ be a $\mathscr{C}^2$ class positive strongly convex function. Let f be a polynomial positive on X. If X is compact, we prove that there exists an exponent $N \geq 1$, such that for any $\xi \in X$, the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is strongly convex on X. If $X = \{\xi \in \mathbb{R}^n : f(\xi) \leq r\}$ is bounded we define a mapping $\kappa_N : X \ni \xi \mapsto \operatorname{argmin}_X \varphi_{N,\xi} \in \mathbb{R}^n$, where $\operatorname{argmin}_X \varphi_{N,\xi}$ is the unique point $x \in X$ at which $\varphi_{N,\xi}$ has a global minimum. We prove that κ_N is a mapping of class $\mathscr{C}^1$ of X onto $Y = \kappa_N(X) \subset X$ and that for any $\xi \in X$ the limit of the iterations $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ exists and belongs to the set $\sum_f$ of critical points of f. If additionally b is logarithmically strongly convex then κ_N is injective and it is defined on $\mathbb{R}^n$, provided f takes only positive values and the leading form of f is positive except of the origin. In the case $b(x) = \exp |x|^2$ and $f|_X$ has only one critical value we prove that the mapping $X \ni \xi \mapsto \lim_{\nu\to\infty} \kappa_N^\nu(\xi) \in \Sigma_f \cap X$ is continuous. Moreover, assuming that $\lim_{\nu\to\infty} \kappa_N^\nu(\xi) = 0$ we study convergence of the sequence of the spherical parts of $\kappa_N^\nu(\xi)$, $\nu \in \mathbb{N}$.

1. Introduction

The first goal of the paper is to study convexification of polynomial functions by a positive strongly convex function $b : \mathbb{R}^n \to \mathbb{R}$ of class $\mathscr{C}^k$, $k \geq 2$. More precisely, we will prove that (see Corollary 5.1): *If a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is positive on a compact and convex set $X \subset \mathbb{R}^n$, then there exists an effectively calculable positive integer N_0 such that for any $N \geq N_0$ the function*

$$\varphi_N(x) = b(x)^N f(x)$$

is strongly convex on X. The exponent N_0 depend on $R = \max\{|x| : x \in X\}$, $S = \max\{b(x) : x \in X\}$, the size of coefficients of the polynomial f and $m > 0$

2010 *Mathematics Subject Classification.* Primary 11E25, 12D15; Secondary 26B25.
Key words and phrases. Polynomial, semialgebraic set, convex function, strongly convex function, logarithmically strongly convex function, critical point.

such that $f(x) \geq m$ for $x \in X$. In case the polynomial f has integer coefficients finding N is fully effective (see Section 7).

A stronger version of the above result we give in Corollary 5.2; *there exists an integer N_0, which can be explicitly estimated, such that for any $N \geq N_0$ the functions*

$$\varphi_{N,\xi}(x) = b(x-\xi)^N f(x), \quad \xi \in X,$$

are strongly convex on X.

The second goal of the paper is to construct a mapping κ_N and investigate its properties. Namely, in the case when $X_{f\leq r} := \{x \in \mathbb{R}^n : f(x) \leq r\} \subset X$, where $r \in \mathbb{R}$ and X is a closed ball, we prove that *the mapping $\kappa_N : X_{f\leq r} \to X_{f\leq r}$ defined by*

$$\kappa_N(\xi) = \operatorname{argmin}_X \varphi_{N,\xi}$$

is of class $\mathscr{C}^{k-1}$ (see Lemma 4.2 and Corollary 5.6). Moreover, *it is a diffeomorphism of class $\mathscr{C}^{k-1}$ provided b is logarithmically strongly convex*, i.e., $\ln b$ is strongly convex (see Lemma 4.3 and Corollary 5.6). For a strongly convex function $g : Y \to \mathbb{R}$ on a closed and convex set Y the unique point $x_0 \in Y$ at which g has a global minimum on Y we denote by $\operatorname{argmin}_Y g$. In Theorem 4.8 we give some properties of the iterations κ_N^ν of the mapping κ_N and prove that: $\kappa_{N,*}(\xi) := \lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ *exists and belongs to the set $X_{f\leq r} \cap \Sigma_f$* of critical points of f in $X_{f\leq r}$. Note that the set of fixed points of κ_N is equal to $X_{f\leq r} \cap \Sigma_f$ (see Lemma 4.5).

Analogous results for unbounded sets we obtain in Section 6 under assumption that b is logarithmically strongly convex and that the *leading form* f_d of f (i.e., a homogeneous polynomial f_d such that $\deg(f - f_d) < \deg f$) satisfy

$$f_d(x) > 0 \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}. \tag{1.1}$$

In Section 8 we give some results on the convergence of the sequence $\kappa_N^\nu(\xi)$, provided $b(x) = \exp|x|^2$. We prove that there is a neighbourhood $U \subset \mathbb{R}^n$ of the set of points, where the function f takes the smallest value such that the mapping assigning to each point $\xi \in U$ the limit point $\kappa_{N,*}(\xi)$ of the proximal algorithm is continuous (see Proposition 8.17). Moreover, we prove that the sequence $\kappa_N^\nu|_U$ uniformly converges to $\kappa_{N,*}|_U$. Without the assumption on U, the assertion of Proposition 8.17 does not hold (see Remark 8.18). We also show that the curve connecting successively the points $\kappa_N^\nu(\xi)$, $\xi \in X$, defined by the formula (8.19), shows a number of properties similar to those of the trajectory of the gradient field $\frac{1}{2N}\nabla(\ln f)$ (see Section 8.2). At the end of the paper we consider the problem of convergence of the sequence of the spherical parts $\kappa_N^\nu(\xi)/|\kappa_N^\nu(\xi)|$ of the sequence $\kappa_N^\nu(\xi)$, provided $\kappa_N^\nu(\xi) \to 0$ as $\nu \to \infty$ (see Fact 8.21).

In the special case when $b(x) = 1 + |x|^2$, a similar results to Corollary 5.1 and Theorem 4.8 are known. In [5, Theorem 5.1] there was proved that: *If a polynomial $f : \mathbb{R}^n \to \mathbb{R}$ is positive on a compact and convex set $X \subset \mathbb{R}^n$, then there exists an effectively calculable positive integer N_0 such that for any integer $N \geq N_0$ the function*

$$\phi_N(x) = (1 + |x|^2)^N f(x)$$

is strongly convex on X. Moreover, a stronger version of [5, Theorem 5.1] was given in [5]; *there exists an effectively calculable positive integer* N_1 *such that for any integer* $N \geq N_1$ *the polynomials* $\phi_{N,\xi}(x) = (1 + |x - \xi|^2)^N f(x)$, $\xi \in X$, *are strongly convex on* X. This is a crucial fact for a construction of a proximal algorithm which for a given polynomial f, positive in the convex compact semialgebraic set X, produces a sequence $\xi_\nu \in X$ starting from an arbitrary point $\xi_0 \in X$, defined by induction: $\xi_\nu = \operatorname{argmin}_X \phi_{N,\xi_{\nu-1}}$. The sequence ξ_ν converges to a lower critical point of f on X (see [5, Theorem 7.5]), i.e., a point $a \in X$ for which there exists a neighborhood $\Omega \subset \mathbb{R}^n$ such that $\langle x - a, \nabla f(a) \rangle \geq 0$ for every $x \in X \cap \Omega$, where ∇f is the gradient of f in the Euclidean norm. In the case of non-compact closed convex set X, under the assumption (1.1) we have that: *if the polynomial* f *is positive on* X *then for any* $R > 0$ *there exists* N_R *such that for any* $\xi \in X$, $|\xi| \leq R$, $N > N_R$ *the polynomial* $\phi_{N,\xi}$ *is strongly convex on* X. Similar results to the above were obtained in [7] for the functions $\psi_{N,\xi}(x) := e^{N|x-\xi|^2} f(x)$ and $\Psi_{N,\xi}(x) := e^{e^{N|x-\xi|^2}} f(x)$.

2. Auxiliary results

2.1. **Convex functions.** Let $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$. The function f is called *convex* if the set X is convex and for any $x, y \in X$ and $0 < t < 1$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y).$$

If the above inequality holds with $<$ for $x \neq y$, the function is called *strictly convex.*

Let f be a real function of class $\mathscr{C}^2$ defined on a neighbourhood of a convex set $X \subset \mathbb{R}^n$.

Denote by $\partial_v f(x)$ the directional derivative of the function f in the direction of a vector $v \in \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$, and by $\partial_v^2 f(x)$ the second order derivative of f in the direction v at x. If $v = (0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is on the ith place, we write traditionally $\partial_v f = \frac{\partial f}{\partial x_i}$. Then the gradient $\nabla f : X \to \mathbb{R}^n$ of f is of the form

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right).$$

For any $a \in X$ and $v \in \mathbb{R}^n$ we put $I_{a,v} = \{t \in \mathbb{R} : a + tv \in X\}$. Obviously, the set $I_{a,v}$ is an interval or a single point. Recall some known facts (cf. [11]).

Fact 2.1. *The following conditions are equivalent:*

(a) *The function* f *is convex.*

(b) *For any vector* $v \in \mathbb{R}^n$ *and any* $a \in X$ *the function* $I_{a,v} \ni t \mapsto \partial_v f(a+tv) \in \mathbb{R}$ *is increasing.*

(c) *For any vector* $v \in \mathbb{R}^n$ *and any* $a \in X$ *we have* $\partial_v^2 f(a) \geq 0$.

Fact 2.2. *The following conditions are equivalent:*

(a) *The function f is strictly convex.*

(b) *For any vector $v \in \mathbb{R}^n$ of positive length and any $a \in X$ the function $I_{a,v} \ni t \mapsto \partial_v f(a+tv) \in \mathbb{R}$ is strictly increasing.*

(c) *The function f is convex and for any vector $v \in \mathbb{R}^n$ of positive length and any $a \in X$ the set $\{t \in I_{a,v} : \partial_v^2 f(a+tv) = 0\}$ is novhere dense in $I_{a,v}$, provided $I_{a,v}$ is an interval.*

A function $g : X \to \mathbb{R}$ is called *strongly convex* or *μ-strongly convex*, $\mu > 0$, if $X \subset \mathbb{R}^n$ is a convex set and for any $x, y \in X$ and $0 < t < 1$,

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y) - t(1-t)\frac{\mu}{2}|x-y|^2,$$

If additionally g is of class $\mathscr{C}^1$ then the above condition is equivalent to

$$g(y) \geq g(x) + \langle y - x, \nabla g(x) \rangle + \frac{\mu}{2}|y-x|^2 \quad \text{for } x, y \in X,$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$. Obviously, any strongly convex function is strictly convex and consequently, it is also convex.

Denote by S^{n-1} the unit sphere in $\mathbb{R}^n$, i.e., $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

Fact 2.3. *Let $\mu > 0$. The following conditions are equivalent:*

(a) *The function f is μ-strongly convex.*

(b) *For any vector $v \in S^{n-1}$ we have $\partial_v^2 f(x) \geq \mu$ at any point $x \in X$.*

(c) *For any $x \in X$ any eigenvalue of the Hessian matrix of f*

$$H(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]_{1 \leq i,j \leq n}$$

is bounded from below by μ.

Fact 2.4. *If $f : \mathbb{R}^n \to \mathbb{R}$ is a strongly convex function then* $\lim_{|x|\to\infty} f(x) = +\infty$.

If $f(x) > 0$ for $x \in X$, the function f we will call *logarithmically convex*, *logarithmically strictly convex* and *logarithmically μ-strongly convex* if $\ln f$ is convex, strictly convex and μ-strongly convex respectively.

Obviously for any μ-strongly convex function $a : \mathbb{R}^n \to \mathbb{R}$ the function $b = \exp a$ is logarithmically strongly convex, for instance $b(x) = \exp(|x|^2)$, $b(x) = \exp(\exp(|x|^2))$,..., are logarithmically strongly convex functions.

Fact 2.5. *If $b : \mathbb{R}^n \to \mathbb{R}$ is a logarithmically strongly convex function then b is also a strongly convex function.*

Proof. Indeed, for any $\beta \in \S^{n-1}$, we have

$$\partial_\beta^2(\ln b(x)) = \frac{b(x)\partial_\beta^2 b(x) - (\partial_\beta b(x))^2}{b(x)^2} \geq \mu \quad \text{for } x \in \mathbb{R}^n,$$

so,

$$\partial_\beta^2 b(x) \geq \eta b(x) + \frac{(\partial_\beta b(x))^2}{b(x)} \geq \mu b(x_0) > 0 \quad \text{for } x \in \mathbb{R}^n$$

and some $\mu > 0$, where $x_0 = \operatorname{argmin}_{\mathbb{R}^n} b$. □

2.2. **Gradient of convex functions.** Let f be a real function of class $\mathscr{C}^2$ defined in a neighbourhood of a convex set $X \subset \mathbb{R}^n$.

From Fact 2.2 we immediately obtain

Corollary 2.6. *If f is a strictly convex function, then the gradient*

$$\nabla f : X \ni x \mapsto \nabla f(x) \in \mathbb{R}^n$$

is injective.

Proof. Indeed, by Fact 2.2, for any $a, b \in X$, $a \neq b$, the function

$$\varphi : I_{a,b-a} \ni t \mapsto \partial_{b-a} f(a + t(b-a)) \in \mathbb{R}$$

is strictly increasing. Moreover, $0, 1 \in I_{a,b-a}$, so

$$\langle \nabla f(a), b-a \rangle = \varphi(0) < \varphi(1) = \langle \nabla f(b), b-a \rangle.$$

Consequently, $\nabla f(a) \neq \nabla f(b)$. □

From Corollary 2.6 we obtain

Corollary 2.7. *If f is an logarithmically strictly convex function, then the mapping*

$$\frac{1}{f}\nabla f : X \ni x \mapsto \frac{1}{f(x)}\nabla f(x) \in \mathbb{R}^n$$

is injective.

Proof. Indeed, by definition, $\ln f$ is strictly convex and $\nabla(\ln f) = \frac{1}{f}\nabla f$. So, Corollary 2.6 gives the assertion. □

Without assuming logarithmically strict convexity of the function f, the above corollary does not hold. This is demonstrated by the following example.

Example 2.8. *Let $f(x) = 1 + x^2$. Then $\frac{f'}{f}(x) = \frac{2x}{1+x^2}$ and obviously this function is not injective. Moreover, the function f is strongly convex.*

Lemma 2.9. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a μ-strongly convex function of class $\mathscr{C}^2$, let $x_0 = \operatorname{argmin}_{\mathbb{R}^n} b$ and let $X \subset \mathbb{R}^n$ be a convex and compact set. If $b(x) > 0$ for $x \in X$ and x_0 is an interior point of the set X then there exists $\varepsilon > 0$ such that*

(i) *the function b is an logarithmically strongly convex in the set $X_{x_0,\varepsilon} = \{x \in X : |x - x_0| \leq \varepsilon\}$.*

(ii) *the function $X_{x_0,\varepsilon} \ni x \mapsto \frac{1}{b(x)}\nabla b(x) \in \mathbb{R}^n$ is injective.*

(iii) *there exists $\delta > 0$ such that for any $x \in X$ such that $\frac{|\nabla b(x)|}{b(x)} < \delta$ we have $|x - x_0| < \varepsilon$.*

Proof. Since $b(x)>0$ for $x\in X$ and b is μ-strongly convex function, for any $x\in X$ and $\beta\in\mathbb{R}^n$, $|\beta|=1$ we have

$$\partial_\beta^2(\ln b)(x)=\frac{\partial_\beta^2 b(x)}{b(x)}-\left(\frac{\partial_\beta b(x)}{b(x)}\right)^2\geq\frac{\mu}{b(x)}-\left(\frac{\partial_\beta b(x)}{b(x)}\right)^2.$$

Since b is of class $\mathscr{C}^2$ and $\partial_\beta b(x_0)=0$ then there exists $\varepsilon>0$ fulfilling (i). The assertion (ii) immediately follows from (i) and Corollary 2.7. Taking

$$\delta=\min\left\{\varepsilon,\inf\left\{\frac{|\nabla b(x)|}{b(x)}:x\in X,\ |x-x_0|\geq\varepsilon\right\}\right\},$$

where $\inf\emptyset=+\infty$, we see that $\delta>0$ and deduce the assertion (iii). □

2.3. **Convexifying functions on compact sets.**

Fact 2.10. *If $b:\mathbb{R}^n\to\mathbb{R}$ is a function of class $\mathscr{C}^2$ such that for any compact and convex set $X\subset\mathbb{R}^n$ there exists $N_0\in\mathbb{N}$ such that for any $N\geq N_0$ the function $x\mapsto b^N(x)$ is strongly convex on X, then b is positive on $\mathbb{R}^n$.*

Proof. Take any compact and convex set $X\subset\mathbb{R}^n$ and let N_0 be such that for any $N\geq N_0$ the function $b^N(x)$ is strongly convex on X. Take $N\geq N_0$. Since b is of class $\mathscr{C}^2$, from Fact 2.3, for any vector $v\in S^{n-1}$ we have

$$\begin{aligned}\partial_v^2 b^N(x)&=N(N-1)b^{N-2}(x)(\partial_v b(x))^2+Nb^{N-1}(x)\partial_v^2 b(x)\\&=Nb^{N-2}(x)\left[(N-1)(\partial_v b(x))^2+b(x)\partial_v^2 b(x)\right]>0\quad\text{for } x\in X.\end{aligned}$$

So, $b(x)\neq 0$ for $x\in\mathbb{R}^n$. Hence, in view of continuity of the functions $x\mapsto b(x)$, $(x,v)\mapsto\partial_v b(x)$, $(x,v)\mapsto\partial_v^2 b(x)$, the Darboux property gives the assertion. □

Example 2.11. *Under assumptions of Fact 2.10 we cannot require that the function b is convex. For example for $b(x)=\sqrt[4]{1+|x|^2}$, $x\in\mathbb{R}^n$, the assertion of Fact 2.10 holds (see* [5, Theorem 5.1]*) but b is not convex. It can not be expected that* $\lim_{|x|\to\infty}b(x)=+\infty$*. For example, for the function $b(x)=\exp x$, $x\in\mathbb{R}$, the assertion of Fact 2.10 holds (see Lemma 3.1 in Section 5.1) but* $\lim_{x\to-\infty}b(x)=0$.

Fact 2.12. *If $b:\mathbb{R}^n\to\mathbb{R}$ is a function of class $\mathscr{C}^2$ such that for any compact and convex set $X\subset\mathbb{R}^n$ there exists $N_0\in\mathbb{N}$ such that for any $N\geq N_0$ the function $x\mapsto b^N(x)$ is logaritmically strongly convex on X, then b is also logarithmically strongly convex on any compact and convex set $X\subset\mathbb{R}^n$.*

Proof. Sine a logarithmically strongly convex function is also strongly convex, by Fact 2.10, the function b is positive on $\mathbb{R}^n$. Take any compact and convex set $X\subset\mathbb{R}^n$. Let N_0 be such that for any $N\geq N_0$ the function $b^N(x)$ is logarithmically strongly convex on X. Then for $N\geq N_0$ the function $\ln b^N(x)=N\ln b(x)$ is strongly convex on X. Consquently, b is logarithmically strongly convex on X. □

2.4. **Polynomials.** Let $f \in \mathbb{R}[x]$ be a polynomial in $x = (x_1, \dots, x_n)$ of the form

$$f = \sum_{j=0}^{d} \sum_{|\nu|=j} a_\nu x^\nu, \tag{2.1}$$

where $a_\nu \in \mathbb{R}$, $x^\nu = x_1^{\nu_1} \cdots x_n^{\nu_n}$ and $|\nu| = \nu_1 + \cdots + \nu_n$ for $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$ (we assume that $0 \in \mathbb{N}$). Assume that $d = \deg f$. Then $f = f_0 + \cdots + f_d$, where f_j is a homogeneous polynomial of degree j or zero, i.e.,

$$f_j := \sum_{|\nu|=j} a_\nu x^\nu, \quad 0 \le j \le d. \tag{2.2}$$

We will call The polynomial f_d the *leading form* of f. Obviously $\deg(f - f_d) < d$.

We set

$$\|f\| := \sum_{|\nu| \le d} |a_\nu|.$$

Then $||f_0|| = |a_0|$ and

$$||f|| = ||f_0|| + \cdots + ||f_d||.$$

Lemma 2.13. *Take any $\beta \in S^{n-1}$. Then for any $x \in \mathbb{R}^n$ we have*

$$|\partial_\beta f(x)| \le \sum_{j=1}^{d} j||f_j|||x|^{j-1}, \qquad |\partial_\beta^2 f(x)| \le \sum_{j=1}^{d} j(j-1)||f_j|||x|^{j-2}. \tag{2.3}$$

In particular if $|x| \ge 1$ then

$$|\partial_\beta f(x)| \le d||f|| \cdot |x|^{d-1}, \qquad |\partial_\beta^2 f(x)| \le d(d-1)||f|| \cdot |x|^{d-2}. \tag{2.4}$$

Proof. Let $\beta = (\beta_1, \dots, \beta_n)$. We have

$$\partial_\beta f(x) = \sum_{j=1}^{d} \sum_{|\nu|=j} a_\nu \partial_\beta x^\nu, \qquad \partial_\beta^2 f(x) = \sum_{j=2}^{d} \sum_{|\nu|=j} a_\nu \partial_\beta^2 x^\nu$$

Take any $\nu = (\nu_1, \cdots, \nu_n) \in \mathbb{N}^n$, $|\nu| = \nu_1 + \cdots + \nu_n = j$. Then

$$|\partial_\beta x^\nu| \le \sum_{k=1}^{n} \nu_k |x_1^{\nu_1} \cdots x_k^{\nu_k - 1} \cdots x_n^{\nu_n}| \le j|x|^{j-1}$$

and consequently,

$$|\partial_\beta^2 x^\nu| \le \sum_{k=1}^{n} \nu_k |\partial_\beta x_1^{\nu_1} \cdots x_k^{\nu_k - 1} \cdots x_n^{\nu_n}| \le j(j-1)|x|^{j-2}.$$

This gives (2.3). Consequently, for $|x| \ge 1$ we have

$$|\partial_\beta f(x)| \le \sum_{j=1}^{d} j|x|^{j-1} \sum_{|\nu|=j} |a_\nu| \le d|x|^{d-1}(||f_1|| + \cdots + ||f_d||) \le d|x|^{d-1} \cdot ||f||.$$

and

$$|\partial_\beta^2 f(x)| \le \sum_{j=2}^{d} j(j-1)|x|^{j-2} \sum_{|\nu|=j} |a_\nu| \le d(d-1)|x|^{d-2} \cdot ||f||,$$

which gives (2.4) and ends the proof. □

From Lemma 2.13 we immediately obtain

Corollary 2.14. *If* $\nabla f(0) = 0$ *then*

$$|\nabla f(x)| \le d\sqrt{n}\|f - f_0\| \cdot |x| \quad \text{for } |x| \le 1.$$

2.5. **Estimation of zeros of a polynomial.** Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1). Put $f_{d*} = \min_{|x|=1} f_d(x)$. Assume that $f_{d*} > 0$ and set

$$K_f(r) := 2\max\left\{ \left(\frac{||f_0|| + r}{f_{d*}}\right)^{1/d}, \max_{1\le j\le d-1} \left|\frac{||f_{d-j}||}{f_{d*}}\right|^{1/j} \right\} \quad \text{for } r > 0.$$

We put $K(f) := K_f(0)$.

Fact 2.15. *For any* $r \ge 0$*,*

$$\{x \in \mathbb{R}^n : f(x) \le r\} \subset \{x \in \mathbb{R}^n : |x| \le K_f(r)\}.$$

Proof. Under notations of Section 2.4,

$$|f_j(\theta)| \le ||f_j|| \quad \text{for} \quad \theta \in S^{n-1}.$$

Take any $x \in \mathbb{R}^n \setminus \{0\}$ and put $r = |x|$ and $\theta = \frac{1}{|x|}x$. Then $x = r\theta$, $r > 0$, $\theta \in S^{n-1}$ and $f(x)$ can be written in the form

$$f(x) = \sum_{j=0}^{d} f_j(\theta) r^j.$$

Since the number

$$2\max_{1\le j\le d} \left|\frac{f_{d-j}(\theta)}{f_0(\theta)}\right|^{1/j}$$

estimate from above the modul of any zero r of the polynomial $f_d(\theta)r^d + f_{d-1}(\theta)r^{d-1} + \cdots + f_0(\theta)$ in r, where $f_d(\theta) \ge f_{d*} > 0$, then the polynomial $f - r$ have no zeros $x \in \mathbb{R}^n$ such that $|x| > K_f(r)$. Since f have positive values for $x \in \mathbb{R}^n$ such that $|x|$ tends to infinity, then we obtain the assertion. □

3. Convexifying functions on compact sets

3.1. **Strongly convex functions.** Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ which is μ-strongly convex, $\mu > 0$, and takes only positive values.

Take any convex and compact set $X \subset \mathbb{R}^n$. Let

$$S := \max\{b(x) : x \in X\}.$$

Obviously $S>0$. Take any function $f:\mathbb{R}^n\to\mathbb{R}$ of class $\mathscr{C}^2$ which is positive on X. Let $m,D\in\mathbb{R}$ be a positive numbers such that

$$f(x)\geq m,\quad |\partial_\beta f(x)|\leq D,\quad |\partial_\beta^2 f(x)|\leq D\quad for\ x\in X \text{ and } \beta\in S^{n-1}.$$

Let

$$N(\mu,S,m,D):=\frac{S}{\mu}\left(\frac{D}{m}+\frac{D^2}{m^2}\right)+1.$$

The following lemma is a version of Lemma 49 from [13] by Klaudia Rosiak.

Lemma 3.1. *For any $N\geq N(\mu,S,m,D)$ the function $\varphi_N(x)=b^N(x)f(x)$ is strongly convex on the set X.*

Proof. Take any $N\geq N(\mu,S,m,D)$ and $x,\beta\in\mathbb{R}^n$, $|\beta|=1$. Then

$$\begin{aligned}\partial_\beta^2\varphi_N(x)=N(N-1)b^{N-2}(x)f(x)\left(\partial_\beta b(x)\right)^2+2Nb^{N-1}(x)\partial_\beta b(x)\partial_\beta f(x)\\ +Nb^{N-1}(x)f(x)\partial_\beta^2 b(x)+b^N(x)\partial_\beta^2 f(x).\end{aligned}$$

Since $b(x)>0$ for $x\in\mathbb{R}^n$, we have

$$\partial_\beta^2\varphi_N(x)=b^N(x)\Lambda(x),$$

where

$$\Lambda(x)=N(N-1)f(x)\left(\frac{\partial_\beta b(x)}{b(x)}\right)^2+2N\frac{\partial_\beta b(x)}{b(x)}\partial_\beta f(x)+\partial_\beta^2 f(x)+Nf(x)\frac{\partial_\beta^2 b(x)}{b(x)}.$$

Since f and b are functions of class $\mathscr{C}^2$, then φ is also class $\mathscr{C}^2$ and it suffices to prove that

$$\Lambda(x)>0\quad\text{for}\ \ x\in X. \tag{3.1}$$

Let now $x\in X$ and put $t=\frac{\partial_\beta b(x)}{b(x)}$. From the assumptions on f and b,

$$\Lambda(x)\geq N(N-1)m|t|^2-2ND|t|-D+Nm\frac{\mu}{S}.$$

The discriminant of the quadratic function in $|t|$ on the right hand of the above inequality is of the form

$$\begin{aligned}\Delta&=4N^2D^2-4N(N-1)m\left(-D+Nm\frac{\mu}{S}\right)\\ &=-\frac{4Nm^2\mu}{S}\left[N\left(N-1-\frac{S}{\mu}\frac{D}{m}-\frac{S}{\mu}\frac{D^2}{m^2}\right)+\frac{S}{\mu}\frac{D}{m}\right]\end{aligned}$$

So, for $N\geq N(\mu,S,m,D)$ we have $\Delta<0$ and consequently

$$N(N-1)m|t|^2-2ND|t|-D+Nm\frac{\mu}{S}>0\quad\text{for}\ \ t\in\mathbb{R}.$$

This gives (3.1) and ends the proof. $\square$

Let

$$S':=\max\{b(x-\xi):x,\xi\in X\}.$$

From Lemma 3.1 we immediately obtain

Corollary 3.2. *For any $N \geq N(\mu, S', m, D)$ and any $\xi \in X$ the function*

$$\varphi_{N,\xi}(x) = b^N(x-\xi)f(x) \tag{3.2}$$

is strongly convex on the set X.

Remark 3.3. *Let $b : \mathbb{R}^n \to \mathbb{R}$ be a μ-strongly convex function, $\mu > 0$, and let $X \subset \mathbb{R}^n$ be a compact and convex set. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ and let $D \in \mathbb{R}$ be a positive number such that*

$$|\partial_\beta^2 f(x)| \leq D \quad for\ x \in X\ and\ \beta \in \mathbb{R}^n,\ |\beta| = 1.$$

Then for any $\xi \in \mathbb{R}^n$ and

$$N > \frac{D}{\mu},$$

the function $\Psi_{N,\xi} : \mathbb{R}^n \to \mathbb{R}$ defined by $\Psi_{N,\xi}(x) = Nb(x-\xi) + f(x)$, $x \in \mathbb{R}^n$, is strongly convex on X (more precisely $(N\mu - D)$-strongly convex).

Indeed, take any $\xi \in \mathbb{R}^n$. Since $N\mu > D$ then for any $\beta \in \mathbb{R}^n$, $|\beta| = 1$ we have

$$\partial_\beta^2 \Psi_{N,\xi}(x) = N\partial_\beta^2 b(x-\xi) + \partial_\beta^2 f(x) \geq N\eta - D > D - D = 0 \quad \text{for } x \in X.$$

This gives the assertion.

3.2. **Logarithmically convex functions.** Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ which is logarithmically μ-strongly convex, $\mu > 0$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^2$ taking only positive values. Take any convex and compact set $X \subset \mathbb{R}^n$. Let $m, D \in \mathbb{R}$ be a positive numbers such that

$$f(x) \geq m, \quad |\partial_\beta f(x)| \leq D, \quad |\partial_\beta^2 f(x)| \leq D \quad for\ x \in X \text{ and } \beta \in S^{n-1}.$$

Let

$$N_{\exp}(\mu, m, D) := \frac{1}{\mu}\left(\frac{D}{m} + \frac{D^2}{m^2}\right).$$

Lemma 3.4. *For any $N > N_{\exp}(\mu, m, D)$ and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is logarithmically strongly convex on the set X.*

Proof. Take any $\xi \in \mathbb{R}^n$. Let $\psi_{N,\xi} = \ln \varphi_{N,\xi}$. Then

$$\psi_{N,\xi}(x) = N \ln b(x-\xi) + \ln f(x), \quad x \in \mathbb{R}^n$$

so for any $\beta \in S^{n-1}$, we have

$$\partial_\beta \psi_{N,\xi}(x) = N\partial_\beta(\ln b(x-\xi)) + \frac{\partial_\beta f(x)}{f(x)}, \quad x \in \mathbb{R}^n,$$

and

$$\partial_\beta^2 \psi_{N,\xi}(x) = N\partial_\beta^2(\ln b(x-\xi)) + \frac{f(x)\partial_\beta^2 f(x) - (\partial_\beta f(x))^2}{f(x)^2}, \quad x \in \mathbb{R}^n.$$

Consequently, for $N > N_{\exp}(\mu, m, D)$ and $x \in X$, we have

$$\partial_\beta^2 \psi_N(x) \geq N\mu - \frac{D}{m} - \frac{D^2}{m^2} > 0, \quad x \in X.$$

Since $\partial_\beta^2\psi_N$ is continuous and X is compact, we obtain the assertion. $\square$

4. Iterations of the mapping $\xi \mapsto \operatorname{argmin}\varphi_{N,\xi}$

Let $f:\mathbb{R}^n\to\mathbb{R}$ be a function of class $\mathscr{C}^k$, $k\geq 2$. Take any $r>0$ and assume that the set

$$X_{f\leq r} := \{x\in\mathbb{R}^n : f(x)\leq r\}$$

is bounded and nonempty. Let $R_{f\leq r}$ be the size of $X_{f\leq r}$, i.e.,

$$R_{f\leq r} := \sup\{|x| : x\in X_{f\leq r}\}.$$

Take any $R>R_{f\leq r}$ and put

$$B_R := \{x\in\mathbb{R}^n : |x|\leq R\}.$$

Since $X_{f\leq r}\neq\emptyset$, we have $R_{f\leq r}\geq 0$ and so, $R>0$.

Let $m_R, D_R\in\mathbb{R}$ be a positive numbers such that

$$f(x)\geq m_R,\quad |\partial_\beta f(x)|\leq D_R,\quad |\partial_\beta^2 f(x)|\leq D_R \quad\text{for}\quad x\in B_R,\quad \beta\in S^{n-1}. \tag{4.1}$$

Let $b:\mathbb{R}^n\to\mathbb{R}$ be a function of class $\mathscr{C}^k$, $k\geq 2$, which is μ-strongly convex, $\mu>0$, and takes only positive values, let (for simplicity of notations),

$$0 = \operatorname{argmin}_{\mathbb{R}^n} b, \tag{4.2}$$

and let

$$S'_{b,R} := \max\{b(x-\xi) : x,\xi\in B_R\}.$$

Let N be an integer number such that

$$N\geq N(\mu, S'_{b,R}, m_R, D_R). \tag{4.3}$$

By Corollary 3.2 for any $\xi\in B_R$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is strongly convex on the set B_R. Let $\kappa_N : B_R\to B_R$ be a mapping defined by

$$\kappa_N(\xi) := \operatorname{argmin}_{B_R}\varphi_{N,\xi}\in B_R \quad \text{for } \xi\in B_R. \tag{4.4}$$

Fact 4.1. $\kappa_N(X_{f\leq r})\subset X_{f\leq r}$.

Proof. Take any $\xi\in B_{f\leq r}$ and let $x=\kappa_N(\xi)$. Then $\varphi_{N,\xi}(x)\leq\varphi_{N,\xi}(\xi)$ and consequently, $b^N(x-\xi)f(x)\leq b^N(0)f(\xi)$. Since, by (4.2), $b(0)\leq b(x-\xi)$, we have $f(x)\leq f(\xi)$ which gives the assertion. $\square$

Lemma 4.2. *The function $\kappa_N|_{X_{f\leq r}}$ is of class $\mathscr{C}^{k-1}$.*

Proof. Take any $\xi\in X_{f\leq r}$. Observe that $x=\kappa_N(\xi)$ satisfies the following system of equations

$$\nabla\varphi_{N,\xi}(x) = 0. \tag{4.5}$$

Indeed, by the choice of R we have $\min\{f(x) : |x|=R\}>r$, so, $X_{f\leq r}\subset\operatorname{Int}B_R$ and by Fact 4.1, $\kappa_N(\xi)\in\operatorname{Int}B_R$. So, x satisfies (4.5). Since the Jacobian (with respect to x) of the system of equations is equal to the Hessian of $\varphi_{N,\xi}$ then the

Jacobian is nonzero at x, because the Hessian matrix has only positive eigenvalues. Then the Implicit function theorem gives the assertion. □

Lemma 4.3. *Let b be μ-logarithmically strongly convex function of class $\mathscr{C}^k$ and let $N > N_{\exp}(\mu, m_R, D_R)$. Then the mapping*

$$\kappa_N|_{X_{f\leq r}} : X_{f\leq r} \to \kappa_N(X_{f\leq r}) \tag{4.6}$$

is a diffeomorphism of class $\mathscr{C}^{k-1}$.

Proof. Take any $\xi \in X_{f\leq r}$ and let $x = \kappa_N(\xi)$. Since $b(x-\xi) > 0$, under notations of the proof of Lemma 4.2 from (4.5) we have

$$N\nabla b(x-\xi)f(x) + b(x-\xi)\nabla f(x) = 0, \tag{4.7}$$

where $\nabla b(x-\xi)$ is the gradient of $b(x-\xi)$ with respect to x. Then

$$\frac{1}{b(x-\xi)}\nabla b(x-\xi) + \frac{1}{Nf(x)}\nabla f(x) = 0. \tag{4.8}$$

So, by Corollary 2.7, the point ξ is uniquely determined by x. Consequently, the mapping (4.6) is bijective and consequently it is a homeomorphism, because $X_{f,R}$ is compact anf κ_N is continuous. To complete the proof it suffices to show that the mapping $\left(\kappa_N|_{X_{f\leq r}}\right)^{-1} : \kappa_N(X_{f\leq r}) \to X_{f\leq r}$ is of class $\mathscr{C}^{k-1}$. For this it is enough to show that the Jacobian with respect to ξ of the system of equations (4.8) is nonzero for any $(x,\xi) \in X_{f\leq r} \times \kappa_N(X_{f\leq r})$ such that $\xi = \kappa_N(x)$. This is due to the fact that the Jacobian with respect to ξ of the system of equations (4.8) is equal to the Hessian of $\ln(\varphi_{N,\xi})$, so it does not zero anywhere in the set $X_{f\leq r}$. Consequently $\left(\kappa_N|_{X_{f\leq r}}\right)^{-1}$ is a mapping of class $\mathscr{C}^{k-1}$, which completes the proof. □

From Lemma 2.9 we obtain an analogous lemma as Lemma 4.3 for strongly convex functions. Unfortunately, this version is not as effective as Lemma 4.3.

Lemma 4.4. *Let b be strongly convex function. Then there exists N_0 such that for any $N > N_0$, the mapping*

$$\kappa_N|_{X_{f\leq r}} : X_{f\leq r} \to \kappa_N(X_{f\leq r}) \tag{4.9}$$

is a diffeomorphism of class $\mathscr{C}^{k-1}$.

Proof. Let $\varepsilon > 0$ and $\delta > 0$ be as in Lemma 2.9. Then there exists N_1 such that for any $N \geq N_1$ we have

$$\frac{1}{Nf(x)}|\nabla f(x)| < \delta \quad \text{for } x \in X_{f\leq r}$$

Then for $N_0 = \max\left\{N_1, N(\mu, S'_{b,R}, m_R, D_R)\right\}$, analogously as in the proof of Lemma 4.3 (by using Lemma 2.9) we obtain the assertion. □

Let Σ_f be the set of critical points of f, i.e. $\Sigma_f := \{\xi \in \mathbb{R}^n : \nabla f(\xi) = 0\}$.

Lemma 4.5. *The set of fixed points of $\kappa_N|_{X_{f\le r}}$ is equal to $\Sigma_f \cap X_{f\le r}$.*

Proof. Let $\xi \in X_{f\le r}$ be a fixed point of $\kappa_N|_{X_{f\le r}}$. Then, analogously as in the proof of Lemma 4.3, we have $\nabla\varphi_{N,\xi}(\xi) = 0$, i.e.,

$$N\nabla b(0)f(\xi) + b(0)\nabla f(\xi) = 0.$$

Since b takes the minimal value at zero we have $\nabla b(0) = 0$, so $\nabla f(\xi) = 0$ and $\xi \in \Sigma_f$. Let now $\xi \in X_{f\le r}$ be a critical point of f and let $x = \kappa_N(\xi)$. Then x is the unique point in $X_{f\le r}$ for which $\nabla\varphi_{N,\xi}(x) = 0$. Since $\nabla\varphi_{N,\xi}(\xi) = 0$, we have $\xi = x$ and ξ is a fixed point of $\kappa_N|_{X_{f\le r}}$. □

Corollary 4.6. *If $\xi \in X_{f\le r} \setminus \Sigma_f$ and $x = \kappa_N(\xi)$, then*

$$\partial_{x-\xi} f(\xi + t(x-\xi)) = \langle \nabla f(\xi + t(x-\xi)), x-\xi\rangle < 0 \quad \textit{for } t \in [0,1], \tag{4.10}$$

$x \notin \Sigma_f$ and the function

$$f_{\xi,x} : [0,1] \ni t \mapsto f(\xi + t(x-\xi)) \in \mathbb{R}$$

is strictly decreasing. In particular, the sequence $f(\kappa_N^\nu(\xi))$, $\nu \in \mathbb{N}$, is strictly decreasing, the sequence $\kappa_N^\nu(\xi)$, $\nu = 0, 1, \ldots$, is injective and

$$\kappa_N^\nu(\xi) \notin \Sigma_f \quad \textit{for } \nu = 0, 1, \ldots.$$

Proof. Since $\xi \notin \Sigma_f$, by Lemma 4.5 we have $x \neq \xi$. Since x is the unique point of $X_{f\le r}$ at which $\varphi_{N,\xi}$ takes the minimal value in $X_{f\le r}$, then (4.7) holds, i.e., $N\nabla b(x-\xi)f(x) + b(x-\xi)\nabla f(x) = 0$. Since $x - \xi \neq 0$, we have $\nabla b(x-\xi) \neq 0$ and, so,

$$\nabla f(x) \neq 0. \tag{4.11}$$

Moreover, the function

$$[0,1] \ni t \mapsto \varphi_{N,\xi}(\xi + t(x-\xi)) \in \mathbb{R}$$

is strongly convex with the minimal value at 1, so it is strictly decreasing and its derivative have no zeroes in $(0,1)$. Consequently, for $\beta = \frac{x-\xi}{|x-\xi|}$ we have

$$\partial_\beta \varphi_{N,\xi}(\xi + t(x-\xi)) < 0 \quad \text{for } t \in (0,1).$$

On the other hand $\partial_\beta b(t(x-\xi)) > 0$ for $t \in (0,1]$ and

$$\partial_\beta \varphi_{N,\xi}(x) = Nb^{n-1}(x-\xi)\partial_\beta b(x-\xi)f(x) + b^N(x-\xi)\partial_\beta f(x),$$

so, $\partial_\beta f(\xi + t(x-\xi)) < 0$ and consequently (4.10) holds. In particular $x \notin \Sigma_f$. Moreover, the function $f_{\xi,x}$ is strictly decreasing. The particular part of the assertion is an easy consequence of the above. □

Remark 4.7. *If $\varphi_{N,\xi}$ is μ-strongly convex function then for any $\xi \in X_{f\le r}$,*

$$f(\xi) - f(\kappa_N(\xi)) \ge \frac{\mu}{2}|\xi - \kappa_N(\xi)|^2.$$

If additionally $\varphi_{N,\xi}$ is logarithmically μ-strongly convex then for any $\xi \in X_{f\le r}$,

$$\frac{f(\xi)}{f(\kappa_N(\xi))} \ge \exp\left(\frac{\mu}{2}|\xi - \kappa_N(\xi)|^2\right).$$

By using the idea from [5, Section 7] we obtain the following proximity algorithm for semialgebraic functions of class $\mathscr{C}^2$ on convex sets (cf [12]).

Theorem 4.8. *If $f:\mathbb{R}^n\to\mathbb{R}$ is a semialgebraic function of class $\mathscr{C}^2$ satisfying* (4.1) *and N satisfies* (4.3), *then for any $\xi\in X_{f\le r}$*

(a) *the limit point $\lim_{\nu\to\infty}\kappa_N^\nu(\xi)$ exists and belongs to $\Sigma_f\cap X_{f\le r}$.*

(b) *the series $\sum_{\nu=0}^\infty|\kappa_N^{\nu+1}(\xi)-\kappa_N^\nu(\xi)|$ is convergent.*

In particular the curve $\gamma_\xi:[0,+\infty)\to X_{f\le r}$ defined by

$$\gamma_\xi(t)=\kappa_N^\nu(\xi)+(t-k)(\kappa_N^{\nu+1}(\xi)-\kappa_N^\nu(\xi))\quad \textit{for } t\in[k,k+1)$$

has finite length and the function $f\circ\gamma_\xi:[0,+\infty)\to\mathbb{R}$ is decreasing. If additionally $\xi\notin\Sigma_f$ then the function $f\circ\gamma_\xi$ is strictly decreasing.

Proof. Take any $\xi\in X_{f\le r}$. The particular part of the assertion immediately follows from (b) and Corollary 4.6, so it suffices to prove (a) and (b).

Put $\xi_0=\xi$ and $\xi_{\nu+1}=\kappa_N^\nu(\xi_0)$ for $\nu=0,1,\dots$. Then $\xi_{\nu+1}=\kappa_N(\xi_\nu)$ for $\nu=0,1,\dots$.

We will quote a sketch of the reasoning used in [5] in the case $X=X_{f\le r}$ and $\xi_0\in X_{f\le r}$. In [5, Theorem 7.5], the assertion was obtained assuming that the function b is of the form $b(x)=1+|x|^2$. Obviously b is strongly convex. In this case we have that (see [5, Lemma 7.1])

$$|\xi_{\nu+1}-\xi_\nu|=\operatorname{dist}(\xi_\nu,f^{-1}(f(\xi_{\nu+1}))).\quad \nu=0,1,\dots \tag{4.12}$$

and the sequence $f(\xi_\nu)$ is decreasing (see [5, Lemma 7.2] and Corollary 4.6). By using the monotonity of the sequence $f(\xi_\nu)$ and the Comparison principle (see [5, Lemma 7.7]) we obtain that the series

$$\sum_{\nu=0}^\infty\operatorname{dist}(\xi_\nu,f^{-1}(f(\xi_{\nu+1}))) \tag{4.13}$$

is convergent. Then, by (4.12), the series

$$\sum_{\nu=0}^\infty|\xi_{\nu+1}-\xi_\nu| \tag{4.14}$$

is convergent and consequently the sequence ξ_ν tends to some ξ_*.

To prove that $\xi_*\in\Sigma_f$, observe that by analogously as in the proof of Lemma 4.3 we have (4.7), i.e.,

$$N\nabla b(\xi_{\nu+1}-\xi_\nu)f(\xi_{\nu+1})+b(\xi_{\nu+1}-\xi_\nu)\nabla f(\xi_{\nu+1})=0\quad\text{for }\nu=0,1,\dots.$$

Since $\nabla b(0)=0$ and ∇b is a Lipschitz mapping on $X_{f\le r}$, there exists $L>0$ such that $|\nabla b(\xi_{\nu+1}-\xi_\nu)-\nabla b(0)|\le L|\xi_{\nu+1}-\xi_\nu|$ for any ν, so,

$$|\nabla f(\xi_{\nu+1})|\le\frac{Nf(\xi_{\nu+1})}{b(\xi_{\nu+1}-\xi_\nu)}L|\xi_{\nu+1}-\xi_\nu|.$$

Hence, by convergence of the series (4.14), we obtain convergence of the series $\sum_{\nu=0}^{\infty} \nabla f(\xi_{\nu+1})$. Moreover, continuity of the gradient ∇f and the necessary condition for series convergence gives $\nabla f(\xi_*) = \lim_{\nu\to\infty} \nabla f(\xi_{\nu+1}) = 0$. This gives the assertion in the case $b(x) = 1 + |x|^2$. Note that the proof of the fact that $\xi_* \in \Sigma_f$ differs from the one in the article [5]. It was carried out without any assumptions about form of the function b, so we proved the assertion (a), provided (b) holds.

Let us return to the proof of the Theorem 4.8. It suffices to prove the part (b) of the assertion.

In the proof of convergence of the series (4.13) the form of the function b was not important, the proof consisted in the use of Comparison principle, semialgebraicity of the function f and monotonity of the sequence $f(\xi_\nu)$. Hence the series (4.13) is convergent. Therefore, taking into account the above considerations, it is enough to prove the convergence of the series (4.14). For this, it is sufficient to show that there is a constant $C > 0$ such that

$$|\xi_{\nu+1} - \xi_\nu| \leq C \operatorname{dist}(\xi_\nu, f^{-1}(f(\xi_{\nu+1}))), \quad \nu = 0, 1, \ldots \tag{4.15}$$

Let $a_\nu \in f^{-1}(f(\xi_\nu))$, $\nu = 1, 2, \ldots$, be such that

$$\operatorname{dist}(\xi_\nu, f^{-1}(f(\xi_{\nu+1})) = |\xi_\nu - a_{\nu+1}|.$$

Then by definition of ξ_ν,

$$b^N(\xi_{\nu+1} - \xi_\nu) f(\xi_{\nu+1}) \leq b^N(a_{\nu+1} - \xi_\nu) f(a_{\nu+1}).$$

Since $f(a_{\nu+1}) = f(\xi_{\nu+1}) > 0$, we have

$$b(\xi_{\nu+1} - \xi_\nu) \leq b(a_{\nu+1} - \xi_\nu).$$

By convergence of the series (4.13) we have $\lim_{\nu\to\infty}(a_{\nu+1} - \xi_\nu) = 0$, and consequently, $\lim_{\nu\to\infty}(\xi_{\nu+1} - \xi_\nu) = 0$, because the origin is the unique point at which the function b takes minimal value. Take the Taylor expansion of the function b at the origin (recal that $\nabla b(0) = 0$),

$$b(x) = b(0) + \frac{1}{2} x^T H_b(0) x + R_3(x),$$

where $H_b(0)$ is the Hessian matrix of b at 0 and $|R_3(x)| \leq M|x|^3$ in a neighbourhood U of the origin for some constant $M > 0$. One can assume that $a_{\nu+1} - \xi_\nu \in U$ and $\xi_{\nu+1} - \xi_\nu \in U$ for $\nu = 0.1. \ldots$. Then

$$\begin{aligned}(\xi_{\nu+1} - \xi_\nu)^T H_b(0)(\xi_{\nu+1} - \xi_\nu) - 2M|\xi_{\nu+1} - \xi_\nu|^3 \\ \leq (a_{\nu+1} - \xi_\nu)^T H_b(0)(a_{\nu+1} - \xi_\nu) + 2M|a_{\nu+1} - \xi_\nu|^3.\end{aligned}$$

Since the matrix $H_b(0)$ is symetric and positively defined, we have

$$|\xi_{\nu+1} - \xi_\nu|^2 \leq C|a_{\nu+1} - \xi_\nu|^2$$

for some constant $C > 0$. Hence $|\xi_{\nu+1} - \xi_\nu| \leq \sqrt{C}|a_{\nu+1} - \xi_\nu|$ which gives (4.15) and ends the proof. □

Remark 4.9. *In the proof of Theorem 4.8 we have shown, inter alia, that if ∇b is a Lipschitz mapping in $X_{f\leq r}$ with a constant $L>0$, then the jump $|\xi_{\nu+1}-\xi_\nu|$ can be estimated from below as follows*

$$|\xi_{\nu+1}-\xi_\nu| \geq \frac{|\nabla f(\xi_{\nu+1})| b(\xi_{\nu+1}-\xi_\nu|)}{LNf(\xi_{\nu+1})}.$$

5. Convexifying of polynomials

5.1. **Convexifying polynomials on compact sets.** Let $f\in\mathbb{R}[x]$ be a polynomial of form (2.1). Assume that $d=\deg f$. Let $X\subset\mathbb{R}^n$ be a compact and convex set.

For any $R>0$ we put

$$D_n(f,R) := \max\left\{ \sum_{j=1}^{d} j||f_j||R^{j-1}; \sum_{j=1}^{d} j(j-1)||f_j||R^{j-2} \right\}. \tag{5.1}$$

From Lemma 2.13, for any $\beta, x\in\mathbb{R}^n$ such that $|\beta|=1$ and $|x|\leq R$ we have

$$|\partial_\beta f(x)| \leq D_n(f,R), \quad |\partial_\beta^2 f(x)| \leq D_n(f,R). \tag{5.2}$$

Let $b:\mathbb{R}^n\to\mathbb{R}$ be a function of class $\mathscr{C}^2$ which is μ-strongly convex, $\mu>0$, and takes only positive values, and let

$$S := \max\{b(x) : x\in X\}.$$

Let

$$R := \max\{|x| : x\in X\}.$$

From Lemma 3.1 we obtain

Corollary 5.1. *If*

$$f(x)\geq m \quad \textit{for } x\in X \tag{5.3}$$

for some positive constant m, then for any

$$N > N(\mu, S, m, D_n(f,R))$$

the function $\varphi_N(x)=b^N(x)f(x)$ is strongly convex on the set X.

Let

$$S' := \max\{b(x-\xi) : x,\xi\in X\}.$$

From Corollary 3.2 we immediately obtain

Corollary 5.2. *If f satisfies* (5.3) *for some positive constant m, then for any $N\geq N(\mu,S',m,D_n(f,R))$ and any $\xi\in X$ the function*

$$\varphi_{N,\xi}(x) = b^N(x-\xi)f(x) \tag{5.4}$$

is strongly convex on the set X.

If additionally we assume that b is logarithmically μ-convex function then from Lemma 3.4 we obtain

Corollary 5.3. *If f satisfies (5.3) for some positive constant m, then for any $N > N_{\exp}(\mu, m, D_n(f,R))$ and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is logarithmically strongly convex on the set X.*

Set

$$||f||_R := \sum_{j=0}^{d} ||f_j|| R^j$$

Then $|f(x)| \le ||f||_R$ and $f(x) + ||f||_R \ge 0$ for $x \in \mathbb{R}^n,\ |x| \le R$. Let

$$\tilde{f} := f + ||f||_R + 1. \tag{5.5}$$

Then $\tilde{f}$ satisfies (5.3) with $m = 1$. So, from Corollaries 5.1 and 5.2 we obtain

Corollary 5.4. *For any*

$$N > N(\mu, S, 1, D_n(f,R) + ||f||_R + 1)$$

the function $\tilde{\varphi}_N(x) = b^N(x)\tilde{f}(x)$ is strongly convex on the set X. For any

$$N \ge N(\mu, S', 1, D_n(f,R) + ||f||_R + 1)$$

and any $\xi \in X$ the function $\tilde{\varphi}_{N,\xi}(x) = b^N(x-\xi)\tilde{f}(x)$ is strongly convex on the set X.

Analogously as in Corollary 5.4, from Corollary 5.3 we obtain

Corollary 5.5. *For any $N > N_{\exp}(\mu, 1, D_n(f,R) + ||f||_R + 1)$ and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)\tilde{f}(x)$ is logarithmically strongly convex on the set X.*

5.2. **Iteration of the mapping $\xi \mapsto \operatorname{argmin} \varphi_{N,\xi}$ for polynomials.** Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1). Assume that $f_{d*} > 0$. Take any $r > 0$ and $R > K_f(r)$ and assume that $X_{f \le r} \ne \emptyset$.

Let $b : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathscr{C}^k$, $k \ge 2$, which is μ-strongly convex, $\mu > 0$, and takes only positive values and the minimal value takes at the point $x = 0$, and let

$$S'_{b,R} := \max\{b(x-\xi) : x, \xi \in B_R\},$$

where $B_R = \{x \in \mathbb{R}^n : |x| \le R\}$.

Let N be an integer number such that

$$N \ge N(\mu, S'_{b,R}, 1, D_n(f,R)). \tag{5.6}$$

If $f(x) \ge 1$ for $x \in \mathbb{R}^n$, by Corollary 5.2 for any $\xi \in B_R$ the function $\varphi_{N,\xi}(x) = b^N(x-\xi)f(x)$ is strongly convex on the set B_R. Let $\kappa_N : B_R \to B_R$ be a mapping defined by (4.4). So, from Lemmas 4.2, 4.3, 4.5 and Theorem 4.8 we obtain

Corollary 5.6. *If $f_{d*} > 0$, $f(x) \geq 1$ for $x \in \mathbb{R}^n$ and N meets the inequality* (5.6) *then:*

(a) $\kappa_N(X_{f\leq r}) \subset X_{f\leq r}$.

(b) *the function $\kappa_N|_{X_{f\leq r}}$ is of class $\mathscr{C}^{k-1}$.*

(c) *the set of fixed points of $\kappa_N|_{X_{f\leq r}}$ is equal to $\Sigma_f \cap X_{f\leq r}$.*

(d) *for any $\xi \in X_{f\leq r}$ the limit point $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ exists and belongs to Σ_f.*

If additionally b is a logarithmically μ-strongly convex function and

$$N > N_{\exp}(\mu, 1, D_n(f, R))$$

then

(e) *the mapping $\kappa_N|_{X_{f\leq r}} : X_{f\leq r} \to \kappa_N(X_{f\leq r})$ is a diffeomorphism of class $\mathscr{C}^{k-1}$.*

Remark 5.7. *To construct a mapping κ_N satisfying the assertion of Corollary 5.6 we do not have to assume that the polynomial f takes only positive values. It is sufficient to assume that $f_{d*} > 0$. More precisely, let $\tilde{f}$ be of form* (5.5)*, i.e., $\tilde{f} = f + ||f||_R + 1$. Then $\tilde{f}(x) \geq 1$ for $|x| \leq R$ and the polynomials f and $\tilde{f}$ have the same set of critical points. So, for suitable N, the mapping $\tilde{\kappa}_N(\xi) = \operatorname{argmin}_{B_R} b^N(x-\xi)\tilde{f}(x) \in B_R$ for $\xi \in B_R$ satisfy the assertion of Corollary 5.6.*

6. Logarithmically convexification of polynomials on unbounded sets

Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1), i.e.,

$$f(x) = \sum_{j=0}^{d} \sum_{|\nu|=j} a_\nu x^\nu. \tag{6.1}$$

Assume that $d = \deg f$. Then $f = f_0 + \cdots + f_d$, where f_j is a homogeneous polynomial of degree j or zero. Assume that $f_{d*} > 0$. Recall that $f_{d*} = \min_{|x|=1} f_d(x)$. Then $\|f\| \geq \|f_d\| \geq f_{d*}$. Put

$$\mathbb{K}(f) := \frac{2\|f\|}{f_{d*}}$$

and

$$c(f) := f_{d*} - \sum_{j=0}^{d-1} \mathbb{K}(f)^{j-d} ||f_j||.$$

Obviously, $\mathbb{K}(f) \geq 2$.

We will need the following lemma (see [7, Lemma 3.4]).

Lemma 6.1. *If $d = \deg f > 0$ and $f_{d*} > 0$, then $c(f) > 0$ and $f(x) \geq c(f)|x|^d$ for any $x \in \mathbb{R}^n$ such that $|x| \geq \mathbb{K}(f)$.*

From Lemmas 6.1 and 2.13 we immediately obtain

Corollary 6.2. *Let f be a polynomial of form* (6.1) *such that $f_{d*} > 0$. Take any $\beta \in S^{n-1}$. Then for any $x \in \mathbb{R}^n$, $|x| \geq \mathbb{K}(f)$ we have*

$$\frac{|\partial_\beta f(x)|}{f(x)} \leq \frac{d||f||}{c(f)} \cdot |x|^{-1} \leq \frac{d||f||}{2c(f)} \tag{6.2}$$

and

$$\frac{|\partial_\beta^2 f(x)|}{f(x)} \leq \frac{d(d-1)||f||}{c(f)} \cdot |x|^{-2} \leq \frac{d(d-1)||f||}{4c(f)}. \tag{6.3}$$

For a polynomial f of form (6.1) such that $f_{d*} > 0$ and for any $\mu > 0$ we put

$$N_{\exp,\infty}(\mu, f) := \frac{d(d+1)||f||}{4\mu c(f)}.$$

Obviously, for any $\beta, x \in \mathbb{R}^n$ such that $|\beta| = 1$ and $|x| \leq R$ we have

$$|\partial_\beta f(x)| \leq D_n(f, R), \quad |\partial_\beta^2 f(x)| \leq D_n(f, R), \tag{6.4}$$

where $D_n(f, R)$ is defined by (5.1).

Let $b : \mathbb{R}^n \to \mathbb{R}$ be logarithmically μ-strongly convex function of class $\mathscr{C}^k$, $k \geq 2$. From Lemma 3.4 and Corollaty 6.2 we obtain

Corollary 6.3. *Let $X \subset \mathbb{R}^n$ be a closed and convex set. Let f be a polynomial of form* (6.1) *such that $f_{d*} > 0$ and there exists $m > 0$ such that $f(x) \geq m$ for $x \in X$. For any*

$$N > \max\{N_{\exp}(\mu, m, D_n(f, \mathbb{K}(f))), N_{\exp,\infty}(\mu, f)\}$$

and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x - \xi) f(x)$ is logarithmically strongly convex on the set X.

Proof. Take any $\xi \in \mathbb{R}^n$. Let $\psi_{N,\xi}(x) = \ln \varphi_{N,\xi}(x)$. Take any $\beta \in S^{n-1}$. By Lemma 3.4 there exists $\mu_1 > 0$ such that $\partial_\beta^2 \psi_{N,\xi}(x) \geq \mu_1$ for $x \in X$, $|x| \leq \mathbb{K}(f)$. Since

$$\partial_\beta^2 \psi_{N,\xi}(x) = N \partial_\beta^2 (\ln b(x - \xi)) + \frac{\partial_\beta^2 f(x)}{f(x)} - \left(\frac{\partial_\beta f(x)}{f(x)}\right)^2, \quad x \in \mathbb{R}^n,$$

then by Corollary 6.2 there exists $\mu_2 > 0$ such that $\partial_\beta^2 \psi_{N,\xi}(x) \geq \mu_2$ for $x \in X$, $|x| \geq \mathbb{K}(f)$. Consequently, $\partial_\beta^2 \psi_{N,\xi}(x) \geq \min\{\mu_1, \mu_2\} > 0$ for $x \in X$. □

From Corollary 6.3 we obtain

Corollary 6.4. *Let $f \in \mathbb{R}[x]$ be a polynomial of form* (6.1). *If $f_{d*} > 0$ and $f(x) \geq m$ for $x \in \mathbb{R}^n$ and some constant $m > 0$, then for any*

$$N > \max\{N_{\exp}(\mu, m, D_n(f, \mathbb{K}(f))), N_{\exp,\infty}(\mu, f)\}$$

and any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}(x) = b^N(x - \xi) f(x)$ is logarithymically strongly convex on $\mathbb{R}^n$ and the mapping $\kappa_N : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\kappa_N(\xi) = \operatorname{argmin}_{\mathbb{R}^n} \varphi_{N,\xi} \in \mathbb{R}^n \quad \textit{for } \xi \in \mathbb{R}^n,$$

is a diffeomorphism of class $\mathscr{C}^{k-1}$. Moreover, for any $\xi \in \mathbb{R}^n$ the limit point $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ exists and belongs to Σ_f.

Proof. By Corollary 6.3 for any $\xi \in \mathbb{R}^n$ the function $\varphi_{N,\xi}$ is logarithmically strongly convex on $\mathbb{R}^n$. So, $\operatorname{argmin}_{\mathbb{R}^n} \varphi_{N,\xi}$ is a critical point of $\varphi_{N,\xi}$ and consequently by analogous argument as in the proof of Theorem 4.8 we obtain the assertion. □

Remark 6.5. *To determine the diffeomorphism, the successive iterations which converge to the critical points of the polynomial f, we do not have to assume that all values of f are positive. It is enough to assume that $f_{d*} > 0$ and take $R = K_f$ and $\tilde{f} = f + ||f||_R + 1$ (see Remark 5.7).*

7. Polynomials with integer coefficients

For applications of the above results it is important to estimate the numbers f_{d*}, $m = \min\{f(x) : x \in X\}$ and $R = \max\{|x| : x \in X\}$ for a polynomial f and a compact and convex set $X \subset \mathbb{R}^n$. In the case when f and polynomials describing X have integer coefficients the above numbers can be effectively estimated. More precisely, let $X \subset \mathbb{R}^n$, $n \geq 2$, be a compact semialgebraic set of the form

$$X = \{x \in \mathbb{R}^n : g_1(x) = 0, \dots, g_l(x) = 0, g_{l+1}(x) \geq 0, \dots, g_k(x) \geq 0\}, \tag{7.1}$$

where $g_1, \dots, g_k \in \mathbb{Z}[x]$. Under the above notations G. Jeronimo, D. Perrucci, E. Tsigaridas in [3] proved that

Theorem 7.1. *Let $f, g_1, \dots, g_k \in \mathbb{Z}[x]$ be polynomials with degrees bound by an even integer d and coefficients of absolute values at most H, and let $\tilde{H} = \max\{H, 2n + 2k\}$. If $f(x) > 0$ for $x \in X$ and X of form (7.1) is compact, then*

$$f(x) \geq \left(2^{4-\frac{n}{2}}\tilde{H}d^n\right)^{-n2^nd^n} \quad \text{for } x \in X.$$

From Theorem 7.1 we immediately obtain

Corollary 7.2. *Let $f \in \mathbb{Z}[x]$ be a homogeneous polynomial with degree bound by an even integer d and coefficients of absolute values at most H, and let $\tilde{H} = \max\{H, 2n + 2\}$. If $f(x) > 0$ for $|x| = 1$. Then*

$$f(x) \geq \left(2^{4-\frac{n}{2}}\tilde{H}d^n\right)^{-n2^nd^n} \quad \text{for } |x| = 1.$$

From Theorems 7.1 we immediately obtain (see [7, Theorem 2.7])

Theorem 7.3. *Let $X \subset \mathbb{R}^n$ be a compact and convex semialgebraic set of form (7.1) and let $f, g_1, \dots, g_k \in \mathbb{Z}[x]$ be polynomials with degrees bound by an even integer d and coefficients of absolute values at most H. Set*

$$\mathfrak{b}(n, d, H, k) = \left(2^{4-\frac{n}{2}} \max\{H, 2n + 2k\}d^n\right)^{-n2^nd^n}$$

and

$$R = \sqrt{\left[\mathfrak{b}(n + 1, \max\{d, 4\}, H, k + 2)\right]^{-1} - 1}, \quad m = \mathfrak{b}(n, d, H, k).$$

Then

$$\max\{|x| : x \in X\} \le R. \tag{7.2}$$

8. The mapping κ_N for $b(x) = \exp|x|^2$

From an IT point of view, it is important to know how fast κ_N^ν converges to its limit. One of the problems that arises here is whether the sequence converges along any direction, that is, whether the spherical part of the sequence (in the polar coordinates) has a limit. It seems to be quite a difficult problem and the methods of solving the gradient conjecture of Rene Thom's used in [4] should be applied. This leads to R. Thom's discrete hypothesis: Does $\kappa_N^\nu/|\kappa_N^\nu|$ have a limit when $\nu \to \infty$. We immediately encounter a difficulty here. While in the case of the gradient field trajectory, the Darboux property holds, it is not the case in the discrete case. We will show in a relatively simple example what are similarities and what are differences in the case of the trajectory and in the case of the sequence.

Let $f \in \mathbb{R}[x]$ be a polynomial of the form

$$f(x) = f_0 + f_k(x) + \cdots + f_d(x), \tag{8.1}$$

where f_j is a homogeneous polynomial of degree j or zero for $j = 0, k, \ldots, d$, $k > 1$, and $f_k \ne 0$, $f_d \ne 0$. Recall that $f_{d*} = \min_{|x|=1} f_d(x)$. Assume that $f_{d*} > 0$ and

$$f(x) \ge 1 \quad \text{for } x \in \mathbb{R}^n. \tag{8.2}$$

Let $g_N : \mathbb{R}^n \to \mathbb{R}$, $N > 0$, be a function defined by

$$g_N(x) := \frac{1}{2N} \ln f(x), \quad x \in \mathbb{R}^n. \tag{8.3}$$

We will assume that

$$b(x) = \exp|x|^2, \qquad x \in \mathbb{R}^n. \tag{8.4}$$

Fact 8.1. *The function b is logarithmically 2-strongly convex in $\mathbb{R}^n$ of class $\mathscr{C}^\infty$. Moreover, $\nabla b^N(x) = 2Nb^N(x) \cdot x$ for $x \in \mathbb{R}^n$.*

Take notations and assumptions from Section 5.2. Let $S'_{b,R} = e^{4R^2}$ and

$$N \ge N(2, S'_{b,R}, 1, D_n(f, R)). \tag{8.5}$$

By Corollary 5.2 the function $\varphi_{N,\xi}(x)$, $\xi \in X_{f\le r}$, is strongly convex on the convex hull of the set $X_{f\le r}$ and the mapping κ_N defined by (4.4) is well defined. By Facts 5.6 and 8.1, analogously as in the proof of Lemma 4.3, from (4.7) we have

Fact 8.2. *The mapping $\kappa_N : X_{f\le r} \to \kappa_N(X_{f\le r})$ is the inverse of*

$$\kappa_N(X_{f\le r}) \ni x \mapsto x + \frac{1}{2Nf(x)}\nabla f(x) \in X_{f\le r}\,, \tag{8.6}$$

so it is an analytic and semialgebraic mapping, i.e., it is a Nash mapping.

Since $\frac{1}{2Nf(x)}\nabla f(x) = \nabla g_N(x)$, so putting $g = g_N$, from Fact 8.2 we have

Fact 8.3. *The Jacobian matrix $J(\kappa_N)$ of κ_N is of the form*

$$J(\kappa_N(\xi)) = (I + H(g)(\kappa_N(\xi)))^{-1},$$

where I is the $n \times n$ unit matrix.

By Fact 8.3 we see that $J(\kappa_N(\xi))$ is a symmetric matrix. So, we have the following corollary suggested by Krzysztof Kurdyka.

Corollary 8.4. *The mapping $\kappa_N : X_{f\le r} \to \kappa_N(X_{f\le r})$ is the gradient of an analytic function $F : X_{f\le r} \to \mathbb{R}$. Moreover, $\xi = \kappa_N(\xi) + \nabla g(\kappa_N(\xi))$ and*

$$\nabla\left(F(\xi) - \frac{|\xi|^2}{2}\right) = -\nabla g(\kappa_N(\xi)).$$

Since we assumed (8.2), from Corollary 6.4 we immediately obtain

Corollary 8.5. *Let $R = K_f$. Assume that $f_{d*} > 0$ and let*

$$N > \max\{N_{\exp}(\mu, 1, D_n(f, \mathbb{K}(f)), N_{\exp,\infty}(\mu, f)\}.$$

Thn the mapping $\kappa_N : \mathbb{R}^m \to \mathbb{R}^m$ is an analytic diffeomorphism. Moreover, for any $\xi \in \mathbb{R}^n$ the limit point $\lim_{\nu\to\infty} \kappa_N^\nu(\xi)$ exists and belongs to $\Sigma_f \cap X_{f\le r}$.

Let $\omega_0 : X_{f\le r} \ni \xi \mapsto \xi \in X_{f\le r}$ be the identity mapping and let $\omega_\nu : X_{f\le r} \to X_{f\le r}$ be mappings defined by

$$\omega_{\nu+1} = \kappa_N(\omega_\nu) \quad \text{for } \nu \ge 0$$

By Fact 5.6 we have that $\omega_\nu(\xi) \in X_{f\le r}$ for any $\xi \in X_{f\le r}$ and $\nu = 1, 2, \ldots$, so the mappings ω_ν are well defined. Obviously $\omega_\nu = \kappa_N^\nu$ for $\nu = 0, 1, \ldots$.

8.1. **Some properties of the sequence $\omega_\nu = \kappa_N^\nu$.** Take any $\xi \in X_{f\le r}$. By [5, Lemma 7.1] (cf., (4.12)),

$$|\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| = \operatorname{dist}(\omega_\nu(\xi), f^{-1}(f(\omega_{\nu+1}(\xi)))), \quad \nu = 0, 1, \ldots, \tag{8.7}$$

and by Theorem 4.8, the sequence

$$\omega_\nu(\xi) \quad \text{has a limit point } \omega_*(\xi) \in \Sigma_f \cap X_{f\le r}, \tag{8.8}$$

the series

$$\sum_{\nu=0}^{\infty} |\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \quad \text{is convergent} \tag{8.9}$$

and the sequence

$$f(\omega_\nu(\xi)) \quad \text{is decreasing.} \tag{8.10}$$

From Lemma 4.5 and Corollary 4.6 we have

Fact 8.6. *The sequence $\omega_\nu(\xi)$ is constant if and only if $\xi \in X_{f\le r} \cap \Sigma_f$. Moreover, for $\xi \in X_{f\le r} \setminus \Sigma_f$ the sequence $\omega_\nu(\xi)$ is injective and $\omega_\nu(\xi) \ne \omega_*(\xi)$ for any ν.*

By Fact 8.2 (or by Fact 8.1, analogously as in the proof of Lemma 4.3, from (4.7)) we have

$$\omega_{\nu+1}(\xi) - \omega_\nu(\xi) = -\frac{1}{2Nf(\omega_{\nu+1}(\xi))}\nabla f(\omega_{\nu+1}(\xi)), \quad \nu \in \mathbb{N}. \tag{8.11}$$

In particular, by (8.9), the series

$$\sum_{\nu=0}^{\infty} |\nabla f(\omega_\nu(\xi))| \quad \text{is convergent.} \tag{8.12}$$

Remark 8.7. *By the Bochnak-Łojasiewicz inequality (see* [2]*),*

$$|f(x) - f(\omega_*(\xi))| \le C|\nabla f(x)||x - \omega_*(\xi)| \tag{BŁ}$$

in a neighbourhood in $\mathbb{R}^n$ of the point $\omega_*(\xi)$ for some positive constant C, so from (8.12) *we obtain that the series*

$$\sum_{\nu=0}^{\infty} \frac{f(\omega_\nu(\xi)) - f(\omega_*(\xi))}{|\omega_\nu(\xi) - \omega_*(\xi)|} \quad \textit{is convergent,}$$

provided $\xi \notin \Sigma_f$.

Remark 8.8. *By the Łojasiewicz gradient inequality (see* [9, 10]*)*

$$|f(x) - f(\omega_*(\xi))|^\varrho \le C|\nabla f(x)| \tag{Ł1}$$

in a neighbourhood in $\mathbb{R}^n$ of the set $f^{-1}(f(\omega_*(\xi)))$ for some constants $0 < \varrho < 1$ and $C > 0$, we have that the series

$$\sum_{\nu=0}^{\infty} (f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^\varrho \quad \textit{is convergent.} \tag{8.13}$$

Note that the Łojasiewicz gradient inequality (Ł1) *was proved in a neighbourhood of a point. Since the set $f^{-1}(f(\omega_*(\xi)))$ is compact, we easily get this inequality around it.*

By the *global Łojasiewicz inequality*:

$$|f(x) - f(y)| \ge C\left(\frac{\operatorname{dist}(x, f^{-1}(f(y)))}{1+|x|^2}\right)^{d(6d-3)^{n-1}} \quad \text{for } x \in \mathbb{R}^n, \tag{Ł2}$$

under fixed y for some positive constant C and $d = \deg f$ (see [6, Corollary 10]), we have

Fact 8.9. *For any neughbourhood $U \subset \mathbb{R}^n$ of the set $f^{-1}(f(\omega_*(\xi)))$ there exists $\varepsilon > 0$ such that*

$$\{x \in \mathbb{R}^n : |f(x) - f(\omega_*(\xi))| < \varepsilon\} \subset U.$$

Moreover, if $f(\omega_{\nu_0}(\xi)) - f(\omega_*(\xi)) < \varepsilon$ then $f(\omega_\nu(\xi)) - f(\omega_*(\xi)) < \varepsilon$ and $\omega_\nu(\xi) \in U$ for any $\nu \ge \nu_0$.

From (8.7), (8.9) and [6, Theorem 1] we obtain

Fact 8.10. *Let C and ϱ be as in* (Ł1). *Then there exists $\delta > 0$ such that for any $\xi \in X_{f\leq r}$ such that $|\omega_\nu(\xi) - \omega_*(\xi)| < \delta$ we have*

$$(8.14)\quad |\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \leq \frac{1}{C(1-\varrho)}\left[(f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^{1-\varrho} - (f(\omega_{\nu+1}(\xi)) - f(\omega_*(\xi)))^{1-\varrho}\right],$$

in particular, there exists ν_0 such that for any $\nu \geq \nu_0$,

$$(8.15)\qquad \operatorname{dist}(\omega_\nu(\xi), f^{-1}(f(\omega_*(\xi)))) \leq \frac{1}{C(1-\varrho)}(f(\omega_\nu(\omega)) - f(\omega_*(\xi)))^{1-\varrho}.$$

Proof. Indeed, for $\omega_\nu(\xi)$ sufficiently close to the origin, from [6, Theorem 1] (more specifically from the proof of this theorem) and (8.7) we obtain (8.14). Since $\lim_{\nu\to\infty}(f(\omega_\nu(\xi)) - f(\omega_*(\xi))) = 0$ and $1 - \varrho > 0$, then

$$\sum_{k=\nu}^{\infty}\left[(f(\omega_k(\xi)) - f(\omega_*(\xi)))^{1-\varrho} - (f(\omega_{k+1}(\xi)) - f(\omega_*(\xi)))^{1-\varrho}\right] = (f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^{1-\varrho}.$$

By (8.8), there exists ν_0 such that foe any $k \geq \nu_0$ the point $\omega_k(\xi)$ is sufficiently close to $\omega_*(\xi)$. So, by (8.7) and (8.9) we have $\operatorname{dist}(\omega_\nu(\xi), f^{-1}(f(\omega_*(\xi)))) \leq \sum_{k=\nu}^{\infty}|\omega_{k+1}(\xi) - \omega_k(\xi)|$. Consequently, the above and (8.14) gives (8.15). ☐

Remark 8.11. *Let $\xi \in X_{f\leq r}$ Take any $\varepsilon > 0$. If N satisfy* (8.5) *and additionally*

$$(8.16)\qquad N \geq \frac{d\sqrt{n}}{2\varepsilon}\|f - f_0\|,$$

then there exists ν_0 such that for any $\nu \geq \nu_0$,

$$|\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \leq \varepsilon|\omega_{\nu+1}(\xi)|.$$

Indeed, by (8.11) *and Corollary 2.14 there exists ν_0 such that for any $\nu \geq \nu_0$,*

$$|\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \leq \frac{d\sqrt{n}}{2Nf(\xi_{\nu+1})}\|f - f_0\| \cdot |\omega_{\nu+1}(\xi)| \leq \frac{d\sqrt{n}}{2N}\|f - f_0\| \cdot |\omega_{\nu+1}(\xi)|.$$

So, (8.16) *givs the assertion.*

Remark 8.12. *By Remark 4.7, there exists $\mu > 0$ such that,*

$$(8.17)\qquad f(\omega_\nu(\xi)) - f(\omega_{\nu+1}(\xi)) \geq \mu|\omega_\nu(\xi) - \omega_{\nu+1}(\xi)|^2 \quad \textit{for any } \nu.$$

Under additional assumption that $0 \in \mathbb{R}^n$ is an isolated singularity of f, there exist positive constants C, α such that

$$(8.18)\qquad |\nabla f(x)| \geq C|x|^\alpha \qquad \text{in a neighbourhood of the origin.}$$

The smallest exponent α is called the *Łojasiewicz exponent of the gradient at the origin* and denoted by $\mathcal{L}_0(\nabla f)$. It is known that $\mathcal{L}_0(\nabla f) \leq (d-1)(6d-9)^{n-1}$, where $d = \deg f$ (see [6, Remark 4]) and (8.18) holds with $\alpha = \mathcal{L}_0(\nabla f)$. Then (8.12) goves that the convergence rate of the sequence $\omega_\nu(\xi)$ is quite fast. Namely, we have the following fact.

Fact 8.13. *Take any $\xi_0 \in X_{f\le r}\setminus\Sigma_f$ and let $\xi_\nu = \omega_\nu(\xi_0)$ for $\nu = 0, 1, \dots,$. Assume that $\omega_*(\xi_0) = 0$. If the origin is an isolated singularity of f then the series*

$$\sum_{\nu=0}^{\infty} |\xi_\nu|^\alpha \qquad \text{is convergent,}$$

where $\alpha = (d-1)(6d-9)^{n-1}$ and $d = \deg f$.

8.2. **Some curves with properties similar to trajectories of the gradient field.** Take any $\xi_0 \in X_{f\le r}\setminus\Sigma_f$ and let $\xi_\nu = \omega_\nu(\xi_0)$ for $\nu = 1, 2, \dots$.

Take a curve $\gamma_{\xi_0} : [0, +\infty) \to X_{f\le r}$ defined by

$$\gamma_{\xi_0}(t) = \xi_\nu + (t-k)(\xi_{\nu+1} - \xi_\nu) \quad \text{for } t \in [\nu, \nu+1). \tag{8.19}$$

The curve γ_{ξ_0} has several similarities to the trajectory of a gradient field. Namely, it has the following properties (see Theorem 4.8 and (8.11)):

Fact 8.14. (i) *The curve γ_{ξ_0} has finite length equal to $\sum_{\nu=0}^{\infty} |\xi_{\nu+1} - \xi_\nu|$.*

(ii) *The function $f \circ \gamma_{\xi_0} : [0, +\infty) \to \mathbb{R}$ is strictly decreasing (recall that we assumed that $\xi_0 \notin \Sigma_f$).*

(iii) *For $t \in (\nu, \nu+1)$, $\nu = 0, 1, \dots$ we have*

$$\gamma'(t) = \xi_{\nu+1} - \xi_\nu = -\frac{1}{2Nf(\xi_{\nu+1})} \nabla f(\xi_{\nu+1}).$$

Condition (iii) does not mean that $\gamma'(t) = -\frac{1}{2Nf(\gamma(t))}\nabla f(\gamma(t))$. This is one of the difficulties in studies of ξ_ν, which does not exist in gradient field trajectory studies.

These curves have another similarity to the trajectories of gradient fields. Namely, we have the following fact.

Proposition 8.15. *Let $0 \in \operatorname{Int} X_{f\le r}$ and let $f(0)$ be the minimal value of f. Then for any $\varepsilon > 0$ there exists $f(0) < \delta < r$ such that for any $\xi_0 \in X_{f\le\delta}$ the length of the curve γ_{ξ_0} does not exceed ε.*

Proof. Let $C > 0$ and $0 < \varrho < 1$ be as in (Ł1). Assume that (Ł1) holds in a meighbourhood U of $f^{-1}(f(0))$, i.e.,

$$|f(x) - f(0)|^\varrho \le C|\nabla f(x)| \quad \text{for } x \in U. \tag{8.20}$$

From Fact 8.9 there exists $c > 0$ such that

$$\{x \in \mathbb{R}^n : (f(x) - f(0))^{1-\varrho} < 2c\} \subset U$$

and $f(0)$ is the unique critical value of $f|_U$.

Take any maximal solution (to the right) $\gamma : [0, \beta) \to U \setminus f^{-1}(f(0))$ of the system of equations

$$x' = -\frac{\nabla f(x)}{|\nabla f(x)|} \qquad \text{in } U \setminus f^{-1}(f(0)). \tag{8.21}$$

From (8.20) we obtain the following *Kurdyka Łojasiewicz inequality* (cf., [6, Proposition 1]):

$$|\nabla(f-f(0))^{1-\varrho}(x)| \geq (1-\varrho)C \quad \text{for } x \in U \setminus f^{-1}(f(0)).$$

Hence it follows that

$$((f-f(0))^{1-\varrho}\circ\gamma)' = -|\nabla(f-f(0))^{1-\varrho})\circ\gamma| \leq -(1-\varrho)C \quad \text{for } x \in U \setminus f^{-1}(f(0))$$

(cf., the proof of [6, Theorem 1]). Consequently, $(f-f(0))^{1-\varrho}\circ\gamma$ and $f\circ\gamma$ are decreasing functions and for any $0 \leq s_1 < s_2$,

$$\begin{aligned}(f-f(0))^{1-\varrho}(\gamma(s_1)) - (f-f(0))^{1-\varrho}(\gamma(s_2)) &= (s_1-s_2)((f-f(0))^{1-\varrho}\circ\gamma)'(t)\\ &\geq (s_2-s_1)(1-\varrho)C.\end{aligned}$$

Since $s_2 - s_1$ is equal to the length of $\gamma|_{[s_1,s_2]}$, we have

$$\text{length}\,\gamma|_{[s_1.s_2]} \leq (f-f(0))^{1-\varrho}(\gamma(s_1)) - (f-f(0))^{1-\varrho}(\gamma(s_2)). \tag{8.22}$$

From the above, for any $s_1 \in [0,\beta)$ we obtain that the length of $\gamma|_{[s_1.\beta)}$ does not exceed $(f-f(0))^{1-\varrho}(\gamma(s_1))$. So, under assumption $(f(\gamma(s))-f(0))^{1-\varrho} < c$ we obtain that the trajectory $\gamma|_{[s_1.\beta)}$ cannot come out of the set U and, consequently, must have a limit point in the set $f^{-1}(f(0))$. This gives that any maximal solution to the right $\gamma : [0,\beta) \to U \setminus f^{-1}(f(0))$ of the system of equations (8.21) with initial condition $(f(\gamma(0))-f(0))^{1-\varrho} < c$ runs in the set $U \setminus f^{-1}(f(0))$ and intersects at exactly one point each level $f^{-1}(y)$, $f(0) < y < f(\gamma(0))$.

Take any $\varepsilon > 0$. Without loss of generality we may assume that $\varepsilon < c$. Put

$$\delta = f(0) + c^{1/(1-\varrho)}.$$

Now suppose that $f(\xi_0) < \delta$. Then $(f(\xi_0)-f(0))^{1-\varrho} < c$ and $(f(\xi_\nu)-f(0))^{1-\varrho} < c$ for any ν (see (8.10)). Take the solution $\gamma : [0,\beta) \to U \setminus f^{-1}(f(0))$ of (8.21) such that $\gamma(0) = \xi_\nu$. By the above there exists $s_1 > 0$ such that $f(\gamma(s_1)) = f(\xi_{\nu+1})$ and by (8.22) and (8.7),

$$|\xi_{\nu+1}-\xi_\nu| \leq \text{length}\,\gamma|_{[0,s_1]} \leq (f-f(0))^{1-\varrho}(\xi_\nu) - (f-f(0))^{1-\varrho}(\xi_{\nu+1}).$$

Since $\lim_{\nu\to\infty}(f(\xi_\nu)-f(0)) = 0$ and $1-\varrho > 0$, then

$$\sum_{k=\nu}^{\infty}\left[(f(\xi_k)-f(0))^{1-\varrho} - (f(\xi_{k+1})-f(0))^{1-\varrho}\right] = (f(\xi_\nu)-f(0))^{1-\varrho}.$$

From this and Fact 8.14 (i) we obtain that the length of γ_{ξ_0} does not exceed ε. $\square$

From Proposition 8.15 and from the proof of this proposition we immediately obtain

Corollary 8.16. *Let $0 \in \operatorname{Int} X_{f\leq r}$ and let $f(0)$ be the minimal value of f. Then for any $\varepsilon > 0$ there exists $f(0) < \delta < r$ such that for any $\xi \in X_{f\leq\delta}$,*

$$|\omega_\nu(\xi) - \omega_*(\xi)| < \varepsilon \quad \textit{for any } \nu.$$

8.3. **Uniform convergence of the sequence ω_ν.** We will show that the sequence of mappings ω_ν has some property similar to a property of the flow of gradient field (cf., [8, 10], see also subsection 8.2).

Proposition 8.17. *Let $0 \in \operatorname{Int} X_{f\leq r}$ and let $f(0)$ be the minimal value of f. Then there exists $f(0) < \delta < r$ such that the sequence ω_ν uniformly convergents to ω_* in the set $U = X_{f\leq\delta}$. In particular the mapping $\omega_* : U \to U \cap \Sigma_f$ is continuous and $\omega_*(\xi) = \xi$ for $\xi \in U \cap \Sigma_f$, i.e., ω_* is a deformation retraction and the set $U \cap \Sigma_f$ is a retract of U.*

Proof. Let C, ϱ be as in (Ł1). Assume that (Ł1) is fulfild in the set $U = X_{f\leq\delta}$ for some $f(0) < \delta < r$ and that the assertiin of Proposition 8.15 holds for any $\xi \in U$.

By the assumption that $f(0)$ is minimal value of f we have that $f(\omega_*(\xi)) = f(0)$ for $\xi \in U$, so, it is a continuous function. Let $0 < \varrho < 1$ and $C > 0$ be constants fulfilling (Ł1) in Remark 8.8. From Corollary 5.6 (b) we see that

$$(f \circ \omega_\nu - f \circ \omega_*)^{1-\varrho} : U \mapsto \mathbb{R}$$

is a sequence of continuous functions and by (8.10) it is decreasing. Obviously, $\lim_{\nu\to\infty}(f \circ \omega_\nu - f \circ \omega_*) = 0$ is a continuous function. So, by Dini's theorem the sequence

$$(f \circ \omega_\nu - f \circ \omega_*)^{1-\varrho} \quad \text{tends uniformly to 0 on } U. \tag{8.23}$$

By the choice of δ, analogously as in the proof of Proposition 8.15, for any $\xi \in U$ we obtain that (cf., Fact 8.10)

$$\begin{aligned}
|\omega_\nu(\xi) - \omega_*(\xi)| &\leq \sum_{k=\nu}^{\infty} |\omega_{\nu+1}(\xi) - \omega_\nu(\xi)| \\
&\leq \frac{1}{C(1-\varrho)} \sum_{k=\nu}^{\infty} \left[(f(\omega_k(\xi)) - f(\omega_*(\xi)))^{1-\varrho} - (f(\omega_{k+1}(\xi)) - f(\omega_*(\xi)))^{1-\varrho}\right] \\
&\qquad = \frac{1}{C(1-\varrho)}(f(\omega_\nu(\xi)) - f(\omega_*(\xi)))^{1-\varrho}.
\end{aligned}$$

This and (8.23) gives the assertion. □

Remark 8.18. *Without assuming that $f(0)$ is the smallest value of the function, the assertion of Proposition 8.17 does not hold. Namely, if the set $X_{f\leq r}$ is connected, and f has at least two critical values in $X_{f\leq r}$, we easily get a contradiction.*

8.4. **Gradient of a polynomial in the polar coordinates.** Let $f \in \mathbb{R}[x]$ be a polynomial of form (2.1). Then f can be written as

$$f(x) = \sum_{j=0}^{d} f_j\left(\frac{1}{|x|}x\right)|x|^j, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Denote

$$r = r(x) = |x| \quad \text{and} \quad \theta = \theta(x) = \frac{1}{|x|}x \quad \text{for } x \neq 0.$$

Then $x = r\theta$, $r > 0$, $\theta \in S^{n-1}$ and f can be written in the polar coordinates

$$f(x) = f(r\theta) = \sum_{j=0}^{d} f_j(\theta) r^j, \quad x \neq 0, \tag{8.24}$$

and

$$\nabla f(x) = \partial_r f(r\theta)\theta + \nabla' f(r\theta), \tag{8.25}$$

where

$$\partial_r f(r\theta) = \frac{\langle \nabla f(r\theta), r\theta\rangle}{r} = \partial_\theta f(r\theta) = \frac{\partial f(r\theta)}{\partial r} = \sum_{j=1}^{d} j f_j(\theta) r^{j-1} \tag{8.26}$$

and

$$\nabla' f(r\theta) = \nabla f(r\theta) - \partial_r f(r\theta)\theta.$$

Obviously,

$$\langle \nabla' f(r\theta), \theta\rangle = 0 \quad \text{for } x = r\theta \neq 0$$

and

$$\nabla' f(x) = \nabla f(x) - \frac{\langle \nabla f(x), x\rangle}{|x|^2} x \quad \text{for } x \neq 0.$$

The vector $\partial_r f(r\theta)\theta$ is called *the radial part* of the gradient $\nabla f(x)$ and $\nabla' f(r\theta)$ – *the spherical part* of $\nabla f(x)$.

From the definition of $\nabla' f$ we immediately obtain the following remark.

Remark 8.19. *Let $e_1, \dots, e_n$ be the standard basis of the linear space $\mathbb{R}^n$, i.e., $e_j = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is on the jth place. Take any $x \in \mathbb{R}^n$, $x \neq 0$. Put*

$$\alpha_j = \frac{\langle e_j, x\rangle}{|x|^2} x, \qquad v_j = e_j - \alpha_j \qquad \textit{for } j = 1, \dots, n.$$

Then $|v_j| = 1 - \frac{x_j^2}{|x|^2}$,

$$v_j = \left(-\frac{x_1 x_j}{|x|^2}, \dots, -\frac{x_{j-1} x_j}{|x|^2}, 1 - \frac{x_j^2}{|x|^2}, -\frac{x_{j+1} x_j}{|x|^2}, \dots, -\frac{x_n x_j}{|x|^2} \right), \quad j = 1, \dots, n$$

and

$$\nabla' f(x) = \sum_{j=1}^{n} \langle \nabla f(x), v_j\rangle v_j.$$

8.5. **Spherical part of the sequence** $\omega_\nu = \kappa_N^\nu$**.** In view of the results of Section 8.2, there is a problem of the convergence of the spherical part of the sequence $\omega_\nu(\xi) = \kappa_N^\nu(\xi)$. We will consider this problem under assumption that $\xi_\nu = \omega_\nu(\xi) \to 0$ as $\nu \to \infty$.

In [4], the key role is played by the sets

$$W_\varepsilon = \{x \in \mathbb{R}^n \setminus \{0\} : \varepsilon|\nabla' f(x)| \leq |\partial_r f(x)|\}, \quad \varepsilon > 0,$$

where $\nabla' f(x)$ is the spherical and $\partial_r f(x)\frac{x}{|x|}$ – the radial part of the gradient $\nabla f(x)$. One of the most important properties was the behavior of the gradient field trajectory when crossing the boundary of such a set (and properties of the so called controlling function). More precisely, the trajectory of the gradient field must run through this set from a certain point and must not leave it. In a discrete case, a sequence can jump into or out of that set without crossing its boundary.

In order for the method from [4] to be applied in a discrete case, the following conjecture would have to hold. Take any $\xi_0 \in X_{f \leq r} \setminus \Sigma_f$ and let $\xi_\nu = \omega_\nu(\xi_0)$ for $\nu = 0, 1, \ldots,$. Assume that $\omega_*(\xi_0) = 0$.

Conjecture 8.20. *There exists a constant* $\varepsilon > 0$ *and* ν_0 *such that for any* $\nu \geq \nu_0$

$$\varepsilon|\xi_{\nu+1} - \xi_\nu| \leq ||\xi_{\nu+1}| - |\xi_\nu||,$$

equivalently, $\varepsilon|\nabla f(\xi_\nu)| \leq |\partial_r f(\xi_\nu)|$*, i.e.,* $\xi_\nu = \omega_\nu(\xi) \in W_\varepsilon$*.*

With fairly strong assumptions, we get that the limit of the spherical part of the sequence ξ_ν exists. Namely, the following fact holds.

Fact 8.21. *Assume that* $f_k(\theta) > 0$ *for* $\theta \in S^{n-1}$*. Then there is the following limit*

$$\lim_{\nu \to \infty} \frac{1}{|\xi_\nu|}\xi_\nu. \tag{8.27}$$

Moreover, the sequence $|\xi_\nu|$ *is strictly decreasing from a certain point.*

Proof. Let's write f in a polar coordinates:

$$f(r\theta) = f_0 + r^k f_k(\theta) + \cdots + r^d f_d(\theta),$$

where $r > 0$ and $\theta \in S^{n-1}$. Then

$$\partial_r f(r\theta) = kr^{k-1} f_k(\theta) + \cdots + dr^{d-1} f_d(\theta), \tag{8.28}$$

$$\nabla' f(r\theta) = r^k \nabla' f_k(\theta) + \cdots + r^d \nabla' f_d(\theta).$$

and

$$\nabla f(r\theta) = \partial_r f(r\theta)\theta + \nabla' f(r\theta)$$

So, from the assumption that $f_k(\theta) > 0$ for $\theta \in S^{n-1}$, there exists $r_0 > 0$ such that

$$\frac{|\nabla' f(r\theta)|}{r} \leq C_1 r^{k-1} \leq C_2 \partial_r f(r\theta) \leq C_3 |\nabla f(r\theta)| \quad \text{for } 0 < r < r_0 \tag{8.29}$$

and some positive constants C_1, C_2, C_3.

Take the curve $\gamma = \gamma_{\xi_0}$ defined by (8.19). By Fact 8.14 (ii) the function $f \circ \gamma$ is strictly decreasing, so we have

$$\gamma(t) \neq 0 \quad \text{for } t \in [0, +\infty),$$

and we may write γ in the polar coordinates $\gamma(t) = r_\gamma(t)\theta_\gamma(t)$, $r_\gamma(t) = |\gamma(t)| > 0$ amd $\theta_\gamma(t) \in S^{n-1}$. Then

$$\gamma'(t) = r'_\gamma(t)\theta_\gamma(t) + r_\gamma(t)\theta'_\gamma(t) \quad \text{for } t \in (\nu, \nu+1),\ \ \nu = 0, 1, \dots,$$

and $\langle \theta_\gamma(t), \theta'_\gamma(t) \rangle = 0$ for $t \in [0, +\infty) \setminus \mathbb{Z}$. On the other hand, by Fact 8.14 (iii),

$$\gamma'(t) = \xi_{\nu+1} - \xi_\nu = -\frac{1}{2Nf(\xi_{\nu+1})} \nabla f(\xi_{\nu+1}) \quad \text{for } t \in (\nu, \nu+1), \quad \nu = 0, 1, \dots.$$

Since $\nabla f(\xi_{\nu+1}) \neq 0$, we may write $\nabla f(\xi_{\nu+1})$ in the polar coordinates, so

$$r'_\gamma(t) = -\frac{1}{2Nf(\xi_{\nu+1})} \partial_r f(\xi_{\nu+1}) \quad \text{for } t \in (\nu, \nu+1), \quad \nu = 0, 1, \dots$$

and

$$r_\gamma(t)\theta'_\gamma(t) = -\frac{1}{2Nf(\xi_{\nu+1})} \nabla' f(\xi_{\nu+1}) \quad \text{for } t \in (\nu, \nu+1), \quad \nu = 0, 1, \dots.$$

So, by (8.28),

$$r'_\gamma(t) = -\frac{1}{2Nf(\xi_{\nu+1})} \left[k r_\gamma^{k-1}(t) f_k(\theta_\gamma(t)) + \cdots + d r_\gamma^{d-1}(t) f_d(\theta_\gamma(t)) \right]$$

for $t \in (\nu, \nu+1)$, $\nu = 0, 1, \dots$. By the assumption that $f_k(\theta) > 0$ for $\theta \in S^{n-1}$ we see that the derivative has a fixed sign $r'_\gamma(t) < 0$ for sufficiently large $t \notin \mathbb{Z}$. Consequently, the sequence $|\xi_\nu|$ is strictly decreasing from a certain point and we proved the moreover part of the assertion. Moreover, $r_\gamma(t)$ tends to 0 as $t \to \infty$ and by (8.29),

$$|\theta'(t)| = \frac{|\nabla' f(\xi_{\nu+1})|}{2Nf(\xi_{\nu+1}) r_\gamma(t)} \leq \frac{C_2}{2Nf(\xi_{\nu+1})} \partial_r f(\xi_{\nu+1}) = C_2 |r'_\gamma(t)| \leq C_3 |\gamma'(t)|$$

for $t \in (\nu, \nu+1)$, and sufficiently large ν. Snce the curve γ has z finite length (see Fact 8.14 (i)), then the above gives that θ_γ also has a finite length. Consequently te curve $\Theta : [0, +\infty) \to \mathbb{R}^n$ defined by

$$\Theta(t) := \theta(\xi_\nu) + (t - k)\left[\theta(\xi_{\nu+1}) - \theta(\xi_\nu)\right] \quad \text{for } t \in [\nu, \nu+1),\ \nu = 0, 1, \dots$$

has a finite length. This gives that exists a limit $\lim_{\nu \to \infty} \theta(\xi_\nu)$ i.e., the limit (8.27) exists. □

Remark 8.22. *In fact, in the proof of Fact 8.21 we proved that W_ε, for some $\varepsilon > 0$, is equal to some neighbourhood of the origin. Moreover, under the assumption of this fact, we proved that $\varepsilon|\nabla' f(r\theta)| \leq r|\partial_r f(r\theta)$ in a neighbourhood of the origin. This is a stronger condition than the fact that ξ_ν belongs to the set W_ϵ. It seems that it is not enough to prove Conjecture 8.20 to show that $\theta(\xi_\nu)$ converges. The sequence ξ_ν should satisfy $\varepsilon|\nabla' f(\xi_\nu)| \leq |\xi_\nu|^\alpha |\partial_r f(\xi_\nu)$ for some positive constants ε and α.*

References

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Analytic and Algebraic Geometry 4
Łódź University Press 2019, 53–60
DOI: https://doi.org/10.18778/8331-092-3.04

ZARISKI MULTIPLICITY CONJECTURE IN FAMILIES OF NON-DEGENERATE SINGULARITIES

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Abstract. We give a new, elementary proof of the Zariski multiplicity conjecture in μ-constant families of non-degenerate singularities.

1. Introduction

One of the most longstanding conjectures in singularity theory is the Zariski multiplicity conjecture [Zar71] that if two hypersurface singularities are embedded topologically equivalent, then their multiplicities (= the orders of reduced functions defining them), are the same. By definition, two hypersurface singularities, not necessarily isolated, $(V, 0) = (V(f), 0)$ and $(W, 0) = (V(\widetilde{f}), 0)$ in $\mathbb{C}^n$ are *embedded topologically equivalent* iff there exists a homeomorphism $\Phi : (U, 0) \to (U', 0)$ of small neighbourhoods of the origin in $\mathbb{C}^n$ which transforms $V \cap U$ onto $W \cap U'$. Fifty years have passed, but the conjecture has been solved only in a few special cases. Information on these particular results one can find in the survey by Ch. Eyral [Eyr07] (up to 2007) and in the monograph by the same author [Eyr16]. One of the general results is that the conjecture is true for plane curve singularities (because in this case, we have complete, discrete characteristics of embedded topological types, for instance so-called Puiseux pairs, and one member of this characteristic is the multiplicity). It seems to be a simpler problem to prove the conjecture for pairs $f, \widetilde{f}$ that are members of a holomorphic family (f_t) of pairwise embedded topologically equivalent singularities. But this last assumption is implied, in the case of isolated singularities, by the fact that (f_t) is μ-constant, i.e., the Milnor number $\mu(f_t)$ at 0 in this family is constant. This follows from the Lê and Ramanujam theorem [LR76]. Because of this, B. Teissier [Tei77] posed the following conjecture

2010 *Mathematics Subject Classification.* Primary 32S05.
Key words and phrases. Zariski multiplicity conjecture; isolated singularity; multiplicity of a singularity; non-degenerate singularity; Newton polyhedron; μ-constant deformation.

Conjecture 1 (B. Teissier)**.** *Let (f_t) be a holomorphic family of isolated singularities. If $\mu(f_t)$ is constant, then (f_t) is equimultiple.*

Until recently, this has been a wide-open problem, except for several special cases that have been settled. In [dBP22], J. F. de Bobadilla and T. Pełka announced a positive solution to Teissier's conjecture. Since, however, this paper counts 80 pages and has not yet been published in a recognized journal, the result still requires independent confirmation. Somewhat earlier, Y. O. M. Abderrahmane [Abd16] proved this conjecture in the case the family is additionally non-degenerate, i.e., all f_t are non-degenerate in the Kushnirenko sense. He proved even more – that the family (f_t) is also topologically trivial. He used advanced results of the singularity theory (characterizations of (c)-regularity and μ-constancy). In the paper, we give a simpler, elementary proof of the Teissier conjecture in the Abderrahmane case, based on the recent result by M. Leyton-Álvarez, H. Mourtada and M. Spivakovsky [LÁMS21] concerning a characterization of the difference of the Newton polyhedra of singularities with the same Newton number.

2. Preliminaries

Let $0 \neq f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function defined by a convergent power series $\sum_{\nu\in\mathbb{N}^n} a_\nu z^\nu$, $z = (z_1, \dots, z_n)$, $\nu = (\nu_1, \dots, \nu_n)$. Let $\mathbb{R}^n_+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ i = 1, \dots, n\}$. We define $\operatorname{supp} f := \{\nu \in \mathbb{N}^n : a_\nu \neq 0\} \subset \mathbb{R}^n_+$ and the *Newton polyhedron* $\Gamma_+(f) \subset \mathbb{R}^n_+$ *of* f as the convex hull of the set $\{\nu + \mathbb{R}^n_+ : \nu \in \operatorname{supp} f\}$. It is a non-compact polyhedron with a finite number of vertices $\operatorname{Vert}(f)$. We say f is *convenient* if $\Gamma_+(f)$ has non-empty intersection with each coordinate x_i-axis, $i = 1, \dots, n$. Let $\Gamma(f)$ be the set of compact boundary faces of any dimension of $\Gamma_+(f)$ – the *Newton boundary* of f. Denote by $\Gamma^k(f)$ the subset of $\Gamma(f)$ of all k-dimensional faces, $k = 0, \dots, n-1$. Then $\Gamma(f) = \bigcup_k \Gamma^k(f)$ and $\Gamma^0(f) = \operatorname{Vert}(f)$. Elements of $\Gamma^1(f)$ we will call *edges.* For each $(n-1)$-dimensional face (compact) $S \in \Gamma^{n-1}(f)$ we denote by $v_S = (v_1, \dots, v_n)$ the unique vector, perpendicular to S with positive, integer coordinates satisfying $\operatorname{GCD}(v_1, \dots, v_n) = 1$. From this we get that the projection of any $S \in \Gamma^{n-1}(f)$ on any coordinate hyperplane $H_i := \{x \in \mathbb{R}^n : x_i = 0\}$ is a linear homeomorphism. For each face $S \in \Gamma(f)$ of any dimension, we define the quasihomogeneous polynomial $f_S := \sum_{\nu\in S} a_\nu z^\nu$. We say f is *non-degenerate on S* if the system of polynomial equations $\partial f_S/\partial z_i = 0$, $i = 1, \dots, n$, has no solution in $(\mathbb{C}^*)^n$; f is *non-degenerate (in the Kushnirenko sense)* if f is non-degenerate on each face $S \in \Gamma(f)$.

For convenient f we define $\Gamma_-(f)$ as $\overline{\mathbb{R}^n_+ \setminus \Gamma_+(f)}$. It is a compact polyhedron (not necessarily convex) which is the union of cones over faces from $\Gamma^{n-1}(f)$ with vertex at 0. We define the Newton number $\nu(f)$ of f as

$$\nu(f) := n!\, V_n - (n-1)!\, V_{n-1} + \cdots + (-1)^{n-1} V_1 + (-1)^n,$$

where V_n is the n-dimensional volume of $\Gamma_-(f)$ and V_i is the sum of the i-dimensional volumes of the intersections of $\Gamma_-(f)$ with all the coordinate hyperplanes of dimension i, $1 \leqslant i \leqslant (n-1)$. The Newton number may also be defined in the non-convenient case, but we will not use this notion. Now we recall two important results. The first one is the formula for the Milnor number of an isolated singularity f in the generic case, expressed in terms of the Newton polyhedron of f.

Theorem 1 (Kushnirenko [Kou76])**.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic, convenient function with an isolated critical point at* 0 *(= an isolated singularity) and $\mu(f)$ be the Milnor number of f at* 0*. Then*

$$\mu(f) \geqslant \nu(f)$$

and the equality holds if f is non-degenerate. Moreover, non-degeneracy is a generic property in the space of coefficients corresponding to integer points of $\Gamma(f)$*.*

The second result is a recent one, by M. Leyton-Álvarez, H. Mourtada and M. Spivakovsky [LÁMS21, Thm. 2.25], giving a characterization of the difference of the Newton polyhedra of isolated singularities with the same Newton number. The same result in the particular case of isolated surface singularities ($n = 3$) was proved in [BKW19]. We will formulate this theorem in a form convenient for us.

Theorem 2. *Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two holomorphic, convenient functions such that $\Gamma_+(f) \subsetneq \Gamma_+(g)$ (equivalently $\Gamma_-(g) \subsetneq \Gamma_-(f)$). Then $\nu(f) = \nu(g)$ if and only if for each vertex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathrm{Vert}(g) \setminus \mathrm{Vert}(f)$:*

1. α lies in one of the coordinate hyperplanes H_i, i.e., there exists $i \in \{1, \ldots, n\}$ such that $\alpha_i = 0$. Denote the set of such i by I.

2. There exists $i_0 \in I$ for which there exists a unique edge $\overline{\alpha\beta'}$ of $\Gamma_+(g)$, $\beta' \in \mathrm{Vert}(g)$, which does not lie in H_{i_0}. Moreover, there exists $\beta = (\beta_1, \ldots, \beta_n) \in \overline{\alpha\beta'} \cap \mathrm{Vert}(f)$ with coordinates $\beta_{i_0} = 1$ and $\beta_i = 0$ for $i \in I \setminus \{i_0\}$.

Remark 3. *The possible configurations for $n = 3$ are illustrated in Fig. 1 (the case $\beta' = \beta$) and Fig. 2 (the case $\beta' \neq \beta$). Notice that in the case $\beta' \neq \beta$ the segment $\overline{\alpha\beta'}$ is an extension of the segment $\overline{\alpha\beta}$.*

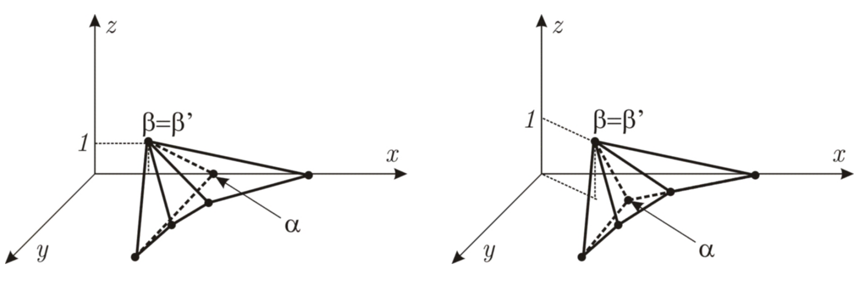

FIGURE 1.

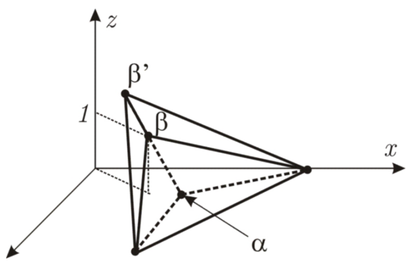

FIGURE 2.

Remark 4. *Geometrically, if* $\mathrm{Vert}(g) \setminus \mathrm{Vert}(f)$ *consists of only one vertex* $\alpha \in H_{i_0}$ *(as in Fig. 1 and Fig. 2), then conditions 1 and 2 in the theorem mean that the difference* $\overline{\Gamma_-(f) \setminus \Gamma_-(g)}$ *is an* n*-dimensional pyramid of height* 1 *with the apex in* β *and the base in* H_{i_0}*, and, moreover,* β *has the same zero coordinates as* α *except for one equal to* 1*.*

We also recall the following monotonicity property (see e.g. [Gwo08]).

Proposition 5. *If* Γ_1*,* Γ_2 *are two convenient Newton polyhedra of holomorphic functions such that* $\Gamma_1 \subset \Gamma_2$*, then*

$$\nu(\Gamma_2) \leqslant \nu(\Gamma_1) < \infty.$$

3. THE MAIN THEOREM

Let $0 \neq f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function defined by a convergent power series $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$. By $\mathrm{ord}\, f$ we denote the order of f. If f is *reduced* in $\mathbb{C}\{z_1, \dots z_n\}$, i.e., has no multiple factors in the factorization into irreducible elements in $\mathbb{C}\{z_1, \dots z_n\}$, then the multiplicity $\mathrm{mult}\, V(f)$ of $V(f)$ is equal to $\mathrm{ord}\, f$. Before the main theorem, we give a geometric lemma that easily follows from properties of the Newton polyhedron of a holomorphic function gathered in Preliminaries. By $\mathrm{Pr}_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$ we denote the projection of $\mathbb{R}^n$ onto $\mathbb{R}^{n-1}$: $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$ and, accordingly, Pr_i $(1 \leqslant i \leqslant n-1)$.

Lemma 6. *If* f *is convenient and we put* $\delta := \bigcup \Gamma^{n-1}(f)$*, the union of compact* $(n-1)$*-dimensional faces of* $\Gamma_+(f)$*, then, for any* $i \in \{1, \dots, n\}$*, we have* $\mathrm{Pr}_i(\delta) = \Gamma_-(f) \cap H_i$ *and the restriction* $\mathrm{Pr}_i|_\delta$ *is a homeomorphism (piecewise linear). In particular, if* $\Gamma^{n-1}(f) = \{S_1, \dots, S_k\}$*, then* $\mathrm{Pr}_i(S_1), \dots, \mathrm{Pr}_i(S_k)$ *are* $(n-1)$*-dimensional convex polyhedra which give a partition of* $\Gamma_-(f) \cap H_i$ *preserving the boundary relation. Moreover, from any point* $\tilde{\alpha} \in \mathrm{Pr}_i(S_j)$ *we "see" all the vertices of* S_j*, i.e., the segments joining* $\tilde{\alpha}$ *with the vertices of* S_j *lie in* $\Gamma_-(f)$*.*

Now we may pass to the main aim of our paper – a new proof of the Abderrahmane theorem.

Theorem 7. *Let (f_t) be a holomorphic family of isolated, non-degenerate singularities, where t is a parameter in a neighbourhood of 0 in $\mathbb{C}$. If $\mu(f_t)$ is constant, then* $\operatorname{ord} f_t$ *is also constant for small t.*

The same assertion holds for any holomorphic family of functions (f_t) if f_0 is convenient and $\nu(f_t) = \mathrm{const}$.

Proof. Notice that the first part of the theorem follows from the second one. Indeed, if (f_t) are non-degenerate, isolated singularities, then if we add to f_t the sum of specific monomials $a_1 z_1^N + \cdots + a_n z_n^N$ with sufficiently large N and generic $a_1, \dots, a_n$, we get a new holomorphic family of *convenient*, isolated singularities, which are also non-degenerate with the same Milnor numbers and the same orders as f_t. By the Kushnirenko theorem, we now have constant ν for this new family; thus, we may assume from the beginning that f_0 is some convenient function and $\nu(f_t) = \mathrm{const}$.

Let us pass to the proof of the second part of the theorem. Because both the Newton number and the multiplicity depend only on the Newton diagram, we may change f_t at will, demanding that $\operatorname{Supp} f_t = \operatorname{Vert} f_t$, for all $|t| \ll 1$; in particular, $\operatorname{Supp} f_0$ is finite. Clearly, $\Gamma_+(f_0) \subset \Gamma_+(f_t)$ and we may assume the containment is strict for $t \neq 0$. Hence, Proposition 5 allows us to reduce the problem further, to the case where $\Gamma_+(f_0)$ and $\Gamma_+(f_t)$ "differ by one point only", i.e., $\Gamma_+(f_t) = \operatorname{conv}(\Gamma_+(f_0), \alpha)$, where $\{\alpha\} = \operatorname{Vert} f_t \setminus \operatorname{Vert} f_0$ $(t \neq 0)$. Accordingly, we may put $f_t := f_0 + t \cdot z^\alpha$. Now, let us note the following

Claim. *We may additionally assume that in f_0 there are no surplus vertices (not on any axis), in the sense that removing any vertex monomial from f_0 changes its Newton number.*

<u>*Proof of Claim.*</u> Indeed, let ι be a vertex of $\Gamma_+(f_0)$ not lying on any axis and let $c_\iota \cdot z^\iota$ be the corresponding monomial with the property that for $\tilde{f}_0 := f_0 - c_\iota \cdot z^\iota$ we have $\Gamma_+(\tilde{f}_0) \subsetneq \Gamma_+(f_0)$ and $\nu(\tilde{f}_0) = \nu(f_0)$. Set $\tilde{f}_t := f_t - c_\iota \cdot z^\iota$. We have

$$\nu(\tilde{f}_0) \geqslant \nu(\tilde{f}_t) \geqslant \nu(f_t) = \nu(f_0) = \nu(\tilde{f}_0),$$

where the inequalities follow from the monotonicity of the Newton number (Proposition 5). Thus, $\nu(\tilde{f}_t) = \nu(\tilde{f}_0)$. Moreover, we obviously have $\operatorname{ord} \tilde{f}_0 \geqslant \operatorname{ord} f_0$, with strict inequality if, and only if, $|\iota| = \operatorname{ord} f_0$ and $c_\iota \cdot z^\iota$ is the only monomial appearing in f_0 and having the degree equal to $\operatorname{ord} f_0$. It follows that if we prove $\operatorname{ord} \tilde{f}_t = \operatorname{ord} \tilde{f}_0$, then $\operatorname{ord} f_t = \min(\operatorname{ord} \tilde{f}_t, |\iota|) = \min(\operatorname{ord} \tilde{f}_0, |\iota|) = \operatorname{ord} f_0$. Note also that we still have $\tilde{f}_t = \tilde{f}_0 + t \cdot z^\alpha$ and $\Gamma_+(\tilde{f}_t) = \operatorname{conv}(\Gamma_+(\tilde{f}_0), \alpha)$, where $\{\alpha\} = \operatorname{Vert} \tilde{f}_t \setminus \operatorname{Vert} \tilde{f}_0$, $t \neq 0$. Hence, we may replace the pair (f_0, f_t) by $(\tilde{f}_0, \tilde{f}_t)$ in our reasoning. Repeating this reduction finitely many times (bounded by the number of elements of $\operatorname{Supp} f_0$), we reach the conclusion of the claim. $\Diamond$

Continuing the main reasoning, we have $\operatorname{ord} f_t = \mathrm{const}$ for $t \neq 0$, and we need to prove $\operatorname{ord} f_t = \operatorname{ord} f_0$. By upper semicontinuity of the order, we have

$$\operatorname{ord} f_t \leqslant \operatorname{ord} f_0.$$

Assume to the contrary that

$$\operatorname{ord} f_t < \operatorname{ord} f_0. \tag{3.1}$$

If $\alpha = (\alpha_1, \dots, \alpha_n)$, then from (3.1)

$$\alpha_1 + \cdots + \alpha_n < \operatorname{ord} f_0. \tag{3.2}$$

By assumption, the family is ν-constant, i.e, $\nu(f_t) = \nu(f_0)$. Since of course $\Gamma_-(f_t) \subsetneq \Gamma_-(f_0)$ for $t \neq 0$, Theorem 2 implies that the vertex α lies in one of the coordinate hyperplanes, say H_n, i.e., $\alpha = (\alpha_1, \dots, \alpha_{n-1}, 0)$, and α is a vertex of the unique edge $\overline{\alpha\beta'}$ of $\Gamma_+(f_t)$ not lying in H_n, which joins α with $\beta' \in \operatorname{Vert}(f_t)$ and for which there exists $\beta = (\beta_1, \dots, \beta_{n-1}, 1) \in \overline{\alpha\beta'} \cap \operatorname{Vert}(f_0)$ satisfying $\beta_i = 0$ if $\alpha_i = 0$ $(i \neq n)$. Since $\alpha \in \Gamma_-(f_t) \cap H_n \subset \Gamma_-(f_0) \cap H_n$, by Lemma 6 we have that $\alpha \in \operatorname{Pr}_n(S)$, for some $S \in \Gamma^{n-1}(f_0)$.

We shall show that the face S has only one vertex, exactly β, not lying in H_n. To this end, we will first exclude vertices outside the set $\{\beta, \beta'\}$. Indeed, suppose there is a vertex $\gamma \notin \{\beta, \beta'\}$ of S not lying in H_n. Since, by Lemma 6, γ is visible from α and the edge $\overline{\alpha\beta'}$ of $\Gamma_+(f_t)$ is the unique one containing α and lying outside H_n, it follows that $\gamma \notin \operatorname{Vert}(f_t)$ for $t \neq 0$. Consider $g_0 := f_0 - c_\gamma z^\gamma$, i.e., f_0 without the monomial corresponding to γ. Note that $\gamma \notin H_n$ cannot lie on any axis; otherwise, γ would still be a vertex of $\Gamma_+(f_t)$ for $t \neq 0$. Hence, g_0 is convenient. By the Claim, we have that $\nu(g_0) > \nu(f_0)$. Putting $g_t := f_t - c_\gamma z^\gamma$, we get $\Gamma_+(g_t) = \Gamma_+(f_t)$ $(t \neq 0)$ because $\gamma \notin \operatorname{Vert}(f_t)$ for $t \neq 0$. Thus, $\infty > \nu(g_0) > \nu(f_0) = \nu(f_t) = \nu(g_t)$. This contradicts Theorem 2 because $\{\alpha\} = \operatorname{Vert}(g_t) \setminus \operatorname{Vert}(g_0) = \operatorname{Vert}(f_t) \setminus \operatorname{Vert}(f_0)$ and $\Gamma_+(g_t) = \Gamma_+(f_t)$ $(t \neq 0)$ so we should have $\nu(g_0) = \nu(g_t)$.

Now, note that for $\beta \neq \beta'$ we must also have $\beta' \notin S$; for, in the opposite case, $\beta \in \overline{\alpha\beta'}$ and we cannot "see" the point β' from α, contrary to Lemma 6.

Summing up, the only vertex of S outside H_n is β, i.e., S is a pyramid with the apex β and the base $T \in \Gamma^{n-2}(f_0)$, where T is an $(n-2)$-dimensional convex polyhedron lying in H_n (see Fig. 3).

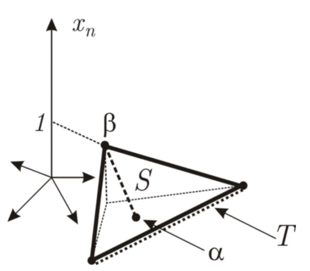

FIGURE 3.

Of course, $\alpha \notin T$ as T is a face of $\Gamma(f_0)$. From (3.2)

$$\alpha_1 + \cdots + \alpha_{n-1} < \operatorname{ord} f_0 \leqslant \beta_1 + \cdots + \beta_{n-1} + 1$$

and, hence,

$$\alpha_1 + \cdots + \alpha_{n-1} \leqslant \beta_1 + \cdots + \beta_{n-1}. \tag{3.3}$$

Consider the hyperplane $\Pi : x_1 + \cdots + x_{n-1} = \beta_1 + \cdots + \beta_{n-1}$ in H_n, treated as $\mathbb{R}^{n-1}$, which passes through $\mathrm{Pr}_n(\beta)$. Then from (3.3), α lies beneath or on Π. Since S is a pyramid with the base T lying in H_n and the apex β, $\mathrm{Pr}_n(S)$ is also a pyramid with the base T and the apex $\mathrm{Pr}_n(\beta)$. Notice $\mathrm{Pr}_n(\beta) \neq \alpha$ because otherwise the edge $\overline{\alpha\beta}$ would be vertical (perpendicular to H_n). Hence the unique line passing through $\mathrm{Pr}_n(\beta)$ and $\alpha \in \mathrm{Pr}_n(S)$ intersects the base T in a point, say $\kappa = (\kappa_1, \ldots, \kappa_{n-1})$. Of course

$$\kappa_1 + \cdots + \kappa_{n-1} \leqslant \alpha_1 + \cdots + \alpha_{n-1} \tag{3.4}$$

as α lies between $\mathrm{Pr}_n(\beta)$ and κ on this line, and by (3.3). Since T is a convex, compact polyhedron and has points lying beneath the hyperplane $\widetilde{\Pi} : x_1 + \cdots + x_{n-1} = \alpha_1 + \cdots + \alpha_{n-1}$ (by (3.4)), it also has a vertex lying beneath $\widetilde{\Pi}$. But such a vertex is in $\mathrm{supp}(f_0)$ and, hence, we obtain a contradiction with (3.1). □

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 61–66
DOI: https://doi.org/10.18778/8331-092-3.05

ON LÊ'S FORMULA IN ARBITRARY CHARACTERISTIC

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Streszczenie. In this note we extend, to arbitrary characteristic, Lê's formula (*Calculation of Milnor number of isolated singularity of complete intersection.* Funct. Anal. Appl. 8 (1974), 127–131).

1. Introduction

Let K be an algebraically closed field of characteristic $p \geqslant 0$. For any power series $f, g \in K[[x,y]]$ we put $i_0(f,g) := \dim_K K[[x,y]]/(f,g)$ and call it the *intersection multiplicity* of f and g. We denote by $[f,g]$ the Jacobian determinant of (f,g), that is $[f,g] = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$.

For any formal power series $f \in K[[x,y]]$ we denote by $\mathrm{ord} f$ the *order* of f. Any power series of order one is called a *regular parameter*.

Let $f \in K[[x,y]]$ be a power series without constant term. The *Milnor number* of $f \in K[[x,y]]$ is $\mu(f) := i_0\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$. Suppose that f is reduced, that is, it has no multiple factors. We put $\mathcal{O}_f = K[[x,y]]/(f)$, $\overline{\mathcal{O}_f}$ the integral closure of $\mathcal{O}_f$ in the total quotient ring of $\mathcal{O}_f$. Let $\mathcal{C}$ be the *conductor* of $\mathcal{O}_f$, that is the largest ideal in $\mathcal{O}_f$ which remains an ideal in $\overline{\mathcal{O}_f}$. We define $c(f) = \dim_K \overline{\mathcal{O}}_f/\mathcal{C}$ (the *degree of conductor*) and $r(f)$ the number of irreducible factors of f.

We define

$$\overline{\mu}(f) := c(f) - r(f) + 1.$$

If the characteristic of K is zero then $\mu(f) = \overline{\mu}(f)$. See [GB-Pł, Proposition 2.1] for other properties of $\overline{\mu}(f)$.

2010 *Mathematics Subject Classification.* Primary 14H20, Secondary 32S05.

Key words and phrases. Milnor number, jacobian determinant.

The first-named author was partially supported by the Spanish Project PID2019-105896GB-I00 funded by MCIN/AEI/10.13039/501100011033.

The main result of this note is to extend Lê's formula (see [L] and [G]) to arbitrary characteristic:

Theorem A (Lê's formula in arbitrary characteristic). *Let l be a regular parameter. Let $f, g \in K[[x,y]]\backslash\{0\}$ be coprime without constant term. Suppose that f is reduced and let $f = f_1 \cdots f_r$ be the factorization of f into irreducible factors. If $i_0(f_i, l) \not\equiv 0 \pmod{p}$ for $i = 1, \dots, r$ then*

$$i_0(f, [f,g]) \geqslant \overline{\mu}(f) + i_0(f,g) - 1. \tag{1}$$

The equality in (1) *holds if and only if $i_0(f_i, g) \not\equiv 0 \pmod{p}$ for $i = 1, \dots, r$.*

Corollary to Theorem A. *If f is irreducible and* $\mathrm{ord} f \not\equiv 0 \pmod{p}$ *then*

$$i_0(f, [f,g]) \geqslant c(f) + i_0(f,g) - 1. \tag{2}$$

The equality in (2) *holds if and only if $i_0(f,g) \not\equiv 0 \pmod{p}$.*

Remark 1.1. *The assumption* $\mathrm{ord} f \not\equiv 0 \pmod{p}$ *in the above corollary is irrelevant (see* [H-R-S1, Corollary 2.4]*).*

2. Proof of Lê's formula

Let t be a variable. A *parametrization* is a pair $(x(t), y(t)) \in K[[t]]^2\backslash\{(0,0)\}$ such that $x(0) = y(0) = 0$. We say that the parametrization $(x(t), y(t))$ is *good* if the field of fractions of the ring $K[[x(t), y(t)]]$ is equal to the field $K((t))$. By the Normalization Theorem (see for example [Pł, Theorem 2.1]), any irreducible power series in $K[[x,y]]$ admits a good parametrization.

The proof of Lê's formula will be a consequence of two lemmas. Let $f, g \in K[[x,y]]\backslash\{0\}$ be without constant term.

Lemma 2.1 (Teissier's lemma in arbitrary characteristic). *Let l be a regular parameter and let $f \in K[[x,y]]$ be a reduced power series with factorization $f = f_1 \cdots f_r$ into irreducible factors. Suppose that $i_0(f_i, l) \not\equiv 0 \pmod{p}$ for $i = 1, \dots, r$. Then*

$$i_0(f, [f,l]) = \overline{\mu}(f) + i_0(f,l) - 1.$$

Proof. See [GB-Pł, Proposition 2.1(iii)]. □

The following lemma generalizes to arbitrary characteristic *Delgado's Formula* (see [D, Proposition 2.1.1] or [Ca, Proposition 2.4.1]).

Lemma 2.2 (Delgado's formula). *Let $f, g \in K[[x,y]]\backslash\{0\}$ be coprime and l be a regular parameter. Suppose that f is reduced and $f = f_1 \cdots f_r$ is its factorization into irreducible factors. If $i_0(f_i, l) \not\equiv 0 \pmod{p}$ for $i = 1, \dots, r$ then*

$$i_0(f, [f,g]) \geqslant i_0(f,g) + i_0(f, [f,l]) - i_0(f,l) \tag{3}$$

with equality if and only if $i_0(f_i, g) \not\equiv 0 \pmod{p}$ for any irreducible factor f_i of f.

Proof. We may assume, without loss of generality, that $l(x,y)=x$, hence $[f,l]=-\frac{\partial f}{\partial y}$ and $i_0(f,l)=\mathrm{ord} f(0,y)$. Let f_i be an irreducible factor of f and let $\gamma(t)=(x(t),y(t))$ be a good parametrization of the curve $\{f_i(x,y)=0\}$. We get

$$\frac{\partial f}{\partial x}(\gamma(t))x'(t)+\frac{\partial f}{\partial y}(\gamma(t))y'(t)=0. \tag{4}$$

Since f and g are coprime, f_i is not a factor of g, that is $g(x(t),y(t))\neq 0$ and

$$\frac{\partial g}{\partial x}(\gamma(t))x'(t)+\frac{\partial g}{\partial y}(\gamma(t))y'(t)=\frac{d}{dt}g(\gamma(t)). \tag{5}$$

Consider the system, in the unknowns U and V:

$$\begin{cases} \frac{\partial f}{\partial x}(\gamma(t))U+\frac{\partial f}{\partial y}(\gamma(t))V=0 \\ \frac{\partial g}{\partial x}(\gamma(t))U+\frac{\partial g}{\partial y}(\gamma(t))V=\frac{d}{dt}g(\gamma(t)). \end{cases} \tag{6}$$

By (4) and (5) the pair $(x'(t),y'(t))$ is a solution of the system (6). By Cramer's identities we get $[f,g](\gamma(t))x'(t)=-\frac{\partial f}{\partial y}(\gamma(t))\frac{d}{dt}g(\gamma(t))$ and taking orders we obtain

$$\mathrm{ord}[f,g](\gamma(t))+\mathrm{ord}x'(t)=\mathrm{ord}\frac{\partial f}{\partial y}(\gamma(t))+\mathrm{ord}\frac{d}{dt}g(\gamma(t)). \tag{7}$$

Since $i_0(f_i,x)\not\equiv 0 \pmod p$ we have $\mathrm{ord}\,x(t)\not\equiv 0 \pmod p$ and consequently $\mathrm{ord}\,x'(t)=\mathrm{ord}\,x(t)-1$. Analogously $\mathrm{ord}\frac{d}{dt}g(\gamma(t))\geqslant \mathrm{ord}g(\gamma(t))-1$, with equality if and only if, $\mathrm{ord}g(\gamma(t))=i_0(f_i,g)\not\equiv 0 \pmod p$. From (7) we get

$$i_0(f_i,[f,g])+i_0(f_i,x)\geqslant i_0(f_i,g)+i_0\left(f_i,\frac{\partial f}{\partial y}\right), \tag{8}$$

with equality if $i_0(f_i,g)\not\equiv 0 \pmod p$. Summing up the inequalities (8), we obtain

$$i_0(f,[f,g])+i_0(f,x)\geqslant i_0(f,g)+i_0\left(f,\frac{\partial f}{\partial y}\right)$$

with equality if $i_0(f_i,g)\not\equiv 0 \pmod p$, for $i=1,\dots,r$. □

Remark 2.3. *A particular case of Delgado's formula in arbitrary characteristic appears in* [H-R-S2, Lemma 3.5].

Proof of Theorem A It is a consequence of Lemmas 2.1 and 2.2. □

3. The case of characteristic zero

If the characteristic of K is zero then we have the following version of Lê's formula.

Theorem B (Lê's formula in zero characteristic)**.** *Let $f,g\in K[[x,y]]\backslash\{0\}$ be without constant term. Then*

$$i_0(f,[f,g])=\mu(f)+i_0(f,g)-1. \tag{9}$$

The left-hand side of (9) *is infinite if and only if the right-hand side is so.*

The following lemma is well-known (see for example [CN-Pł]):

Lemma 3.1. *Let $f, g \in K[[x,y]]\backslash\{0\}$ be without constant term.*

(1) $i_0(f,g) = +\infty$ *if and only if f and g are not coprime.*
(2) $\mu(f) = +\infty$ *if and only if f is not reduced.*

The following general property also will be useful:

Property 3.2. *Let $h(x,y) \in K[[x,y]]$ be an irreducible power series. Let $\gamma(t) = (x(t), y(t))$ be a good parametrization of $h(x,y)$. Then $\frac{\partial h}{\partial x}(\gamma(t)) \neq 0$ or $\frac{\partial h}{\partial y}(\gamma(t)) \neq 0$.*

Proof. Suppose that $\frac{\partial h}{\partial x}(\gamma(t)) = 0$ and $\frac{\partial h}{\partial y}(\gamma(t)) = 0$. This implies $\frac{\partial h}{\partial x} \equiv 0 \pmod{h}$ or $\frac{\partial h}{\partial y} \equiv 0 \pmod{h}$. Hence $\operatorname{ord}\frac{\partial h}{\partial x} \geqslant \operatorname{ord} h$ and $\operatorname{ord}\frac{\partial h}{\partial y} \geqslant \operatorname{ord} h$. This is a contradiction since if the characteristic of K is zero then $\operatorname{ord}\frac{\partial h}{\partial x} = \operatorname{ord} h - 1$ or $\operatorname{ord}\frac{\partial h}{\partial y} = \operatorname{ord} h - 1$. □

Proof of Theorem B If $\mu(f) + i_0(f,g)$ is finite then $\mu(f)$ is also. Hence, by the second part of Lemma 3.1, f is reduced and $\mu(f) = \overline{\mu}(f)$. Therefore, in this case, Theorem B follows from Theorem A.

The case where one of the two sides of (9) is infinite follows from the following proposition, which is equivalent to [Sz, Theorem 3.6].

Proposition 3.3. *Let $f, g \in K[[x,y]]\backslash\{0\}$ be without constant term. The following two conditions are equivalent:*

(1) $\mu(f) = +\infty$ *or* $i_0(f,g) = +\infty$.
(2) $i_0(f, [f,g]) = +\infty$.

Proof. Suppose that $\mu(f) = +\infty$ or $i_0(f,g) = +\infty$. In order to prove the equality $i_0(f,[f,g]) = +\infty$, we distinguish two cases.

Case 1: $\mu(f) = +\infty$. There is an irreducible power series $h \in K[[x,y]]$ such that $f \equiv 0 \pmod{h^2}$. Therefore $\frac{\partial f}{\partial x} \equiv 0 \pmod{h}$ and $\frac{\partial f}{\partial y} \equiv 0 \pmod{h}$ which implies $[f,g] \equiv 0 \pmod{h}$. Since $f \equiv 0 \pmod{h}$ we conclude $i_0(f,[f,g]) = +\infty$ by properties of the intersection multiplicity.

Case 2: $i_0(f,g) = +\infty$. There exists an irreducible power series $h \in K[[x,y]]$ such that $f \equiv 0 \pmod{h}$ and $g \equiv 0 \pmod{h}$. Let $f = a \cdot h$ and $g = b \cdot h$ for some $a, b \in K[[x,y]]$. Observe that $\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} \equiv ab\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} \equiv \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \pmod{h}$, hence $[f,g] \equiv 0 \pmod{h}$. Since h is an irreducible factor of f and $[f,g]$ we conclude $i_0(f,[f,g]) = +\infty$.

Suppose now that $i_0(f,[f,g]) = +\infty$. There is an irreducible power series $h \in K[[x,y]]$ such that $f \equiv 0 \pmod{h}$ and $[f,g] \equiv 0 \pmod{h}$. If $f \equiv 0 \pmod{h^2}$ then, by the second part of Lemma 3.1, $\mu(f) = +\infty$. Suppose that $f = hf_1$ for some $f_1 \in K[[x,y]]$ with $f_1 \not\equiv 0 \pmod{h}$. We have $[f,g] = [hf_1, g] = h[f_1,g] + f_1[h,g]$. Since h is an irreducible factor of $[f,g]$, we get $f_1\,[h,g] \equiv 0 \pmod{h}$. Let $\gamma(t) := (x(t), y(t))$ be a good parametrization of h. By Property 3.2 we may assume, without loss of generality, that $\frac{\partial h}{\partial y}(\gamma(t)) \neq 0$.

From the identity $h(\gamma(t))=0$ we get

$$\frac{\partial h}{\partial x}(\gamma(t))x'(t)+\frac{\partial h}{\partial y}(\gamma(t))y'(t)=0. \tag{10}$$

On the other hand, since h is an irreducible factor of $[h,g]$, we get $[h,g](\gamma(t))=0$, hence

$$\frac{\partial h}{\partial x}(\gamma(t))\frac{\partial g}{\partial y}(\gamma(t))+\frac{\partial h}{\partial y}(\gamma(t))\left(-\frac{\partial g}{\partial x}(\gamma(t))\right)=0. \tag{11}$$

From (10) and (11) we get that the pair $\left(\frac{\partial h}{\partial x}(\gamma(t)),\frac{\partial h}{\partial y}(\gamma(t))\right)$ is a nonzero solution of the system, in the unknowns U and V:

$$\begin{cases} x'(t)U+y'(t)V=0 \\ \frac{\partial g}{\partial y}(\gamma(t))U+\left(-\frac{\partial g}{\partial x}(\gamma(t))\right)V=0. \end{cases} \tag{12}$$

Hence the determinant of the matrix associated with system (12) equals $\frac{d}{dt}g(\gamma(t))=0$ so $g(\gamma(t))=0$. Given that h is a common factor of f and g, we conclude that $i_0(f,g)=+\infty$. $\square$

Remark 3.4. *Proposition 3.3 does not hold when the characteristic p of the field K is positive: consider $f(x,y)=y^p+x^{p+1}$ and $g(x,y)=x+y$, then $\mu(f)=+\infty$ but $i_0(f,[f,g])=p^2$.*

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 67–73
DOI: https://doi.org/10.18778/8331-092-3.06

LEFSCHETZ NUMBERS AND ASYMPTOTIC PERIODS

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Streszczenie. In this note we prove several results linking Lefschetz numbers with asymptotic behaviour of the orbit in flows. With the aid of the Lefschetz fixed point theorem and the presence of a non-trivial limit set we prove the existence of asymptotically non-periodic orbits.

1. Introduction

The study of dynamical systems is divided into the variety of categories. In this article we want to utilize classic topological methods, going back to Lefschetz [8] and his well–known fixed point theorem.

The Lefschetz fixed point theorem has many applications in mathematics [2, 4], especially in the fixed point theory, but also, surprisingly, in digital topology [3]. The Lefschetz formula and the Euler characteristic are another tolls that have a wide application in algebraic topology and dynamical systems.

In this article, we link the Lefschetz numbers with the so called G-asymptotic period. In Section 3 we, among others, prove that if the limit set of some point x has non-zero Euler characteristic, then x cannot be G-asymptotically periodic. We also provide several examples of flows that justify the assumptions of our results.

2010 *Mathematics Subject Classification.* 37C25, 32S50, 37B20.
Key words and phrases. Lefschetz number, Euler characteristic, dynamical system, asymptotic period.

2. Preliminaries

Let us start by introducing fundamental definitions used in the entire paper.

2.1. **Dynamical systems.** Let (X,d) be a metric space. A *dynamical system* (a flow) ϕ is a continuous mapping $\phi\colon \mathbb{R}\times X \to X$ such that $\phi(0,x)=x$ and for any x,s and t we have $\phi(t,\phi(s,x))=\phi(t+s,x)$. We call X a *phase space* of ϕ. A *motion through* x is the mapping $t\mapsto \phi(t,x)$. We will identify properties of the motion through x with properties of x. Given dynamical system ϕ and $x\in X$, the set $o(x)=\phi(\mathbb{R},x)$ is the *orbit* of x and $o^+(x)=\phi([0,+\infty),x)$ is the *positive orbit* of x. A point x is *stationary* if $x=\phi(t,x)$ for any $t\in\mathbb{R}$. If for some $T>0$ we have $x=\phi(T,x)$ and x is not stationary, then x is *periodic*. If $T>0$ is the smallest such that $x=\phi(T,x)$, then we say that x is T*-periodic* and we call T the *period* of x. The ω*-limit set* $\omega(x)$ consists of all points $y\in X$ such that there exists a strictly increasing and diverging to $+\infty$ sequence $(t_n)_{n\in\mathbb{N}}$ of times with the property: $\phi(t_n,x)\to y$. For more definitions and properties related to dynamical systems see [1, 13].

The following notion is a generalization of periodicity and it relies on the asymptotic behaviour of the orbit outside of a small neighbourhood of a point belonging to the positive orbit of x. This idea was introduced in [5]. We briefly introduce the necessary notation.

Let ϕ be a flow on X. Fix $x\in X$ and $\varepsilon>0$, and define

$$A(x,\varepsilon) := \{t\geqslant 0 \mid d(\phi(t,x),x)>\varepsilon\}.$$

This set is the union of at most countably many pairwise disjoint and open intervals denoted by (q_i,r_i). Define

$$w_{x,\varepsilon}(t) := \begin{cases} 0, & t\notin A(x,\varepsilon),\\ \operatorname{diam}(q_i,r_i), & t\in (q_i,r_i).\end{cases}$$

The set $W_{x,\varepsilon} := \{w_{x,\varepsilon}(t) \mid t\geqslant 0\}$ contains at most countably many different non-negative real numbers, including $+\infty$ if necessary. We call the elements of that sequence *return times*. Set

$$W(x,\varepsilon) := \limsup_{t\to+\infty} w_{x,\varepsilon}(t).$$

Definition 2.1. The *G-asymptotic period* of x (of the orbit of x) is defined as

$$\text{G-AP}(x) := \lim_{\varepsilon\to 0}\limsup_{t\to+\infty} W(\phi(t,x),\varepsilon).$$

If $\text{G-AP}(x)=0$, then x is called *G-asymptotically fixed*. If x has a finite asymptotic period, then it is called *G-asymptotically periodic*. If $\text{G-AP}(x)=+\infty$, then x is called *G-asymptotically non-periodic*.

See also [5, 6, 7] for more properties of G-asymptotically periodic orbits.

2.2. **Homotopies and ENRs.** Let X be a topological space. For any mapping $f\colon X \to X$, we say that f has the *fixed point property* if f has a fixed point, i.e., there exists $x_0 \in X$ such that $f(x_0) = x_0$. Define the set

$$\mathrm{Fix}(f) = \{x \in X \mid f(x) = x\}.$$

Suppose $f\colon X \to X$ and $g\colon X \to X$ are continuous functions. We say that f is *homotopic to* g and we denote this relation by $f \sim g$, if there is a continuous mapping $h\colon [0,1] \times X \to X$ such that $h(0,\cdot) = f$ and $h(1,\cdot) = g$.

We say that X has the *weak fixed point property* if for any $f\colon X \to X$ which is homotopic to Id_X (the identity function on X) we have $\mathrm{Fix}(f) \neq \emptyset$.

We call the space X *euclidean neighborhood retract* (ENR) if there exists an open set $V \subset \mathbb{R}^n$ and continuous functions $r\colon V \to X$ and $s\colon X \to V$ such that $r \circ s = \mathrm{Id}_X$.

2.3. **Lefschetz numbers.** Let X be a compact ENR and let $f\colon X \to X$ be continuous. Let H denote the singular homology functor with rational coefficients. Let $H(f)\colon H(X) \to H(X)$ be the induced homomorphism.

Definition 2.2. The number

$$L(f) = \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{tr} H_n(f) \in \mathbb{Z}$$

is called the *Lefschetz number* of f. Here, $\mathrm{tr}H_n(f)$ is the trace of the endomorphism $H_n(f)\colon H_n(X) \to H_n(X)$.

If $f = \mathrm{Id}_X$, then $\chi(X) = L(\mathrm{Id}_X)$ is called the *Euler characteristic* of X. It can also be defined as

$$\chi(X) = \sum_{n=0}^{+\infty} (-1)^n \dim H_n(X).$$

The above definitions are well-defined since it is well-known that compact ENRs have only finitely many non-zero homologies $H_n(f)$ and they are all of finite dimension. It is also well-known, that if $f \sim g$, then $L(f) = L(g)$. See [2] for more information related to the topic.

3. Main results

We shall use the following lemma. It is a variation of Proposition III 4.8 in [2]. See also [12].

Lemma 3.1. *If ϕ is a flow on a compact metric space (X,d) and X has the weak fixed point property, then ϕ has a stationary point.*

Dowód. For each $t \in \mathbb{R}$ we let ϕ_t denote the map $X \ni x \mapsto \phi(t,x) \in X$. Then $\phi_t \sim \mathrm{Id}_X$; the homotopy is defined via relation

$$h(s,x) = \phi(st,x).$$

Each ϕ_t has a fixed point by the weak fixed point property. We define the sets

$$A_n = \{x \in X \mid \phi(2^{-n}, x) = x\}.$$

Each of the sets A_n is not empty, closed and therefore compact. Furthermore, since

$$x = \phi(2^{-(n+1)}, x) = \phi(2^{-(n+1)}, \phi(2^{-(n+1)}, x)) = \phi(2^{-n}, x)$$

for any $x \in A_{n+1}$, we have

$$A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots .$$

Since X is compact and the family $\{A_n\}_{n\in\mathbb{N}}$ has a finite intersection property, we can take the set

$$A = \bigcap_{n\in\mathbb{N}} A_n \neq \emptyset.$$

Take any $z \in A$. Then $\phi(2^{-n}, z) = z$ for all n. We claim that for all n and all integers m we also have

$$\phi(m \cdot 2^{-n}, z) = z.$$

Since $z \in A_0$, we have for any natural number k,

$$\phi(k, z) = \phi(k - 1, \phi(1, z)) = \phi(k - 1, z) = \cdots = \phi(1, z) = z,$$

$$\phi(-k, z) = \phi(-k, \phi(1, z)) = \phi(-k + 1, z) = \cdots = \phi(-k + k, z) = \phi(0, z) = z,$$

thus for any $m \in \mathbb{Z}$,

$$\phi(m \cdot 2^{-n}, z) = \phi(m \cdot 2^{-n} \bmod 1, z)$$

and it is enough to prove the claim in the case $0 < m \cdot 2^{-n} < 1$.

Suppose $0 < m \cdot 2^{-n} < 1$ and let $m = \sum_{i=0}^{M} m_i \cdot 2^i$ be the binary representation of m. Then $i - n \leqslant 0$ for each $i = 0, \ldots, M$. Note that $z \in A_n$ and $\phi(2^{-n}, z) = z$, hence

$$\begin{aligned}\phi(m \cdot 2^{-n}, z) &= \phi\Big(\sum_{i=0}^{M} m_i \cdot 2^{i-n}, z\Big) = \phi\Big(\sum_{i=1}^{M} m_i \cdot 2^{i-n}, \phi(m_0 \cdot 2^{-n}, z)\Big)\\ &= \phi\Big(\sum_{i=1}^{M} m_i \cdot 2^{i-n}, z\Big)\end{aligned}$$

(if $m_0 = 0$, then $\phi(0, z) = z$, otherwise $\phi(m_0 \cdot 2^{-n}, z) = \phi(2^{-n}, z) = z$). The claim now follows from the induction on i (note that the induction terminates after finitely many steps for any m).

Since the set $\{m2^{-n} \mid m \in \mathbb{Z}, n \in \mathbb{N}\}$ is dense in $\mathbb{R}$ and ϕ is continuous, this implies that $\phi(t, z) = z$ for all $t \in \mathbb{R}$. □

A great example of a space with the weak fixed point property is a connected polyhedron.

Lemma 3.2 (See Proposition III 4.6 in [2])**.** *Any connected polyhedron K with $\chi(K) \neq 0$ has the weak fixed point property. Any flow on such polyhedron has a stationary point.*

The following lemma shows that if the limit set of the orbit has a stationary point and at least one other point, then the G-asymptotic period need to be infinite. Recall that a metric space is *proper* if all closed balls are compact sets.

Lemma 3.3 (see also [5]). *Assume that* (X, d) *is a proper metric space and* ϕ *is a flow on* X*. If* $x \in X$ *has* $\#\omega(x) > 1$ *and* $\omega(x)$ *contains a stationary point, then* $\text{G-AP}(x) = +\infty$*.*

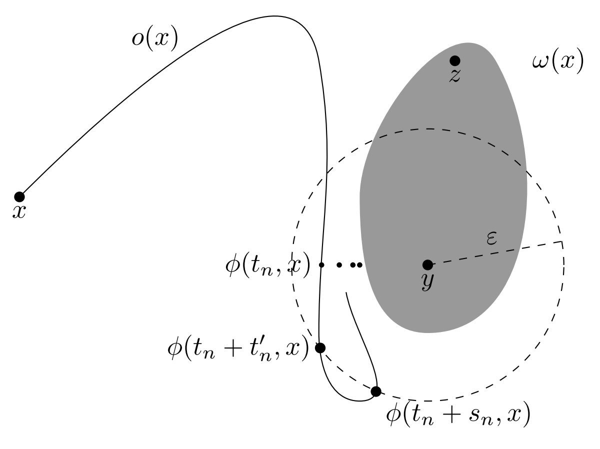
Dowód. Suppose $y \in \omega(x)$ is stationary. It is sufficient to show that the return times of x to $B(y, \varepsilon)$ cannot be bounded, and hence $\text{G-AP}(x) = +\infty$. Indeed, if that were the case, then take $\varepsilon' < \varepsilon$ and t' such that $B(\phi(t', x), \varepsilon') \subset B(y, \varepsilon)$. Then since the return times in the former case are not bounded, they are not bounded in the latter case, thus implying $\text{G-AP}(x) = +\infty$.

Suppose the opposite is true and let K be the bound. Pick $z \in \omega(x) \setminus \{y\}$ and $\varepsilon > 0$ so that $d(y, z) > \varepsilon$. There is a sequence $(t_n)_{n\in\mathbb{N}}$ such that $\phi(t_n, x) \to y$ and $d(\phi(t_n, x), y) < \varepsilon$ for all n.

Let t'_n be the infimum of all $u > 0$ such that $\phi(t_n + u, x) \notin B(y, \varepsilon)$. Such an u exists since z is an element of $\omega(x)$ and $d(y, z) > \varepsilon$. Let s_n be the infimum of all $v > t'_n$ such that $\phi(t_n + v, x) \in B(y, \varepsilon)$ (see Figure 1). The sequence s_n is bounded by K. We can assume without loos of generality that it is convergent. Let $s = \lim_{n\to+\infty} s_n$. Then, since the space is proper, $\phi(t_n + s_n, x) \to w$ for some $w \in X$ and $w \notin B(y, \varepsilon)$ On the other hand,

$$\phi(t_n + s_n, x) = \phi(s_n, \phi(t_n, x)) \to \phi(s, y) = y$$

which is a contradiction. □



Rysunek 1. Sketch of the proof of Lemma 3.3.

With the aid of the above lemmas, we can formulate the following theorem.

Theorem 3.4. *Suppose ϕ is a flow on a proper metric space (X,d). Let $x \in X$ be such that $\omega(x) = S$ is a compact ENR with the weak fixed point property. If $\#S > 1$, then* G-AP$(x) = +\infty$.

Dowód. The set S is compact, therefore by Lemma 3.1 there is a stationary point in S. Then, by Lemma 3.3 we have G-AP$(x) = +\infty$. □

The assumption that the limit set S is an ENR is actually not needed for the proof, however it was added since the later results require the set to be an ENR.

Recall the famous Lefschetz fixed point theorem [8, 9, 10, 11].

Theorem 3.5. *Suppose X is a compact ENR and $f\colon X \to X$ is continuous. If $L(f) \neq 0$, then* $\mathrm{Fix}(f) \neq \emptyset$.

We have the immediate.

Corollary 3.6. *If X is a compact ENR with $\chi(X) \neq 0$, then any flow ϕ on X has a stationary point.*

Dowód. Indeed, since the Lefschetz numbers are homotopy invariant,

$$\chi(X) = L(\mathrm{Id}_X) = L(\phi(t,\cdot))$$

for any t. Thus by Lefschetz fixed point theorem, each map $x \mapsto \phi(t,x)$ has a fixed point. The rest follows from the proof of Lemma 3.1. □

Example 3.7. Consider n-dimensional spheres. Then

$$\chi(\mathbb{S}^{2k}) = 2, \qquad \chi(\mathbb{S}^{2k+1}) = 0.$$

It is now clear that any flow on $\mathbb{S}^{2k}$ must have a stationary point. On the other hand, each odd-dimensional sphere $\mathbb{S}^{2k+1}$ admits a flow with no stationary points.

Indeed, let $z = (z_1, \dots, z_{k+1}) \in \mathbb{S}^{2k+1}$ with $z_i \in \mathbb{C}$. Then the function

$$\phi(t,z) = ze^{it} = (z_1e^{it}, \dots, z_{k+1}e^{it})$$

defines a flow on $\mathbb{S}^{2k+1}$ with no stationary point.

A variation of Theorem 3.4 is presented below.

Theorem 3.8. *Suppose ϕ is a flow on a proper metric space (X,d) and $\omega(x) = S$ is a compact ENR (or a connected polyhedron) for some $x \in X$. If $\#S > 1$ and $\chi(S) \neq 0$, then* G-AP$(x) = +\infty$.

Example 3.9. If we take $S = \mathbb{S}^{2k}$ in Theorem 3.8, then G-AP$(x) = +\infty$. In particular, even-dimensional sphere cannot be a limit set of G-asymptotically periodic point. On the other hand, the unit circle $\mathbb{S}^1$ is the limit set of all points in $\mathbb{R}^2 \setminus \{(0,0)\}$ of the flow in $\mathbb{R}^2$ generated by the equations

$$\begin{cases} r' = r(1-r), \\ t' = 1. \end{cases}$$

This in turn implies that the assumption about the Euler characteristic cannot be relaxed.

Finally, in view of Theorem 3.8, by constructing a flow which has $\omega(x) = \mathbb{T}$ (the two-dimensional surface of the torus - one such construction was provided in [5]), we can show that the condition G-AP$(x) = +\infty$ does not imply $\chi(S) \neq 0$,

Acknowledgements

The Author is grateful to the Organizers of the "Analytic and Algebraic Geometry" conference and workshops, held in Łódź (Poland) for the event.

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 75–82
DOI: https://doi.org/10.18778/8331-092-3.07

ON THE NEARLY FREE SIMPLICIAL LINE ARRANGEMENTS WITH UP TO 27 LINES

MAREK JANASZ

Streszczenie. In the present note we provide a complete classification of nearly free (and not free simultaneously) simplicial arrangements of $d \leqslant 27$ lines.

1. Introduction

The theory of line arrangements is a classical subject of studies in many branches of contemporary mathematics. In the recent years, many authors wanted to understand possible linkages between combinatorial and geometric properties of line arrangements. Let us recall that the famous Terao's conjecture predicts that the so-called freeness of a given arrangement of lines $\mathcal{A}$ is determined by the intersection poset of $\mathcal{A}$. It is very difficult to predict whether Terao's conjecture is true, and in order to approach this problem Dimca and Sticlaru in [6] defined a new class curves which is called nearly free. This class is designed as a natural generalization of free curves and it is important in the context of a potential counterexample to Terao's conjecture. It seems that the class of nearly free arrangements is more accessible, and it is definitely much wider. In the present note, which can be considered as an appendix to works devoted to simplicial line arrangements in the real projective plane, we want to understand which sporadic examples of simplicial line arrangements in the real projective plane are nearly free and not free. Even if the classification problem of simplicial line arrangements is open in its whole generality, we will use a great result due to M. Cuntz which provides a complete classification of simplicial arrangements up to 27 lines and, in this way, we provide a complete classification result of nearly free sporadic simplicial arrangements up to 27 lines. Our main result, surprising to us, can be formulated as follows.

2010 *Mathematics Subject Classification.* 14N20, 14C20.
Key words and phrases. hypersurface arrangements, freeness.

Main Theorem. *A sporadic simplicial line arrangement $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{R}}$ is nearly free if and only if $\mathcal{A} = \mathcal{A}(17,6)$ according to Cuntz's catalogue.*

Remark 1.1. More precisely, $\mathcal{A}(17,6)$ is a sporadic simplicial line arrangement consisting of 17 lines and it has 16 double, 15 triple, 10 quadruple, and one sixtuple intersection point.

It means that the class of free sporadic simplicial line arrangements is barely different from the class of nearly free sporadic simplicial line arrangements provided that we restrict our attention to $d \leqslant 27$ lines.

In order to prove Main Theorem, we will use combinatorial properties of the singular points of sporadic simplicial line arrangements. This allows us to determine all those sporadic arrangements for which the total Milnor number is determined exclusively by a polynomial equation of degree 2 that depends only on the number of lines and the minimal degree of the syzygies between partial derivatives of the defining polynomial. In the last step, using cohomological methods, we are able to determine those arrangements which are purely nearly free.

The structure of the paper goes as follows. In Section 2, we provide all necessary definitions and tools related to simplicial and nearly free line arrangements. In Section 3, we provide our proof of Main Theorem. All necessary symbolic computations were performed with use of `Singular` [3].

2. Preliminaries

In the section, we recall all necessary notations and definitions. For more information in this area please consult [4, 9].

Let $\mathbb{K}$ be any field and consider $S := \mathbb{K}[x,y,z]$ the graded polynomial ring over $\mathbb{K}$.

Definition 2.1. A finite collection of d lines $\mathcal{L} = \{\ell_1, ..., \ell_d\} \subset \mathbb{P}^2_{\mathbb{K}}$ is called an arrangement of lines in the projective plane over $\mathbb{K}$.

For an arrangement $\mathcal{L} = \{\ell_1, ..., \ell_d\}$ we denote by $\operatorname{Sing}(\mathcal{L})$ the set of all intersection points among the lines, i.e., points in the plane where at least two lines from $\mathcal{L}$ meet, and for such an intersection point $p \in \operatorname{Sing}(\mathcal{L})$ we denote by mult_p its multiplicity, i.e., the number of lines passing through the point p. Following Hirzebruch's convention, we denote by t_r the number of all intersection points of multiplicity $r \geqslant 2$.

We define the class of simplicial line arrangements in the real projective plane via Melchior's result [7].

Definition 2.2. Let $\mathcal{L} = \{\ell_1, ..., \ell_d\} \subset \mathbb{P}^2_{\mathbb{R}}$ of $d \geqslant 3$ lines such that $t_d = 0$. Then $\mathcal{L}$ is a simplicial line arrangement if and only if

$$t_2 = 3 + \sum_{r \geqslant 4} (r-3)t_r.$$

Classically, a simplicial line arrangement $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{R}}$ is an arrangement for which all connected components of the complement $M(\mathcal{L}) := \mathbb{P}^2_{\mathbb{R}} \setminus \mathcal{L}$ are **open triangles**. It is worth recalling that simplicial line arrangements were studied, for may years, by Grünbaum, and he discovered three infinite families of such arrangements and around 90 additional examples which are nowadays called **sporadic**. The collection of the three infinite families and around 90 sporadic examples is called in the literature as Grünbaum's catalogue. One of the most important conjectures related to simplicial line arrangements is motivated by a strong claim of Grünbaum [8, p. 4].

Conjecture 2.3. *Except only finitely many corrections, Grünbaum's catalogue is complete.*

In other words, one expects that there are only three infinite families of simplicial line arrangements. A stronger conjecture, proposed by Cuntz and Geis in [2, Conjecture 1.6], predicts even more.

Conjecture 2.4. *Let $\mathcal{L}$ be a sporadic simplicial line arrangement in $\mathbb{P}^2_{\mathbb{R}}$ of d lines. Then $d \leqslant 37$.*

The main aim of the present note is to understand the homological properties of Jacobian ideals given by simplicial line arrangements. In order to do so, let recall some crucial definitions. For a reduced curve $C \subset \mathbb{P}^2_{\mathbb{C}}$ of degree d given by $f = 0$ we denote by $J_f = \langle \partial_x f, \partial_y f, \partial_z f \rangle$ the Jacobian ideal and by $\mathfrak{m} = \langle x, y, z \rangle$ the irrelevant ideal. Consider the graded S-module $N(f) = I_f / J_f$, where I_f is the saturation of J_f with respect to $\mathfrak{m}$.

Definition 2.5. We say that a reduced plane curve C is *free* if $N(f) = 0$.

Definition 2.6. We say that a reduced plane curve C is *nearly free* if $N(f) \neq 0$ and for every k one has $\dim N(f)_k \leqslant 1$.

Recall that for a curve C given by $f \in S$ we define the Milnor algebra as $M(f) = S/J_f$. The description of $M(f)$ for nearly free curves comes from [6] as follows.

Theorem 2.7 (Dimca-Sticlaru)**.** *If C is a nearly free curve of degree d given by $f \in S$, then the minimal free resolution of the Milnor algebra $M(f)$ has the following form:*

$$0 \to S(-b - 2(d-1)) \to S(-d_1 - (d-1)) \oplus S^2(-d_2 - (d-1)) \\ \to S^3(-d+1) \to S$$

for some integers d_1, d_2, b such that $d_1 + d_2 = d$ and $b = d_2 - d + 2$. In that case, the pair (d_1, d_2) is called the set of exponents of C.

The nearly freeness can be also studied via the following result due to Dimca [5, Theorem 1.3], and this result is a vital technical tool for our proposes.

Theorem 2.8 (Dimca). *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of d lines and let $f = 0$ be its defining equation. Denote by r the minimal degree among all the Jacobian relations, i.e., the minimal degree r for the triple $(a, b, c) \in S^3_r$ such that $a \cdot \partial_x(f) + b \cdot \partial_y(f) + c \cdot \partial_z(f) = 0$. Assume that $r \leqslant d/2$, then $\mathcal{L}$ is nearly free if and only if*

$$r^2 - r(d-1) + (d-1)^2 = \mu(\mathcal{L}) + 1, \tag{1}$$

where $\mu(\mathcal{C})$ is the total Milnor number of $\mathcal{L}$, i.e.,

$$\mu(\mathcal{L}) = \sum_{p \in \mathrm{Sing}(\mathcal{L})} (\mathrm{mult}_p - 1)^2.$$

Finally, let us also present a cohomological description of free arrangements, see [6] for details.

Theorem 2.9. *Let $C \subset \mathbb{P}^2_{\mathbb{C}}$ be a reduced curve of degree d and let $f = 0$ be its defining equation. Then C is free if and only if then the minimal free resolution of the Milnor algebra $M(f)$ has the following form:*

$$0 \to S(-d_1 - (d-1)) \oplus S(-d_2 - (d-1)) \to S^3(-d+1) \to S$$

with $d_1 + d_2 = d - 1$. The pair (d_1, d_2) is called the set of exponents of C.

3. Proof of Main Result

Dowód. Here we want to present the main idea standing behind our proof. First of all, the table below presents all known sporadic simplicial line arrangements in the real projective plane having at most 27 lines. We have, according to Cuntz's catalogue, around 70 such arrangements. In the table below we provide additionally the total Milnor number of a given arrangement $\mathcal{A}(x, y)$ (here x denotes the number of lines in the given arrangement and y its type), the discriminant $\triangle_r$ for (1) computed with respect to r as variable, and we provide information about the roots of (1) computed with respect to r.

Here is the outline of our strategy:

- Among all sporadic simplicial line arrangements we detect those for which $\sqrt{\triangle_r}$ is an integer.
- For those line arrangements with an integral value of $\sqrt{\triangle_r}$, we extract all arrangements for which (1), computed with respect to r, has integral roots.
- Finally, after the above two-step process, we compute the minimal free resolutions of Milnor algebras, minimal degrees of the Jacobian relations and, based on that information, we detect those sporadic arrangements which are nearly free and not free.

We start with the aforementioned table.

Tabela 1: The list of sporadic simplicial line arrangements up to 27 lines

$\mathcal{A}(n,k)$	$(t_2, t_3, \ldots)$	$\mu(\mathcal{L})$	$\triangle_r$	roots
$\mathcal{A}(7,1)$	$(3,6)$	27	> 0	$r_1 = 2,\ r_2 = 4$

Tabela 1 – continued from the previous page

$\mathcal{A}(n,k)$	$(t_2,t_3,\ldots)$	$\mu(\mathcal{L})$	$\triangle_r$	roots
$\mathcal{A}(9,1)$	$(6,4,3)$	49	>0	real
$\mathcal{A}(10,2)$	$(6,7,3)$	61	>0	real
$\mathcal{A}(10,3)$	$(6,7,3)$	61	>0	real
$\mathcal{A}(11,1)$	$(7,8,4)$	75	>0	$r_1=4,\ r_2=6$
$\mathcal{A}(12,2)$	$(8,10,3,1)$	91	>0	real
$\mathcal{A}(12,3)$	$(9,7,6)$	91	>0	real
$\mathcal{A}(13,2)$	$(12,4,9)$	109	>0	real
$\mathcal{A}(13,3)$	$(10,10,3,2)$	109	>0	real
$\mathcal{A}(13,4)$	$(6,18,3)$	105	<0	complex
$\mathcal{A}(14,2)$	$(11,12,4,2)$	127	>0	real
$\mathcal{A}(14,3)$	$(9,16,4,1)$	125	<0	complex
$\mathcal{A}(14,4)$	$(10,14,4,0,1)$	127	>0	real
$\mathcal{A}(15,1)$	$(15,10,0,6)$	151	>0	real
$\mathcal{A}(15,2)$	$(13,12,6,2)$	147	>0	$r_1=6,\ r_2=8$
$\mathcal{A}(15,3)$	$(12,13,9)$	145	<0	complex
$\mathcal{A}(15,4)$	$(12,14,6,0,1)$	147	>0	$r_1=6,\ r_2=8$
$\mathcal{A}(15,5)$	$(9,22,0,3)$	145	<0	complex
$\mathcal{A}(16,2)$	$(14,15,6,1,1)$	169	>0	real
$\mathcal{A}(16,3)$	$(15,13,6,3)$	169	>0	real
$\mathcal{A}(16,4)$	$(15,15,0,6)$	171	>0	real
$\mathcal{A}(16,5)$	$(14,16,3,4)$	169	>0	real
$\mathcal{A}(16,6)$	$(15,12,9,0,1)$	169	>0	real
$\mathcal{A}(16,7)$	$(12,19,6,0,1)$	167	<0	complex
$\mathcal{A}(17,2)$	$(16,16,7,0,2)$	193	>0	real
$\mathcal{A}(17,3)$	$(18,12,7,4)$	193	>0	real
$\mathcal{A}(17,4)$	$(16,16,7,0,2)$	193	>0	real
$\mathcal{A}(17,5)$	$(16,18,1,6)$	193	>0	real
$\mathcal{A}(17,6)$	$(16,15,10,0,1)$	191	0	$r_0=8$
$\mathcal{A}(17,7)$	$(13,22,7,0,1)$	189	<0	complex
$\mathcal{A}(17,8)$	$(14,20,7,2)$	189	<0	complex
$\mathcal{A}(18,2)$	$(18,18,6,3,1)$	217	>0	real
$\mathcal{A}(18,3)$	$(19,16,6,5)$	217	>0	real
$\mathcal{A}(18,4)$	$(18,19,3,6)$	217	>0	real
$\mathcal{A}(18,5)$	$(18,19,3,6)$	217	>0	real
$\mathcal{A}(18,6)$	$(18,16,12,0,1)$	215	<0	complex
$\mathcal{A}(18,7)$	$(18,18,6,3,1)$	217	>0	real
$\mathcal{A}(18,8)$	$(16,22,6,2,1)$	215	<0	complex
$\mathcal{A}(19,1)$	$(21,18,6,0,4)$	247	>0	real
$\mathcal{A}(19,2)$	$(21,18,6,6)$	243	>0	$r_1=8,\ r_2=10$
$\mathcal{A}(19,3)$	$(24,12,6,6,1)$	247	>0	real
$\mathcal{A}(19,4)$	$(20,20,6,4,1)$	243	>0	$r_1=8,\ r_2=10$
$\mathcal{A}(19,5)$	$(20,20,6,4,1)$	243	>0	$r_1=8,\ r_2=10$
$\mathcal{A}(19,6)$	$(20,20,6,4,1)$	243	>0	$r_1=8,\ r_2=10$
$\mathcal{A}(19,7)$	$(21,15,15,0,1)$	241	<0	complex
$\mathcal{A}(20,2)$	$(25,15,10,6)$	271	>0	real
$\mathcal{A}(20,3)$	$(21,24,6,4,0,1)$	271	>0	real
$\mathcal{A}(20,4)$	$(23,20,7,5,1)$	271	>0	real
$\mathcal{A}(20,5)$	$(20,26,4,4,0,0,1)$	273	>0	real
$\mathcal{A}(21,2)$	$(30,10,15,6)$	301	>0	real
$\mathcal{A}(21,3)$	$(24,24,9,0,4)$	301	>0	real
$\mathcal{A}(21,4)$	$(22,28,6,4,0,0,1)$	301	>0	real

Tabela 1 – continued from the previous page

$\mathcal{A}(n,k)$	$(t_2, t_3, \ldots)$	$\mu(\mathcal{L})$	$\triangle_r$	roots
$\mathcal{A}(21,5)$	$(26,20,9,4,2)$	301	>0	real
$\mathcal{A}(21,6)$	$(25,20,15,2,1)$	297	<0	complex
$\mathcal{A}(21,7)$	$(24,22,15,3)$	295	<0	complex
$\mathcal{A}(22,2)$	$(24,30,12,3,1)$	325	<0	complex
$\mathcal{A}(22,3)$	$(27,28,0,12)$	331	>0	real
$\mathcal{A}(22,4)$	$(27,25,9,3,3)$	331	>0	real
$\mathcal{A}(22,5)$	$(12,58,0,0,3)$	319	<0	complex
$\mathcal{A}(23,1)$	$(27,32,10,4,2)$	359	<0	complex
$\mathcal{A}(23,2)$	$(16,56,2,0,1,2)$	355	<0	complex
$\mathcal{A}(24,2)$	$(32,32,0,12,0,0,1)$	401	>0	real
$\mathcal{A}(24,3)$	$(31,32,9,5,3)$	395	<0	complex
$\mathcal{A}(24,4)$	$(20,54,4,0,0,2,1)$	393	<0	complex
$\mathcal{A}(25,2)$	$(36,28,15,0,6)$	433	>0	real
$\mathcal{A}(25,3)$	$(30,40,15,6)$	421	<0	complex
$\mathcal{A}(25,4)$	$(36,30,9,6,4)$	433	>0	real
$\mathcal{A}(25,5)$	$(36,32,0,8,4,0,1)$	441	>0	real
$\mathcal{A}(25,6)$	$(36,30,9,6,4)$	433	>0	real
$\mathcal{A}(25,7)$	$(33,34,12,2,3,0,1)$	433	>0	real
$\mathcal{A}(25,8)$	$(24,52,6,0,0,0,3)$	433	>0	real
$\mathcal{A}(26,2)$	$(35,40,10,11)$	461	<0	complex
$\mathcal{A}(26,3)$	$(37,36,9,6,3,1)$	469	>0	real
$\mathcal{A}(26,4)$	$(35,39,10,4,3,0,1)$	469	>0	real
$\mathcal{A}(27,1)$	$(40,40,6,14,1)$	503	<0	complex
$\mathcal{A}(27,2)$	$(39,40,10,6,2,2)$	507	>0	$r_1=12,\ r_2=14$
$\mathcal{A}(27,3)$	$(39,40,10,6,2,2)$	507	>0	$r_1=12,\ r_2=14$
$\mathcal{A}(27,4)$	$(38,42,9,6,3,0,1)$	507	>0	$r_1=12,\ r_2=14$

Based on what we have seen so far, we can check directly that the following arrangements pass the first two steps of our selection, namely:

$$\mathcal{A}(7,1), \mathcal{A}(11,1), \mathcal{A}(15,2), \mathcal{A}(15,4), \mathcal{A}(17,6), \mathcal{A}(19,2), \mathcal{A}(19,4), \mathcal{A}(19.5), \mathcal{A}(19,6),$$
$$\mathcal{A}(27,2), \mathcal{A}(27,3), \mathcal{A}(27,4).$$

Now, according to Step 3, we present a detailed discussion regarding nearly freeness and freeness of the extracted arrangements.

$\mathcal{A}(7,1)$: The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-9) \to S^3(-6) \to S,$$

which means that $\mathcal{A}(7,1)$ is free.

$\mathcal{A}(11,1)$: The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-15) \to S^3(-10) \to S,$$

so $\mathcal{A}(11,1)$ is free.

$\mathcal{A}(15,2)$: The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-21) \to S^3(-14) \to S,$$

so $\mathcal{A}(15,2)$ is free.

$\mathcal{A}(15,4):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-21) \to S^3(-14) \to S.$$

so $\mathcal{A}(15,4)$ is free.

$\mathcal{A}(17,6):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S(-26) \to S^2(-25) \oplus S(-24) \to S^3(-16) \to S.$$

Since the minimal degree of the Jacobian relations r is equal to 8 and it satisfies Equation (1), **then $\mathcal{A}(17,6)$ is nearly free**.

$\mathcal{A}(19,2):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-27) \to S^3(-18) \to S,$$

so $\mathcal{A}(19,2)$ is free.

$\mathcal{A}(19,4):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-27) \to S^3(-18) \to S,$$

so $\mathcal{A}(19,4)$ is free.

$\mathcal{A}(19,5):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-27) \to S^3(-18) \to S,$$

so $\mathcal{A}(19,5)$ is free.

$\mathcal{A}(19,6):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-27) \to S^3(-18) \to S,$$

so $\mathcal{A}(19,6)$ is free.

$\mathcal{A}(27,2):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S(-51) \to S(-49) \oplus S(-41) \oplus S(-39) \to S^3(-26) \to S,$$

so according to Theorem 2.7 arrangement $\mathcal{A}(27,2)$ is not nearly free.

$\mathcal{A}(27,3):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S(-51) \to S(-49) \oplus S(-41) \oplus S(-39) \to S^3(-26) \to S,$$

so according to Theorem 2.7 arrangement $\mathcal{A}(27,3)$ is not nearly free.

$\mathcal{A}(27,4):$ The minimal free resolution of the Milnor algebra has the following form

$$0 \to S^2(-39) \to S^3(-26) \to S,$$

so $\mathcal{A}(27,4)$ is free.

This completes the proof. $\square$

Acknowledgments

I would like to thank Piotr Pokora for his guidance during the project and useful suggestions.

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 83–93
DOI: https://doi.org/10.18778/8331-092-3.08

REALIZABILITY OF SOME BÖRÖCZKY ARRANGEMENTS OVER THE RATIONAL NUMBERS

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Streszczenie. In this paper, we study the parameter spaces for Böröczky arrangements B_n of n lines, where $n < 12$. We prove that up to $n = 12$, there exist only one arrangement nonrealizable over the rational numbers, that is B_{11}.

1. Introduction

Recently, some considerations about realizability of Böröczky configurations over the rational numbers have shown up, especially in algebra and combinatorics. An excuse for such research is the problem of containment relations between the symbolic and ordinary powers of homogeneous ideals. The Böröczky arrangement of 12 lines was the first counterexample for some hypothesis in this area over the reals. In [9], using the parameter space, it was shown that this arrangement is relizable over the rational numbers and also that 12 lines is the minimal number of Böröczky lines, where intersection points give a similar counterexample.

In this context, some new results appeared with references to the higher number of lines. The aim of this paper is to complete the picture for number of lines between 3 and 11 in Böröczky arrangements and to establish the realizability of these configurations over the rational numbers.

According to [2], the Böröczky configurations were originally introduced in connection with the orchard problem. Böröczky described his construction to some mathematicians but he never published this results. In [2], these configurations are concidered in a relation to the celebrated Sylvester-Gallai Theorem. In [7, 8] they appear as configurations with a large number of ordinary lines.

2010 *Mathematics Subject Classification.* 52C30, 14N20, 05B30.
Key words and phrases. line arrangement, parameter space, singular points, Böröczky configuration.

The interest was recently renewed with a linkaye to the containment problem studied in commutative algebra (see details in [3] and [5]). Our research are inspired by papers [9] and [6], where the parameter spaces of some Böröczky arrangements were considered.

Let us denote by B_n the configuration of n lines arranged with Böröczky construction. Up to now, there were published such results for configurations B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{18} and B_{24}. The Böröczky arrangement of 12 lines is up to now the only known Böröczky configuration realizable over the rational numbers. We mean by this that there exists a configuration of 12 lines with the same incidences between the lines and the intersection points, which all the points have coordinates being the rational numbers. Since, in connection with the containment problem, there were considered only arrangements with at least 12 lines, we fill the gap in picture for $3 \leqslant n \leqslant 11$.

The Böröczky configurations B_n were described in [7]. Following this, we present here the construction.

Consider an $2n$-gon inscribed in a circle. Let us fix one of the $2n$ points and denote it by Q_0. By Q_α we mean the point arising by the rotation of Q_0 around the center of a circle by some angle α.

We take the following n lines:

$$B_n = \left\{ Q_\alpha Q_{\pi-2\alpha}, \text{ where } \alpha = \frac{2k\pi}{n} \text{ for } k = 0, 1, \ldots, n-1 \right\}.$$

If $\alpha \equiv (\pi - 2\alpha)(mod\ 2\pi)$, then $Q_\alpha Q_{\pi-2\alpha}$ is the tangent to the circle at the point Q_α.

These configurations have the maximal numbers of triple intersection points estimated in [8], with reference to n, namely

$$t_3 = 1 + \left\lfloor \frac{n(n-3)}{6} \right\rfloor.$$

2. Realization of line configurations

By a configuration we mean an ordered pair $A = (S, L)$, where a set L is a finite family of lines and by S we denote the set of all their intersection points. The realizability problem for configurations is intensively studied during the last few decades. Sturmfels in [10] establishes a connection between the realizability of projective configurations and some polynomial identity, so called *final polynomial.* Instead of this, we consider a system of equations, which are the generators of some standard basis connected with the configuration.

Following [10], we recall some basic notions necessary in the future considerations.

Let $\mathbb{K}$ be an arbitrary field of characteristic 0 and let $\varphi\colon S \longrightarrow \mathbb{K}^3$ be a mapping such that

$$s \longmapsto r_s = (r_{s,1}, r_{s,2}, r_{s,3})^T.$$

We call φ a *realization* of a configuration A over a field $\mathbb{K}$ if the following conditions are equivalent:

- $det(r_i, r_j, r_k) = 0$,
- $i, j, k \in S$ are contained in some line of A.

If $|S| = n$, then every realization of A can be though as a $3 \times n$ matrix, which columns are the coordinates of points of S. We call such matrix as *points matrix of A*.

Directly from definition, the 3×3 minors of points matrix are 0 iff their collumns are the cordinates of collinear points. Hence the realizations of A correspond to labeled subsets of the projective plane $\mathbb{P}^2(\mathbb{K})$ which satisfy the given incidence structure. The subset $\mathbb{F} \subset \mathbb{K}$ corresponding to a realization of configuration A (i.e. entries of matrix are the elements of $\mathbb{F}$) is called the *realization space of A*.

Realizability of configuration can be expressed in the language of polynomials.

Theorem 2.1 ([10], Theorem 3.2)**.** *The following problems are polynomially equivalent:*

- *Do the polynomials of the set $\{f_1, \dots, f_m\}$ have the common zero in $\mathbb{K}^n$?*
- *Is a configuration A realizable over $\mathbb{K}$?*

A parametrization of the realization space can be found by an analysis of polynomials of the standard basis for some polynomials connected with the configuration.

Let M_A be a points matrix of A and let $\{f_1, \dots, f_m\}$ be a subset of polynomials in $\mathbb{K}[x_1, \dots, x_n]$ with no common zero in $\overline{\mathbb{K}^n}$, where f_i are minors of collinear points of A. We define the auxiliary polynomials

$$\widehat{f_i}(x_1, \dots, x_n, t_1, \dots, t_m) := f_i(x_1, \dots, x_n) - t_i$$

with the slack variables t_i. Thus $t_i = 0$ if and only if the proper points are collinear. Let $\widehat{G}$ be a Gröbner basis of the set $\{\widehat{f_1}, \dots, \widehat{f_m}\}$ with pure lexicographic order

$$t_1 < \dots < t_m < x_1 < \dots < x_n. \tag{1}$$

Then the generators of $\widehat{G}$ designate the realization space of A (compare to [10], Theorem 6.2). Order (1) assures that the variables $x_1, \dots, x_n$ appears in generators of $\widehat{G}$ with relatively low powers, comparing to variables $t_1, \dots, t_m$. It is the main reason why we introduce these additional variables. Taking into consideration that finally $t_i = 0$ for collinear points, we obtain emphatically simpler conditions involving coordinates x_i, than computing a Gröbner basis of the set $\{f_1, \dots, f_m\}$ directly.

It leads to the explicit algorithm allowing us to conclude a realizability of some configurations. An algorithm is based on general ideas of Sturmfels [10] combined with methods established in [9].

We carry out the construction in the following way:

Step 1 : We fix matrix $M_A = (r_1, r_2, \ldots, r_s) \in M_{3\times s}$ of triple points of the configuration.

Step 2 : We establish the family of equations $\mathfrak{F} = \{f_1, f_2, \ldots, f_k\}$, where f_i are the 3×3 minors of M_A with 3 collinear points as the columns.

Step 3 : We define the family of auxiliary equations $\hat{\mathfrak{F}} = \{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_k\}$ with slack variables.

Step 4 : We compute a Gröbner basis $\widehat{G}$ of $\hat{\mathfrak{F}}$ in the following way. We divide the set $\hat{\mathfrak{F}}$ into finite number of subsets (not necessary disjoint), which sum is all $\hat{\mathfrak{F}}$. We take the ideals of these sets and compute their sum. Finally, the basis of sum of ideal is the basis of $\hat{\mathfrak{F}}$. We substitute $t_i := 0$.

Step 5 : (Optionally) We use one of conditions determined by the elements of $\widehat{G}$ (with no variables t_i) to eliminate some of variables $x_1, \ldots, x_n$. After such substitution we repeat **Steps** $1-4$ for matrix M_A with reduced number of variables.

Step 6 : We repeat all algorithm step by step until we obtain condition clearly designating the realization space of configuration (or eventually we obtain condition excluding realization of configuration over some taken field).

3. Realizability of Böröczky configurations over the rationals

Below we present detailed algorithm for Böröczky configurations B_8 and B_{11}. We establish in this way, which of them are realizable over the rationals.

From now on, if there is no additional informations about fixed point, we assume $P_i = (x_i : y_i : z_i)$. General idea in the first step of algorithm is to introduce as many parameters as necessary and reduce considerably necessary parameters, using some obvious incidences.

Example 3.1. (Configuration of 8 lines)

Step 1:

We start with finding the matrix M_A. We fix the first four appropriate points in the arrangement as the fundamental points: $P_1 = (1 : 0 : 0)$, $P_2 = (0 : 1 : 0)$, $P_3 = (0 : 0 : 1)$ and $P_4 = (1 : 1 : 1)$. They give as the beginning five lines of the construction, namely P_1P_2, P_2P_3, P_1P_3, P_3P_4 and P_2P_4 (lines distinguished with bold solid line in the Figure 1).

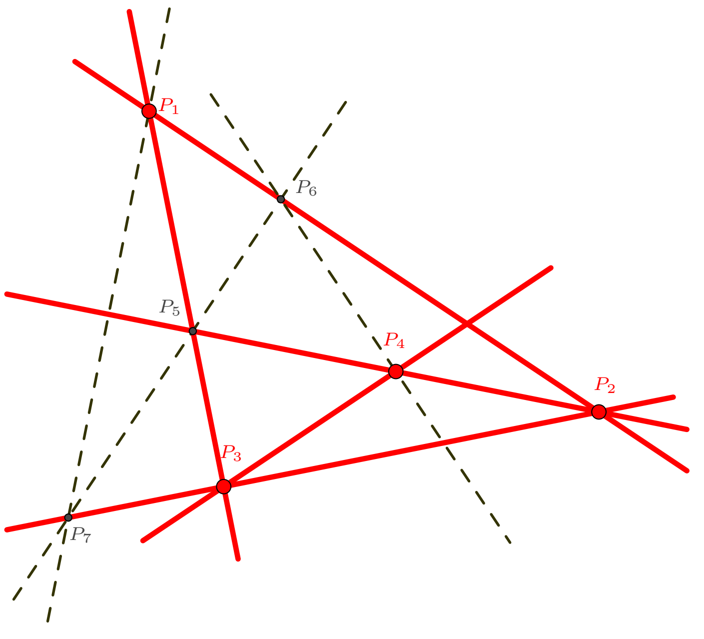

FIGURE 1

Automatically we obtain one more point:

$$P_5 = P_1P_3 \cap P_2P_4 = (1 : 0 : 1).$$

The last two points of the configuration are taken as some free points on the fixed lines and they are expressed with parameters:

$$P_6 = (x_6 : 1 : 0) \in P_1P_2,$$
$$P_7 = (0 : y_7 : 1) \in P_2P_3.$$

Thus the points matrix of the configuration is the following

$$M_A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & x_6 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & y_7 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}.$$

The remaining three lines of the construction are P_4P_6, P_1P_7, P_5P_6 (distinguished as the dashed lines).

Step 2:

The lines P_3P_4, P_1P_7, P_4P_6 contain only two from points $\{P_1, \ldots, P_7\}$. Remaining five lines contain exactly three of them. The points are grouped on the lines as follows:

$\{P_1, P_2, P_6\}$, $\{P_2, P_3, P_7\}$, $\{P_2, P_4, P_5\}$, $\{P_5, P_6, P_7\}$, $\{P_1, P_3, P_5\}$.

The only collinearity demanding to check is for points P_5, P_6, P_7. The rest of them are automatically satisfied. Thus

$$\mathcal{F} = \{det(P_5, P_6, P_7)\}.$$

Step 3:

We have only one auxiliary equation with slack variable t_1

$$\hat{f}_1 = x_6y_7 + 1 - t_1.$$

Step 4:

The basis of an ideal $< x_6y_7 + 1 - t_1 >$ with $t_1 = 0$ is

$$\widehat{G} = \{x_6y_7 + 1\}.$$

Step 5:

Not applicable.

Step 6:

Since condition $x_6y_7 + 1 = 0$ may be fulfilled by infinitely many pairs of rational numbers (x_6, y_7), the configuration B_8 can be realized over rationals.

Analogously we may easily check, that all remaining Böröczky configurations B_n with $3 \leqslant n \leqslant 10$ are realizable over the rational numbers. In [9], there was proved that also B_{12} may be realized over rationals.

In fact for $n \leqslant 12$ there exist only one configuration in this family, which can not be obtained over the field of rational numbers, namely B_{11}. We prove it in Example 3.2 by showing the resulting of algorithm in this case.

Example 3.2. (Configuration of 11 lines)

Step 1:

We start with finding the matrix M_A. As a core of configuration, we fix the first four appropriate points in the arrangement as the fundamental points:

$$P_1 = (1:0:0), \quad P_2 = (0:1:0), \quad P_3 = (0:0:1), \quad P_4 = (1:1:1).$$

They give us the beginning five lines of the construction, namely P_1P_2, P_2P_3, P_1P_3, P_3P_4 and P_2P_4 (distinguished with bold solid lines in the Figure 2). Automatically we obtain two more points:

$$P_5 = P_1P_3 \cap P_2P_4 = (1:0:1),$$

$$P_{15} = P_1P_2 \cap P_3P_4 = (1:1:0).$$

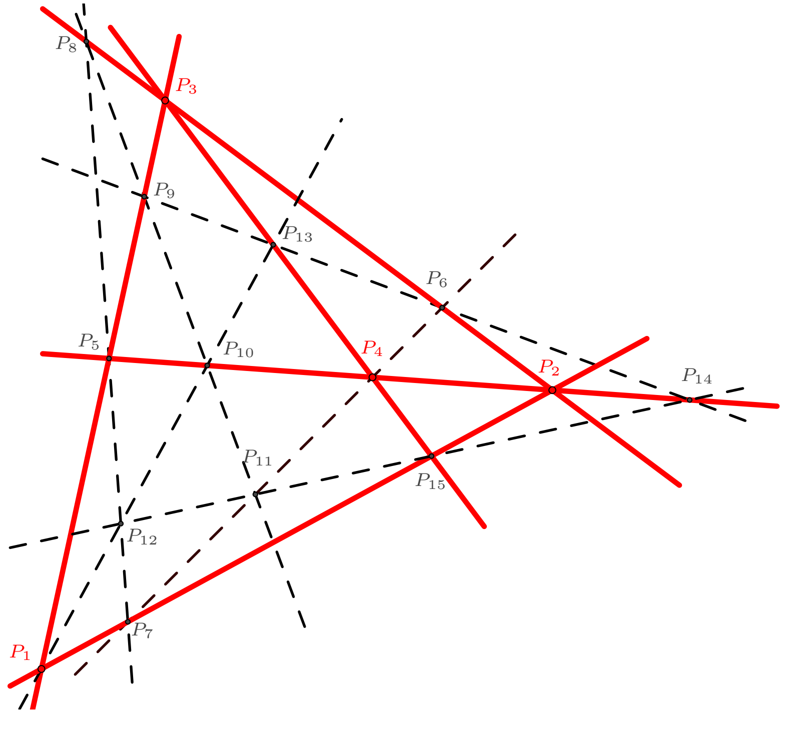

FIGURE 2

Remaining points of the configuration are taken as some free points on the fixed lines and they are expressed with parametres:

$$\begin{aligned}
P_6 &= (0 : y_6 : 1) \in P_2P_3,\\
P_7 &= (1 : 1 - y_6 : 0) \in P_1P_2 \cap P_4P_6,\\
P_8 &= (0 : y_6 - 1 : 1) \in P_2P_3 \cap P_5P_7,\\
P_9 &= (x_9 : 0 : 1) \in P_1P_3,\\
P_{10} &= (1 : y_{10} : 1) \in P_2P_4,\\
P_{11} &= (1 : y_6 \cdot z_{11} - y_6 + 1 : z_{11}) \in P_4P_6,\\
P_{12} &= (1 : 1 - y_6 + z_{12}(y_6 - 1) : z_{12}) \in P_5P_7,\\
P_{13} &= (1 : 1 : z_{13}) \in P_3P_4,\\
P_{14} &= (1 : y_{14} : 1) \in P_2P_4.
\end{aligned}$$

Thus the points matrix of configuration in this case is the following

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & x_9 & 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 1 & 0 & 1 & 0 & y_6 & 1-y_6 & y_6-1 & 0 & y_{10} & y_6\cdot(z_{11}-1)+1 & (z_{12}-1)(y_6-1) & 1 & y_{14} & 1\\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & z_{11} & z_{12} & z_{13} & 1 & 0 \end{pmatrix}.$$

The remaining six lines of the construction are P_4P_6, P_5P_7, P_8P_9, P_1P_{10}, P_6P_9 and $P_{11}P_{15}$.

Step 2:

Triple points $P_1, \ldots, P_{15}$ are grouped on the lines in the following sets (compare with Figure 2):

$\{P_1, P_2, P_7, P_{15}\}$, $\{P_2, P_3, P_6, P_8\}$,
$\{P_2, P_4, P_5, P_{10}, P_{14}\}$,
$\{P_3, P_4, P_{13}, P_{15}\}$, $\{P_1, P_3, P_5, P_9\}$,
$\{P_1, P_{10}, P_{12}, P_{13}\}$,
$\{P_8, P_9, P_{10}, P_{11}\}$, $\{P_5, P_7, P_8, P_{12}\}$,
$\{P_4, P_6, P_7, P_{11}\}$,
$\{P_6, P_9, P_{13}, P_{14}\}$, $\{P_{11}, P_{12}, P_{14}, P_{15}\}$.

Some of these collinearities results directly from the construction (for example P_6 is taken as a point on the line P_2P_3). Remaining collinearities generate the family of polynomials $\mathcal{F}$, where the polynomials are the following determinants:

$$\begin{aligned}
f_1 &= det(P_8, P_9, P_{10}), & f_2 &= det(P_8, P_9, P_{11}),\\
f_3 &= det(P_1, P_{10}, P_{12}), & f_4 &= det(P_1, P_{10}, P_{13}),\\
f_5 &= det(P_6, P_9, P_{13}), & f_6 &= det(P_6, P_9, P_{14}),\\
f_7 &= det(P_{12}, P_{14}, P_{15}), & f_8 &= det(P_{11}, P_{14}, P_{15}).
\end{aligned}$$

Step 3:

We introduce the auxiliary variables $t_1, \ldots, t_8$ and we define the family of equations $\hat{\mathfrak{F}} = \{\hat{f}_1, \hat{f}_2, \ldots, \hat{f}_8\}$, where

$$\hat{f}_i = f_i - t_i.$$

Step 4:

We consider the ideals $I =< \hat{f}_1, \ldots \hat{f}_7 >$ and $J =< \hat{f}_4, \ldots, \hat{f}_8 >$. We take $I + J$ and computing with Singular [4] we obtain its basis. Substituting $t_i := 0$ we have

$$\widehat{G} = \{z_{12}^2 \cdot z_{13}^2 - z_{12} \cdot z_{13}^2 - z_{12} + z_{13}, z_{11} \cdot z_{13} - 1, z_{11} \cdot z_{12} - z_{12}^2 \cdot z_{13} + z_{12} \cdot z_{13} - 1,$$
$$y_{14} - z_{12} \cdot z_{13} + z_{13} - 1, y_{10} - z_{11}, y_6 \cdot z_{13} - y_6 + z_{12} \cdot z_{13} - z_{13},$$
$$y_6 \cdot z_{12} - y_6 - z_{12}^2 \cdot z_{13} + z_{12} \cdot z_{13} - z_{12}, y_6 \cdot z_{11} - y_6 - z_{12} + 1, x_9 - z_{12}\}.$$

Step 5:

We make substitution using condition $x_9 - z_{12} = 0$. We repeat all algorithm for matrix M_1 without variable z_{12}. We obtain a new Gröbner basis

$$\widehat{G}_1 = \{z_{11} \cdot z_{13} - 1, y_{14}^2 - y_{14} \cdot z_{11} + y_{14} \cdot z_{13} - 2 \cdot y_{14} + z_{11}, y_{10} - z_{11}, y_6 \cdot z_{13} - y_6 + y_{14} - 1,$$
$$y_6 \cdot z_{11} - y_6 - y_{14} \cdot z_{11} + z_{11}, y_6 \cdot y_{14} - y_6 - y_{14} \cdot z_{11} - y_{14} + z_{11}, x_9 - y_{14} \cdot z_{11} + z_{11} - 1\}.$$

We make new substitution using condition $y_{10} - z_{11} = 0$. We obtain the following basis, independent of variable z_{11}:

$$\widehat{G}_2 = \{y_{14}^2 \cdot z_{13} + y_{14} \cdot z_{13}^2 - 2 \cdot y_{14} \cdot z_{13} - y_{14} + 1, y_{10} \cdot z_{13} - 1,$$
$$y_{10} \cdot y_{14} - y_{10} - y_{14}^2 - y_{14} \cdot z_{13} + 2 \cdot y_{14}, y_6 \cdot z_{13} - y_6 + y_{14} - 1, y_6 \cdot y_{14} - y_6 - y_{14}^2 - y_{14} \cdot z_{13} + y_{14},$$
$$y_6 \cdot y_{10} - y_6 - y_{14}^2 - y_{14} \cdot z_{13} + 2y_{14}, x_9 - y_{14}^2 - y_{14} \cdot z_{13} + 2y_{14} - 1\}.$$

Step 6:

Let us focus on the condition:

$$y_{14}^2 \cdot z_{13} + y_{14} \cdot z_{13}^2 - 2 \cdot y_{14} \cdot z_{13} - y_{14} + 1 = 0.$$

It is a plane cubic in variables y_{14} and z_{13}. To make further considerations more transparent, we substitute $y_{13} := u$ and $z_{14} := v$. Thus we have curve

$$C : u^2 v + uv^2 - 2uv - u + 1 = 0.$$

By homogenization we obtain plane projective cubic

$$\widetilde{C} : u^2 v + uv^2 - 2uvw - uw^2 + w^3 = 0.$$

Using *Magma* computations ([1]), we verify that $\widetilde{C}$ is an elliptic curve with only five rational points, namely

$$(1 : 1 : 1), \quad (1 : 0 : 1), \quad (0 : 1 : 0), \quad (1 : 0 : 0), \quad (-1 : 1 : 0).$$

Only first two of these points can be applied to the curve C. Remaining points are the points at infinity, while C is an affine plane cubic. But if $y_{14} = 1$, the configuration is degenerated. More precisely, $P_4 = P_{14}$. Analogously, when $z_{13} = 1$

or $z_{13} = 0$. Thus $P_{13} = P_4$ or $P_{13} = P_{15}$. We conclude that configuration of 11 Böröczky lines can not be realizable over the rational numbers.

Corollary 3.3. *The configurations B_n for $n \leqslant 12$ can be realizable over the rationals, except the case of $n = 11$.*

The proof of case $n = 12$ reader can find in [9].

Remark 3.4. We believe that, among all Böröczky arrangements B_n with $n > 10$, arrangement B_{12} is the only one realizable over the rationals. In [9], the authors consider B_{12} and B_{15} arrangements and they explain why B_{12} arrangement can be realized over the rationals. Furthermore, in [6], another set of authors consider cases with $n \in \{13, 14, 16, 18, 24\}$. In all these cases, it is directly proved that B_ns are not realizable over the rationals, or there is no evidence that any realization over rationals would not degenerate the whole construction, i.e., available tools do not allow us to decide the existence of another such realizations. We want to reveal additionally here some additional unpublished results for other values of $n < 30$. In these cases, arrangements are not realizable over the rationals.

Our aim is to understand in deep the case $n = 12$ in order to find some combinatorial features that can potentially give some evidence about the speciality of B_{12}. We hope to come back to such a discussion in a forthcoming article soon.

Acknowledgements. We would like to thank Grzegorz Malara and Piotr Pokora for helpful remarks on this text and for assistance with programming of some symbolic computations.

We also thank the knowledgeable referee for helpful remarks, which improved our paper.

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 95–119
DOI: https://doi.org/10.18778/8331-092-3.09

EFFECTIVE PROOF OF GUSEĬN-ZADE THEOREM THAT BRANCHES MAY BE DEFORMED WITH JUMP ONE

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Abstract. Let $f \in \mathbb{C}\{X,Y\}$ be a reduced series which defines a singular branch $f=0$ in a neighbourhood of zero in $\mathbb{C}^2$. Let $\mathbf{h} \geq 1$ be the number of characteristic exponets of a Puiseux root $y(X) \in \mathbb{C}\{X\}^*$ of the equation $f=0$. For any $k \in \{1,\ldots,\mathbf{h}\}$ we define the series $f_k \in \mathbb{C}\{X,Y\}$ generated by all terms of the series $y(X)$ with orders strictly smaller than the k-th characteristic exponent. We consider a deformation $F_t = f + tX^{\omega_0} f_1^{\omega_1} \ldots f_{\mathbf{h}}^{\omega_{\mathbf{h}}}$ ($t \neq 0$, small) where $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}}$ are nonnegative integers. Using a version of the Newton algorithm proposed by Cano we show how to choose exponents $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}}$ to obtain the Milnor number of the deformation F_t smaller by one than the Milnor number of the branch f. We prove a version of Kouchnirenko theorem which is useful in computation the Milnor number.

1. Introduction

Let $f \in \mathbb{C}\{X,Y\}$ be a reduced series which defines an isolated singularity in the neighbourhood of $0 \in \mathbb{C}^2$ and let $F \in \mathbb{C}\{T,X,Y\}$ be a series such that $F(0,X,Y) = f(X,Y)$ and $F_t \in \mathbb{C}\{X,Y\}$ are isolated singularities for small $t \in \mathbb{C}$. The series F is called a deformation of the singularity of f. For any series $g,h \in \mathbb{C}\{X,Y\}$ the intersection multiplicity $(g,h)_0$ is defined as the $\mathbb{C}$ codimension of the ideal generated by g and h in $\mathbb{C}\{X,Y\}$. We consider the Milnor number $\mu(f) = (\partial f/\partial X, \partial f/\partial Y)_0$. At Arnold's seminar they asked what happened with the Milnor number of the singularity after deformation ([1], e.g. 1975–15, 1982–12). The semi-continuity of the Milnor number implies that $\mu(f) \geq \mu(F_t)$ (see: e.g. [9]). A basic notion that can be studied in this context is the minimal jump

2010 *Mathematics Subject Classification.* Primary 32S30; Secondary 14B07.
Key words and phrases. plane curve singularity, Milnor number, deformations of singularities, Newton algorithm.

of the Milnor number $\mu(f) - \mu(F_t)$ where F_t runs over all deformations of singularity. In [10] Guseĭn-Zade showed that there exist reducible singularities which the minimal jump greater than one. Moreover, he proved that this jump equals one for branches. The proofs of the above mentioned results are not effective. The effective proof of the second result is the aim of this note. The effective proof of the first type result was obtained by Brzostowski and Krasiński in [3]. Many results concerning deformations of homogeneous singularities can be found in [4].

Bodin in [2] used the Kouchnirenko theorem [14] in order to obtain an effective construction of the deformation. We recall the Kouchnirenko theorem in dimension two. For any series $f = \sum c_{\alpha\beta} X^\alpha Y^\beta$ we consider its Newton diagram $\Delta(f)$ which is the convex hull of the union of the sets $(\alpha, \beta) + \mathbb{R}_+^2$ where (α, β) runs over all nonzero coefficients of f; $\mathbb{R}_+ = \{x : x \geq 0\}$. Assume that the Newton diagram has the vertex $(a, 0)$ on the horizontal axis and the vertex $(0, b)$ on the vertical axis. Note that if f is singular then $a, b \geq 2$. Let P denotes the area of the polygon bounded by the boundary of the diagram $\Delta(f)$ and by the coordinate axes. The Kouchnirenko theorem states that $\mu(f) \geq 2P - a - b + 1$.

In order to describe the equality case in the formula of Kouchnirenko we need the notion of nondegeneracy. We consider the Newton polygon $\mathcal{N}(f)$ which is the set of compact boundary faces (pairwise nonparallel) of the Newton diagram $\Delta(f)$. For any face (segment) $S \in \mathcal{N}(f)$ we define the initial form $\mathrm{in}(f, S)$ as the sum of all monomials $c_{\alpha\beta} X^\alpha Y^\beta$ of f such that $(\alpha, \beta) \in S$. We say that the series f is nondegenerate on S if the initial form $\mathrm{in}(f, S) \in \mathbb{C}[X, Y]$ has only single factors different from the powers of variables X or Y. We say that the series f is nondegenrate (in Kouchnirenko sense) if it is nondegenerate on every segment of the Newton polygon $\mathcal{N}(f)$. In the case of nondegeneracy we have the equality in the formula of Kouchnirenko. The opposite implication is true in dimention two (see e.g. [7]).

For any coprime integers p and q such that $p > q \geq 2$ let us consider a nondegenerate singularity $f = X^p + Y^q$. In the mentioned paper, Bodin proposed the deformation $F_t = X^p + Y^q + tX^{\tilde{\alpha}} Y^{\tilde{\beta}}$. Using the elementary number theory it is possible to choose $(\tilde{\alpha}, \tilde{\beta})$ below the segment joining $(0, q)$ and $(p, 0)$ such that $0 < \tilde{\alpha} < p$, $0 < \tilde{\beta} < q$ and the area of the triangle with vertices $(0, q)$, $(\tilde{\alpha}, \tilde{\beta})$, $(p, 0)$ equals $\frac{1}{2}$. By Kouchnirenko Theorem we get $\mu(F_t) = \mu(f) - 1$ for $t \neq 0$. This idea was developed by Michalska and Walewska in [21]. They showed for the considered singularity that every number from 1 to $r(q - r)$ can be the jump of the Milnor number of f where r is the remainder of division p by q.

The main result of our note (Theorem 1.1) may be treated as a generalization of the mentioned above observation of Bodin. Before presenting the result let us recall the ring of Puiseux sereis $\mathbb{C}\{X\}^* = \bigcup_{N \geq 1} \mathbb{C}\{X^{1/N}\}$ [23, 22, 20, 18]. For any positive integer number v_0 we consider a series $y \in \mathbb{C}\{X^{1/v_0}\}$. For nonzero y we can write

$$y = a_1 X^{v_1/v_0} + a_2 X^{v_2/v_0} + \dots, \quad a_1, a_2, \dots \neq 0, \tag{1}$$

$0 < v_1 < v_2 < \dots$ integers. We call $(v_0, v_1, v_2, \dots)$ a sequence associated with y. With $y = 0$ we associate the sequence (v_0). The elements of every two sequences associated with $y \in \mathbb{C}\{X\}^*$ are proportional. Therefore, there exists exactly one sequence associated with y for which the greatest common divisor of its elements equals 1. Let $\mathcal{G}(v_0)$ denotes the group of unity roots of degree v_0. For every $\tau \in \mathcal{G}(v_0)$ we define the action

$$\tau * y = a_1 \tau^{v_1} X^{v_1/v_0} + a_2 \tau^{v_2} X^{v_2/v_0} + \dots \; . \tag{2}$$

Let τ be a primitive root of $\mathcal{G}(v_0)$. The series $\tau^0 * y, \tau^1 * y, \dots, \tau^{v_0 - 1} * y$ are called the conjugations of y in $\mathbb{C}\{X^{1/v_0}\}$. The conjugation of the zero series equals itself. The number of different conjugations of y equals $N = v_0/\mathrm{GCD}(v_0, v_1, \dots)$ (see: e.g. [20]). We obtain them for $i = 0, 1, \dots, N-1$. The different conjugations form the so-called cycle of series y. The number N and the cycle depend only on the series y. We write $N = N(y)$ for the number of elements and cycle(y) for the cycle.

By using Newton-Puiseux theorem (see e.g. [23], [18]) we conclude that for every branch f coprime to X there exists a series $y \in \mathbb{C}\{X^{1/v_0}\}$ with N-elemental cycle $\{\tau^0 * y, \tau^1 * y, \dots, \tau^{N-1} * y\}$, $N = v_0/\mathrm{GCD}(v_0, v_1, \dots)$, $\tau \in \mathcal{G}(v_0)$ a primitive root, such that the equality

$$f(X, Y) = \prod_{i=0}^{N-1} (Y - \tau^i * y) \tag{3}$$

is satisfied up to a unit factor. An argument of Galois theory shows that fractional powers do not appear on the right side of (3) [22]. We can assume this unit factor to be one without loss of generality.

By definition, a characteristic exponent of the series $y \in \mathbb{C}\{X\}^*$ is an exponent which can appear as the order of difference between the series y and its conjugation (e.g. [20]). The exponent v_ℓ/v_0 $(\ell = 1, 2, \dots)$ is characteristic if and only if

$$\mathrm{GCD}(v_0, \dots, v_{\ell-1}) > \mathrm{GCD}(v_0, \dots, v_\ell) \; . \tag{4}$$

The number $\mathbf{h} = \mathbf{h}(y)$ of characteristic exponents is less than or equal to $N(y) - 1$. Moreover, $\mathbf{h}(y) = 0 \Leftrightarrow N(y) = 1$ Let $\ell_1 < \dots < \ell_{\mathbf{h}}$ denote the characteristic positions and let $w^* = w^*(y) = \mathrm{GCD}(v_0, v_1, \dots)$. We define the Puiseux characteristic $(b_0, b_1, \dots, b_{\mathbf{h}})$ as $b_0 := v_0/w^*$, $b_1 := v_{\ell_1}/w^*$, $\dots$, $b_{\mathbf{h}} := v_{\ell_{\mathbf{h}}}/w^*$, the first sequence of divisors $e_k := \mathrm{GCD}(b_0, b_1, \dots, b_k)$ $(k = 0, 1, \dots, \mathbf{h})$ and the second sequence of divisors $n_k = e_{k-1}/e_k$ $(k = 1, \dots, \mathbf{h})$. We put $N_0 := 1$ and $N_k := n_1 \dots n_k$ for $k = 1, \dots, \mathbf{h}$. We have $N_k = b_0/e_k$ for $k = 0, 1, \dots, \mathbf{h}$. Classical characteristics of branches are described in [25].

For every $k \in \{1, 2, \dots, \mathbf{h}\}$ we define the series y_k as the sum of all terms of y of order strictly less then b_k/b_0. The cycle of y_k has N_{k-1} elements. We put

$$f_k(X, Y) = \prod_{i=0}^{N_{k-1}-1} (Y - \tau^i * y_k) \in \mathbb{C}[X, Y] \tag{5}$$

where $\tau \in \mathcal{G}(v_0)$ is a primitive root. The following theorem is the main result of this paper.

Theorem 1.1. *Let $f \in \mathbb{C}\{X,Y\}$ be a singular branch and let $y \in \mathbb{C}\{X\}^*$ be a Puiseux root of the equation $f=0$. Let $\mathbf{h}=\mathbf{h}(y)$ be the number of characteristic exponents ($\mathbf{h} \geq 1$) and let $f_1,\dots,f_{\mathbf{h}}$ be the series generated from y by cutting below the characteristic exponents. Then there exist nonnegative integers $\omega_0,\omega_1,\dots,\omega_{\mathbf{h}}$ such that the Milnor number of the deformation $F_t = f + tX^{\omega_0}f_1^{\omega_1}\dots f_{\mathbf{h}}^{\omega_{\mathbf{h}}}$ ($t \neq 0$, small) equals $\mu(f)-1$.*

In chapter 2 we present the Newton algorithm in version of Cano [5, 19]. In chapter 3 we present a variant of the Kouchnirenko theorem adopted to the Newton algorithm. In the last chapter of this note we prove Theorem 1.1.

2. The Newton Algorithm

Let us introduce some usefull notions. For any segment S of the Newton polygon we consider its *inclinaction* which is a rational number $|S|_{\mathbf{H}}/|S|_{\mathbf{V}}$ where $|S|_{\mathbf{H}}$ (resp. $|S|_{\mathbf{V}}$) is the lenght of the projection of S on the horizontal axis (resp. on the vertical axis). For a nonzero series $y \in \mathbb{C}\{X\}^*$ we define its initial form $\operatorname{in} y = aX^\theta$ ($a \neq 0$) as the term with the minimal order. By convention we put $\operatorname{in} 0 = 0$ and $\operatorname{ord} 0 = +\infty$. Let $f \in \mathbb{C}\{X,Y\}$ be a nonzero series and let $y \in \mathbb{C}\{X\}^*$ be a series of a positive order such that $\operatorname{in} y = aX^\theta$. Isaac Newton (in the letter to Odenburg) presented an observation that if y is a nonzero root of the series f (i.e. $f(X,y(X))=0$ in $\mathbb{C}\{X\}^*$) then there exists a segment S of the Newton polygon $\mathcal{N}(f)$ of inclination θ such that the initial form $\operatorname{in} y = aX^\theta$ is a nonzero root of the initial form $\operatorname{in}(f,S)$ in $\mathbb{C}\{X\}^*$. Therefore, the Newton polygon gives us the information about the orders of all nonzero solutions (of positive order). Moreover, we can read the number of such solutions from the shape of $\mathcal{N}(f)$. We denote by $\delta(f)$ the distance between the diagram $\Delta(f)$ and the horizontal axis. The zero solution $y=0$ appears if and only if $\delta(f)>0$.

The information of iniatial forms of solutions $y \in \mathbb{C}\{X\}^*$ of the equation $f=0$ may be expressed by using systems (see: notion of symmetric power [24]). For elements $a_1,\dots,a_p$ of a given set by the system $\mathcal{A} = \langle a_1,\dots,a_p\rangle$ we mean the sequence $a_1,\dots,a_p$ treated as unordered. We put $\deg\mathcal{A}=p$. Instead of

$$\langle \underbrace{a_1,\dots,a_1}_{m_1 \text{ times}},\dots,\underbrace{a_p,\dots,a_p}_{m_p \text{ times}}\rangle$$

we write $\langle a_1 : m_1,\dots,a_p : m_p\rangle$. For $\mathcal{A} = \langle a_1,\dots,a_p\rangle$ and $\mathcal{B} = \langle b_1,\dots,b_q\rangle$ we have a natural addition $\mathcal{A}\oplus\mathcal{B} = \langle a_1,\dots,a_p,b_1,\dots,b_q\rangle$ with the neutral element $\langle\ \rangle$ (empty system). By convention $\langle a:0\rangle = \langle\ \rangle$.

Let $f \in \mathbb{C}\{X,Y\}$ be a series such that $p := (f,X)_0 = \operatorname{ord} f(0,Y) \geq 1$. Let us denote by $\operatorname{Zer} f$ the system $\langle y_1,\dots,y_p\rangle$ of solutions of the equation $f=0$ in $\mathbb{C}\{X\}^*$. For $S \in \mathcal{N}(f)$ let $\operatorname{in}(f,S)^\circ$ denotes the form $\operatorname{in}(f,S)$ divided by the maximal possible powers of variables X and Y.

Theorem 2.1. (Newton-Puiseux) *Then*

(i) $\langle \operatorname{ord} y_1, \ldots, \operatorname{ord} y_p\rangle = \bigoplus_{S\in\mathcal{N}(f)} \langle |S|_{\mathbf{H}}/|S|_{\mathbf{V}} : |S|_{\mathbf{V}}\rangle \oplus \langle +\infty : \delta(f)\rangle$,

(ii) $\langle \operatorname{in} y_1, \ldots, \operatorname{in} y_p\rangle = \bigoplus_{S\in\mathcal{N}(f)} \operatorname{Zer}\operatorname{in}(f,S)^\circ \oplus \langle 0 : \delta(f)\rangle$,

(iii) $p = |\mathcal{N}(f)| + \delta(f)$.

Now, let aX^θ be a nozero root of an intial form $\operatorname{in}(f,S)$, $S\in\mathcal{N}(f)$. By Isaac Newton observation aX^θ is the first term of a Puiseux solution of $f=0$ in $\mathbb{C}\{X\}^*$. In order to find the second term Cano [5] consider the substitution

$$\tilde f = f(X, aX^\theta + Y)\ . \tag{6}$$

Observing the Newton diagram $\Delta f(X, aX^\theta + Y)$ he look for the boundary segments $S\in\mathcal{N}(\tilde f)$ with the inclination stricly greater then θ. Then he choose the second term as a nonzero root of $\operatorname{in}(\tilde f, S)$. He continue the process to construct all nonzero terms of all nonzero solutions.

In order to deal with substitutions of the type (6) we apply the ring $\mathbb{C}\{X^*, Y\} = \sum_{N\ge 1}\mathbb{C}\{X^{1/N}, Y\}$ and we analogously define all neccessary notions. In comparison to the classical algorithm, Cano's approche allows to analyze every step of the algorithm in the same coordinate system. The Newton algorithm is closely related to the Kuo-Lu tree technique (see [15]). The Newton diagram of the substitution of the type $f(X, z+Y)$, $f\in\mathbb{C}\{X,Y\}$, $z\in\mathbb{C}\{X\}^*$ is analized in [13], [16]. The first author of this note applied the Newton algorithm in Cano's version to determine the so-called polar quotients with their multiplicities [19]. A survey of results concerning polar invariants (quotients) is given in [12]. The more general are the so-called jacobian quotients [17].

Now, let us introduce some definitions and facts similar to that from [19]. Let us consider the ring of Pusiseux polynomials $\mathbb{C}[X]^* = \bigcup_{N\ge 0}\mathbb{C}[X^{1/N}]$. For any $\varphi\in\mathbb{C}[X]^*$ we have $\deg\varphi < +\infty$. We put $\deg 0 = 0$. Since we consider only polynomials of positive orders this convention does not lead to a contradiction. Let $f\in\mathbb{C}\{X,Y\}$ be a reduced series such that the number $p = \operatorname{ord} f(0,Y) = (f,X)_0$ is finite and positive. We denote $f_\varphi := f(X, \varphi + Y)\in\mathbb{C}\{X^*, Y\}$. For any polynomial φ of positive order the diagram $\Delta f(X, \varphi+Y)$ has the vertex $(0,p)$ lying on the horizontal axis.

We denote by $\mathcal{N}(f,\varphi)$ the subset of the polygon $\mathcal{N}(f_\varphi)$ which consists segments with inclinations strictly greater than $\deg\varphi$. We define the hight of the polygon $|\mathcal{N}(f,\varphi)|$ as the sum of lengths of the projections of its segments on the vertical axis. The number of solutions $y\in\operatorname{Zer} f$ of the form $y = \varphi + \ldots$ (equivalently $\operatorname{ord}(y-\varphi) > \deg\varphi$) equals $|\mathcal{N}(f,\varphi)| + \delta(f_\varphi)$. If f is reduced then $\delta(f_\varphi)\in\{0,1\}$.

Definition 2.2. We define the set $T(f,X)$ of tracks of the Newton algorithm for f as the minimal subset (in the sense of inclusion) of $\mathbb{C}[X]^*$ such that the following conditions are satisfied:

(I) $0 \in T(f,X)$,
(II) for any $\varphi \in T(f,X)$, if there exists $S \in \mathcal{N}(f,\varphi)$ then for every nonzero root aX^θ of the initial form $\mathrm{in}(f_\varphi, S)$ we have $\varphi + aX^\theta \in T(f,X)$.

We have the following two equivalent characterizations of the set $T(f,X)$. Let

$$T'(f,X) = \{\varphi \in \mathbb{C}[X]^* : \exists y \in \mathrm{Zer}\, f \text{ such that } \mathrm{ord}(y-\varphi) > \deg \varphi\}$$

and let

$$T''(f,X) = \{\varphi \in \mathbb{C}[X]^* : |\mathcal{N}(f,\varphi)| + \delta(f_\varphi) > 0\}\ .$$

Proposition 2.3. ([19], Proposition 3.11) $T(f,X) = T'(f,X) = T''(f,X)$.

The following notions are useful in the proof of main result in the last chapter. Now, let us assume that $f \in \mathbb{C}\{X,Y\}$ is reduced and singular. Let $\varphi \in T(f,X)$. Let us introduce a symbol for the system of initial forms of solutions corresponding to $\mathcal{N}(f,\varphi)$ and $\delta(f_\varphi)$. For $\varphi = 0$ such system appears in Theorem 2.1 (ii). We put

$$\mathcal{I}(f,\varphi) = \bigoplus_{S \in \mathcal{N}(f,\varphi)} Zer\, \mathrm{in}(f_\varphi, S)^\circ + \langle 0 : \delta(f_\varphi)\rangle\ . \tag{7}$$

Clearly $\deg \mathcal{I}(f,\varphi) = |\mathcal{N}(f,\varphi)| + \delta(f_\varphi)$.

Definition 2.4. We say that a solution $y \in \mathrm{Zer}\, f$ is counted by a track $\varphi \in T(f,X)$ if all the conditions are satisfied:

(1) $\deg \mathcal{I}(f,\varphi) \geq 2$,
(2) $\mathrm{ord}(y-\varphi) > \deg \varphi$,
(3) $\mathrm{in}(y-\varphi) \in \mathcal{I}(f,\varphi)$,
(4) $\mathrm{in}(y-\varphi)$ has the multiplicity one in $\mathcal{I}(f,\varphi)$.

The following property is important.

Property 2.5. *Every $y \in \mathrm{Zer}\, f$ is counted by the unique $\varphi \in T(f)$.*

We denote this unique track by $\varphi = \varphi_f(y)$.

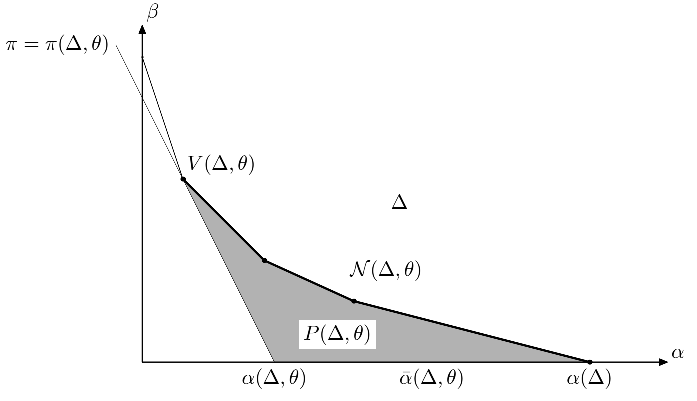
Example 2.6. Let $f = Y(Y-X)(Y-X-X^2)$. We have $\mathrm{Zer}\, f = \langle 0, X, X+X^2\rangle$ and $\varphi_f(0) = 0$, $\varphi_f(X) = X$, $\varphi_f(X+X^2) = X$.

3. Version of Kouchnirenko theorem

In this chapter we compute the Milnor number by using the Newton algorithm in Cano's version. Our main reference is [19]. Analogous results were obtained by Pi. Cassou-Noguès and Płoski in [6] (they applied the classical Newton algorithm) and by Gwoździewicz [11] who used the toric modification technique.

Let Δ be the Newton diagram of a nonzero series of $\mathbb{C}\{X^*, Y\}$ and let $\mathcal{N} = \mathcal{N}(\Delta)$ be the Newton polygon of this diagram. Let us denote by $\delta(\Delta)$ the distance between Δ and the horizontal axis. We consider only diagrams touching the vertical axis and with $\delta(\Delta) \leq 1$. With the above assumptions we have $\delta(\Delta) \in \{0,1\}$. For $\theta \geq 0$

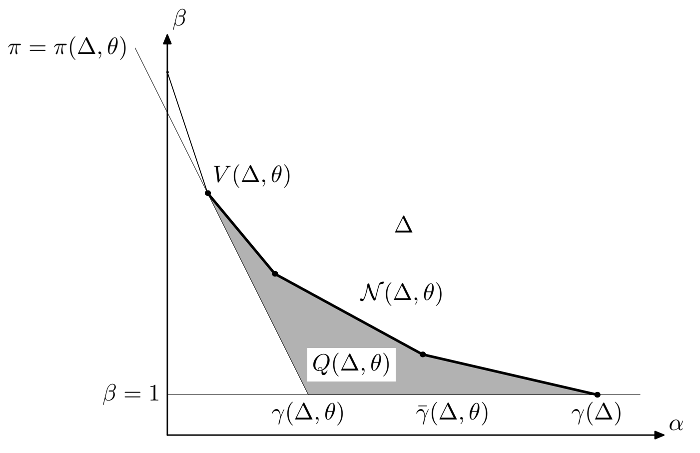
we define the stright line π with inclinaction θ that supports the diagram Δ. We denote this line by $\pi = \pi(\Delta, \theta)$. Let V be the commont point of π with Δ of the minimal possible ordinate. We denote this point by $V = V(\Delta, \theta)$; V must be a vertex of the diagram Δ.



The line $\pi(\Delta, \theta)$ crosses the horizontal axis at the point with abscissa $\alpha(\Delta, \theta) \geq 0$. Let $\mathcal{N} = \mathcal{N}(\Delta, \theta)$ denotes the subset of these segments of the Newton polygon $\mathcal{N}$ that have the inclinations strictly greater than θ. If the diagram Δ touches the horizontal axis ($\delta(\Delta) = 0$) then we define $\alpha(\Delta)$ as the minimal possible abscissa of the points of the diagram Δ that lie on the horizontal axis. Clearly $\alpha(\Delta, \theta) \leq \alpha(\Delta)$. We put

$$\bar{\alpha}(\Delta, \theta) = \alpha(\Delta) - \alpha(\Delta, \theta) .$$

If $\bar{\alpha}(\Delta, \theta) > 0$ then we define $P(\Delta, \theta)$ as the area of the polygon bounded by the line $\pi(\Delta, \theta)$, the polygon $\mathcal{N}(\Delta, \theta)$ and the horizontal axis. Otherwise, we put $P(\Delta, \theta) = 0$.



If the diagram Δ does not touch the horizontal axis ($\delta(\Delta) = 1$) then the line $\pi(\Delta, \theta)$ crosses the line $\beta = 1$ at the point with abscissa $\gamma(\Delta, \theta) \geq 0$. We define $\gamma(\Delta)$ to be the minimal abscissa of the points of the diagram Δ lying on the line $\beta = 1$. Clearly $\gamma(\Delta, \theta) \leq \gamma(\Delta)$. We put

$$\bar{\gamma}(\Delta, \theta) = \gamma(\Delta) - \gamma(\Delta, \theta) .$$

If $\bar{\gamma}(\Delta, \theta) > 0$ then we define $Q(\Delta, \theta)$ as the area of the polygon bounded by the line $\pi(\Delta, \theta)$, the polygon $\mathcal{N}(\Delta, \theta)$ and by the line $\beta = 1$. Otherwise, we put $Q(\Delta, \theta) = 0$. If $\delta(\Delta) = 0$ then the numbers $\gamma(\Delta)$, $\gamma(\Delta, \theta)$, $\bar{\gamma}(\Delta, \theta)$ and $Q(\Delta, \theta)$ can be also defined assuming that the ordinate of the vertex $V(\Delta, \theta)$ is greater or equal to 1. Using the formula for area of triangle, we check that

$$2P(\Delta, \theta) - \bar{\alpha}(\Delta, \theta) = 2Q(\Delta, \theta) + \bar{\gamma}(\Delta, \theta) .$$

Now, let us discuss the notions introduced above in the context of the Newton algorithm. We assume that the series $f \in \mathbb{C}\{X, Y\}$ is reduced and that the number $p = (f, X)_0$ is finite and greater then one. We put

$$\hat{\mu}(f, \varphi) = \begin{cases} 2\,P(\Delta f_\varphi, \deg \varphi) - \bar{\alpha}(\Delta f_\varphi, \deg \varphi) & \text{if } \delta(f_\varphi) = 0 \,, \\ 2\,Q(\Delta f_\varphi, \deg \varphi) + \bar{\gamma}(\Delta f_\varphi, \deg \varphi) & \text{if } \delta(f_\varphi) = 1 \,. \end{cases}$$

Theorem 3.1. *With the above assumptions on f*

(a) *for almost all $\varphi \in T(f, X)$ the number $\hat{\mu}(f, \varphi)$ equals zero,*

(b) $\mu(f) = 1 - p + \sum_{\varphi \in T(f,X)} \hat{\mu}(f, \varphi)$.

Proof. Let us recall few notions [19]. For a series $g \in \mathbb{C}\{X^*, Y\}$ and for a segment S of its Newton polygon we denote by $t(g, S)$ the number of different roots of initial form $\mathrm{in}(g, S)$ in $\mathbb{C}\{X\}^*$. Let $r_1, \ldots, r_s$ denote the multiplicities of nonzero roots among all these roots ($t - 1 \leq s \leq t$). Clearly $r_1 + \cdots + r_s = |S|_{\mathbf{V}}$. We put $d(g, S) = (r_1 - 1) + \cdots + (r_s - 1)$ and we call $d(g, S)$ the degeneracy of g on S. The condition $d(g, S) = 0$ means that every nonzero root is a single root (nondegeneracy). We have

$$t(g, S) - 1 + d(g, S) = |S|_{\mathbf{V}} + \varepsilon(S) \tag{8}$$

where $\varepsilon(S) = -1$ for a segment S touching the horizontal axis and $\varepsilon(S) = 0$ for segments that do not touch the horizontal axis. The number $\alpha(S)$ equals the abscissa of point where the line containing segment S crosses the horizontal axis.

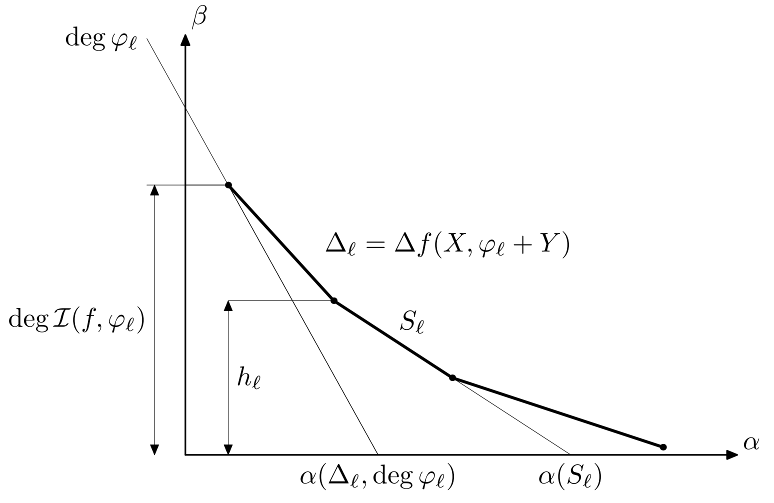
We apply the following fact.

Proposition 3.2. ([19], Proposition 3.9) *Let us assume that $\varphi \in \mathbb{C}[X]^*$ is a polynomial such that the polygon $\mathcal{N}(f, \varphi)$ is nonempty. Let $S \in \mathcal{N}(f, \varphi)$ and let aX^θ be a nonzero root of the form* $\mathrm{in}(f_\varphi, S)$. *Then*

$$\deg \mathcal{I}(f, \varphi + aX^\theta) = \textit{multiplicity of } aX^\theta \textit{ as a root of the form } \mathrm{in}(f_\varphi, S) .$$

Proof of (a). We base on [19]. Let $y = a_1X^{\theta_1} + a_2X^{\theta_2} + \ldots$ ($a_1, a_2, \ldots$ nonzero, $0 < \theta_1 < \theta_2 < \ldots$) be a Puiseux solution of the equation $f = 0$ in $\mathbb{C}\{X\}^*$. Without loss of generality it suffices to consider a solution with infinite number of terms. We define tracks $\varphi_1 = 0$ and $\varphi_\ell = a_1X^{\theta_1} + \cdots + a_{\ell-1}X^{\theta_{\ell-1}}$ for $\ell = 2, 3, \ldots$. Let

$\Delta_\ell := \Delta f(X, \varphi_\ell + Y)$. Let us fix $\ell \in \{1, 2, \dots\}$. According to the Newton algorithm there exists a segment S_ℓ of the polygon $\mathcal{N}(f, \varphi_\ell)$ such that $a_\ell X^{\theta_\ell}$ is a root of the form $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$. We denote $h_\ell = \deg_Y \mathrm{in}(f_{\varphi_\ell}, S_\ell)$.



Let r_ℓ be the multiplicity of the root. Obviously $h_\ell \geq r_\ell$. By Proposition 3.2 $r_\ell = \deg \mathcal{I}(f, \varphi_\ell + a_\ell X^{\theta_\ell}) = |\mathcal{N}(f, \varphi_{\ell+1})| + \delta(f_{\varphi_{\ell+1}}) \geq \deg_Y \mathrm{in}(f_{\varphi_{\ell+1}}, S_{\ell+1}) = h_{\ell+1}$. This construction gives the infinite sequence of positive integers $h_1 \geq r_1 \geq h_2 \geq r_2 \geq \dots$ that must stabilize. The stable value is the multiplicity of y as a root of f. For the reduced series it equals one. Let us note that the equality $h_\ell = r_\ell$ means that the segment S_ℓ touches the horizontal axis and that $a_\ell X^{\theta_\ell}$ is the unique root of the initial form $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$. Then $t(f_{\varphi_\ell}, S_\ell) = 1$ which will be important in the proof of part (b). Moreover, from the step where stability is reached, we will have $|\mathcal{N}(f, \varphi_\ell)| = 1$. Then we get $\hat{\mu}(f, \varphi_\ell) = 0$ for such terms.

Proof of (b). Applying the Teissier Lemma (cited and proved e.g. in [6]) we have

$$\mu(f) = 1 - p + \left(f, \frac{\partial f}{\partial Y}\right)_0 = 1 - p + \sum_{j=1}^{p-1} \mathrm{ord}\, f(X, z_j(X)) \tag{9}$$

where $z_1, \dots, z_{p-1} \in \mathbb{C}\{X\}^*$ is the sequence of solutions of the equation $(\partial f/\partial Y) = 0$. The system

$$\langle \mathrm{ord}\, f(X, z_1(X)), \dots, \mathrm{ord}\, f(X, z_{p-1}(X))\rangle$$

giving the so-called polar quotients was described in [19] (Theorem 2.1). Using this result we can write the equality (9) as

$$\mu(f) = 1 - p + \sum_{\varphi \in T(f,X)} \sum_{S \in \mathcal{N}(f,\varphi)} \alpha(S)[t(f_\varphi, S) - 1]\,. \tag{10}$$

In the proof of part (a) we checked that almost all components in the above sum equal zero. Now, to finish the proof it suffices to show that (10) equals the right side in the statement (b) of the theorem. In order to simplify notation we hide the dependence of f as in the following table.

new symbol	instead of
$\mathcal{N}_\varphi$	$\mathcal{N}(f,\varphi)$
$\hat{\mu}_\varphi$	$\hat{\mu}(f,\varphi)$
δ_φ	$\delta(f_\varphi)$
π_φ	$\pi(\Delta f_\varphi, \deg\varphi)$
V_φ	$V(\Delta f_\varphi, \deg\varphi)$
P_φ	$P(\Delta f_\varphi, \deg\varphi)$
Q_φ	$Q(\Delta f_\varphi, \deg\varphi)$

Let n_φ be the number of segments of polygon $\mathcal{N}_\varphi$ ($n_\varphi \geq 0$). We number the segments of $\mathcal{N}_\varphi$ from up to down:

$$S_\varphi^{(1)}, \ldots, S_\varphi^{(n_\varphi)} .$$

For $i = 1, \ldots, n_\varphi$ we put $t_\varphi^{(i)} := t(f_\varphi, S^{(i)})$, $d_\varphi^{(i)} := d(f_\varphi, S^{(i)})$, $\alpha_\varphi^{(i)} := \alpha(S^{(i)})$, $\varepsilon_\varphi^{(i)} := \varepsilon(S^{(i)})$. Moreover $\alpha_\varphi^{(0)} := \alpha(\Delta f_\varphi, \deg\varphi)$ and $\bar{\alpha}_\varphi = \alpha_\varphi^{(n_\varphi)} - \alpha_\varphi^{(0)}$.

Applying (8) and denoting $b_\varphi^{(i)} = |S_\varphi^{(i)}|_{\mathbf{V}}$ ($i = 1, \ldots, n_\varphi$) we can write

$$t_\varphi^{(i)} - 1 + d_\varphi^{(i)} = b_\varphi^{(i)} + \varepsilon_\varphi^{(i)} \text{ for } i = 1, \ldots, n_\varphi . \tag{11}$$

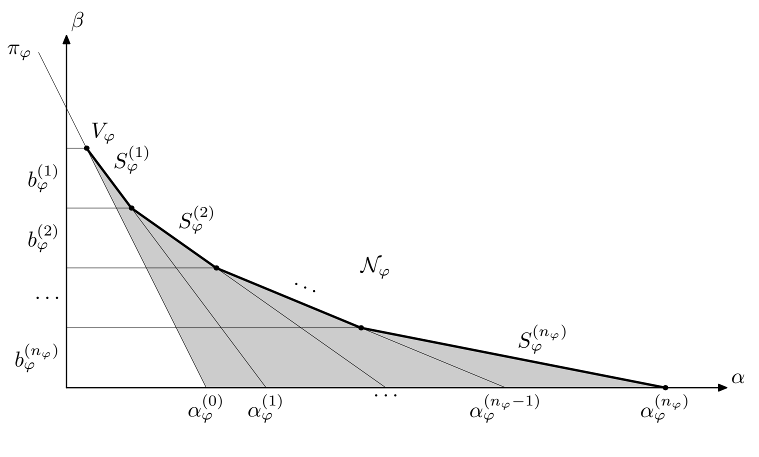
The formula (10) can be rewritten as

$$\mu(f) = 1 - p + \sum_{\varphi \in T(f,X)} \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} (t_\varphi^{(i)} - 1) . \tag{12}$$

Let us fix $\varphi \in T(f,X)$. We are going to prove that

$$\sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} (t_\varphi^{(i)} - 1) = \hat{\mu}_\varphi + \alpha_\varphi^{(0)} (|\mathcal{N}_\varphi| + \delta_\varphi - 1) - \sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} d_\varphi^{(i)} . \tag{13}$$

First, we consider the case $\delta_\varphi = 0$.



We have

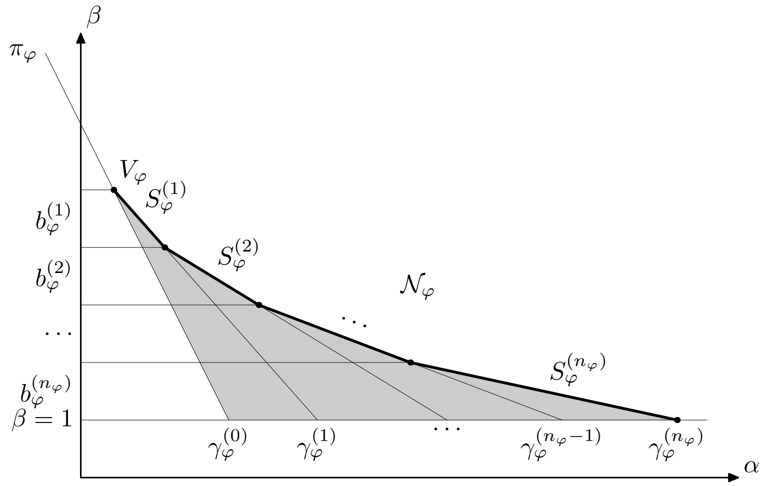
$$2\,P_\varphi = \sum_{i=1}^{n_\varphi}(\alpha_\varphi^{(i)} - \alpha_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) = -\alpha_\varphi^{(0)}|\mathcal{N}_\varphi| + \sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}b_\varphi^{(i)}\ . \tag{14}$$

By (11) and (14) we can write

$$\begin{aligned}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(t_\varphi^{(i)}-1) &= \sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(b_\varphi^{(i)}+\varepsilon_\varphi^{(i)}-d_\varphi^{(i)}) = \sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(b_\varphi^{(i)}-d_\varphi^{(i)}) - \alpha_\varphi^{(n_\varphi)}\\ &= \sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}b_\varphi^{(i)} - \sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)} - \alpha_\varphi^{(n_\varphi)}\\ &= \hat{\mu}_\varphi + \alpha_\varphi^{(0)}(|\mathcal{N}_\varphi|-1) - \sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}\end{aligned}$$

which gives (13).

Now, let us check the case $\delta_\varphi = 1$. For $i = 1, \ldots, n_\varphi$ we put $\gamma_\varphi^{(i)} := \gamma(S^{(i)})$. Moreover $\gamma_\varphi^{(0)} = \gamma(\Delta f_\varphi, \deg\varphi)$ and $\bar{\gamma}_\varphi = \gamma_\varphi^{(n_\varphi)} - \gamma_\varphi^{(0)}$.



Let us note that for $i = 1, \ldots, n_\varphi$ (by the formula for area of triangle)

$$(\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) + (\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)}) = (\alpha_\varphi^{(i)} - \alpha_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)})\ .$$

By using the above observation we get

$$\begin{aligned}\hat{\mu}_\varphi &= 2Q_\varphi + \bar{\gamma}_\varphi = \sum_{i=1}^{n_\varphi}(\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) + \sum_{i=1}^{n_\varphi}(\gamma_\varphi^{(i)} - \gamma_\varphi^{(i-1)})\\ &= \sum_{i=1}^{n_\varphi}(\alpha_\varphi^{(i)} - \alpha_\varphi^{(i-1)})(b_\varphi^{(i)} + \cdots + b_\varphi^{(n_\varphi)}) = -\alpha_\varphi^{(0)}|\mathcal{N}_\varphi| + \sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}b_\varphi^{(i)}\ .\end{aligned}$$

Now, we compute

$$\begin{aligned}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(t_\varphi^{(i)}-1)&=\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(b_\varphi^{(i)}-d_\varphi^{(i)})=\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}b_\varphi^{(i)}-\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}\\&=\hat{\mu}_\varphi+\alpha_\varphi^{(0)}|\mathcal{N}_\varphi|-\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}\ ,\end{aligned}$$

which also gives (13).

Applying (12) and by using (13) we get

$$\begin{aligned}\mu(f)&=1-p+\sum_{\varphi\in T(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}(t_\varphi^{(i)}-1)\\&=1-p+\sum_{\varphi\in T(f,X)}\hat{\mu}_\varphi+\sum_{\varphi\in T(f,X)}\alpha_\varphi^{(0)}(|\mathcal{N}_\varphi|+\delta_\varphi-1)-\sum_{\varphi\in T(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}\ .\end{aligned}$$

Therefore, to the finish of the proof it suffices to show that

$$\sum_{\varphi\in T(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}=\sum_{\varphi\in T(f,X)}\alpha_\varphi^{(0)}(|\mathcal{N}_\varphi|+\delta_\varphi-1)\ . \tag{15}$$

We denote by $T_\ell(f,X)$ the set of all tracks with the lenght ℓ ($\ell=0,1,2,\dots$). These sets are finite. The set $T_0(f,X)$ contains only zero track. For $\varphi=0$ we have $\alpha_\varphi^{(0)}=0$. Hence, the component on the right side of the formula (15) corresponding to zero track equals zero. Therefore, it is enough to show

$$\sum_{\varphi\in T_\ell(f,X)}\sum_{i=1}^{n_\varphi}\alpha_\varphi^{(i)}d_\varphi^{(i)}=\sum_{\varphi\in T_{\ell+1}(f,X)}\alpha_\varphi^{(0)}(|\mathcal{N}_\varphi|+\delta_\varphi-1) \tag{16}$$

for $\ell=0,1,2,\dots$. Let us fix $\varphi\in T_\ell(f,X)$. To this track we can assign the tracks of the form $\varphi+aX^\theta\in T_{\ell+1}(f,X)$ taking as aX^θ all different nonzero roots of all forms

$$\mathrm{in}(f_\varphi,S_\varphi^{(1)}),\dots,\mathrm{in}(f_\varphi,S_\varphi^{(n_\varphi)})\ .$$

We write these roots as $a_{ij}X^{\theta_i}$ ($j=1,\dots,s_\varphi^{(i)}$, $i=1,\dots,n_\varphi$), remembering about the dependence of coefficients and exponents on φ; $s_\varphi^{(i)}:=t_\varphi^{(i)}-1-\varepsilon_\varphi^{(i)}$ stands for the number of different nonzero roots of the form $\mathrm{in}(f_\varphi,S_\varphi^{(i)})$. For $\varphi\in T_\ell(f,X)$ ($\ell\geq 0$) we can write

$$T_{\ell+1}(\varphi)=\{\varphi+a_{ij}X^{\theta_i}\,:\,i=1,\dots,n_\varphi\,j=1,\dots,s_\varphi^{(i)}\}\ .$$

If $T_\ell(f,X)=\{\varphi_1,\dots,\varphi_m\}$ ($m\geq 1$), then $T_{\ell+1}(f,X)=T_{\ell+1}(\varphi_1)\cup\dots\cup T_{\ell+1}(\varphi_m)$. Hence, it suffices to check (16) taking into consideration fixed track $\varphi\in T_\ell(f,X)$ on the left side, while on the right side the set $T_{\ell+1}(\varphi)$. The appropriate formula has the form

$$\sum_{i=1}^{n_\varphi} \alpha_\varphi^{(i)} d_\varphi^{(i)} = \sum_{i=1}^{n_\varphi} \sum_{j=1}^{s_\varphi^{(i)}} \alpha^{(0)}_{\varphi + a_{ij}X^{\theta_i}} (|\mathcal{N}_{\varphi+a_{ij}X^{\theta_i}}| + \delta_{\varphi+a_{ij}X^{\theta_i}} - 1) \ . \tag{17}$$

The property of the Newton algorithm implies that $\alpha_\varphi^{(i)} = \alpha^{(0)}_{\varphi+a_{ij}X^{\theta_i}}$. Therefore for the proof of the above equality it is enough to show that

$$d_\varphi^{(i)} = \sum_{j=1}^{t_\varphi^{(i)}} (|\mathcal{N}_{\varphi+a_{ij}X^{\theta_i}}| + \delta_{\varphi+a_{ij}X^{\theta_i}} - 1) \ . \tag{18}$$

Let $r_\varphi^{(i,j)}$ be the multiplicity of $a_{ij}X^{\theta_j}$ as a root of the form $\mathrm{in}(f_\varphi, S_\varphi^{(i)})$. Then

$$d_\varphi^{(i)} = \sum_{j=1}^{s_\varphi^{(i)}} (r_\varphi^{(i,j)} - 1) \ .$$

Therefore, for the proof of (18) it suffices to know that

$$r_\varphi^{(i,j)} = |\mathcal{N}_{\varphi+a_{ij}X^{\theta_i}}| + \delta_{\varphi+a_{ij}X^{\theta_i}} \ ,$$

but it follows directly from Proposition 3.2. □

4. Proof of Gusein-Zade Theorem

Let $f \in \mathbb{C}\{X, Y\}$ be a reduced and singular series. In analogy to the set $T(f, X)$ of tracks of the Newton algorithm discussed in Section 2 we define below a new set $T_*(f, X) \subset T(f, X)$ which is finite an can be applied to compute the Milnor number by Theorem 3.1.

Definition 4.1. We define the set $T_*(f, X)$ of multiple tracks of the Newton algorithm for f as the minimal subset (in the sense of inclusion) of $\mathbb{C}[X]^*$ such that the following conditions are satisfied:

(I) $0 \in T_*(f, X)$,
(II) for any $\varphi \in T_*(f, X)$, if there exists $S \in \mathcal{N}(f, \varphi)$ then for every nonzero multiple root aX^θ of the initial form $\mathrm{in}(f_\varphi, S)$ we have $\varphi + aX^\theta \in T_*(f, X)$.

In analogy to $T(f, X)$ the set $T_*(f, X)$ has also two equivalent characterizations. Let

$$T'_*(f, X) = \{\varphi \in \mathbb{C}[X]^* : \exists y^{(1)} \neq y^{(2)} \in \mathrm{Zer}\, f \text{ that } \mathrm{ord}(y^{(i)} - \varphi) > \deg \varphi,\ i = 1, 2\}$$

and let

$$T''_*(f, X) = \{\varphi \in \mathbb{C}[X]^* : |\mathcal{N}(f, \varphi)| + \delta(f_\varphi) > 1\} \ .$$

Proposition 4.2. $T_*(f, X) = T'_*(f, X) = T''_*(f, X)$.

The proof is analogous to the proof of Proposition 2.3.

Proposition 4.3. *Let $\varphi \in T(f,X)$. Then $\varphi \in T_*(f,X)$ if and only if $\hat{\mu}(f,\varphi) > 0$.*

Below we present the steps of construction a deformation

$$F_t = f + tX^{\omega_0} f_1^{\omega_1} \dots f_{\mathbf{h}}^{\omega_{\mathbf{h}}} \tag{19}$$

where f has the form (3) and f_k $(k = 1, \dots, \mathbf{h})$ are defined in (5). All what can be controlled are nonegative integers $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$. Applying Treorem 3.1 and Proposition 4.3 we can write

$$\mu(f) = 1 - (f,X)_0 + \sum_{\varphi \in T_*(f,X)} \hat{\mu}(f,\varphi)\ , \tag{20}$$

$$\mu(F_t) = 1 - (F_t,X)_0 + \sum_{\varphi \in T_*(F_t,X)} \hat{\mu}(F_t,\varphi)\ . \tag{21}$$

Since we want $\mu(f)$ and $\mu(F_f)$ to be close, the idea in choosing $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$ is to obtain many common elements in both (20) and (21). To have equality $(f,X)_0 = (F_f,X)_0$ it suffices that $\omega_0 > 0$. Moreover, we want to have as many common tracks as possible. For example, the equality holds $T_*(f,X) = T_*(F_t,X) = \{0\}$ in Bodin's deformation from Introduction. In our construction we will obtain $T_*(F_t,X) \subset T_*(f,X)$. Unfortunately, the inclusion may be strict.

As in Introduction we apply that f is generated by a cycle of $y \in \mathbb{C}\{X\}^*$ (1) in the sense of (3). On the basis of y we can define tracks: $\varphi_1 := 0$, $\varphi_\ell := a_1 X^{v_1/v_0} + \dots + a_{\ell-1} X^{v_{\ell-1}/v_0}$ $(\ell = 2, 3, \dots)$. By Proposition 2.3 we have $T(f,X) = \mathrm{cycle}(\varphi_1) \cup \mathrm{cycle}(\varphi_2) \cup \dots$. In order to determine $T_*(f,X)$ let us recall a description of the Newton polygon $\mathcal{N}(f,\varphi_\ell)$ from [19]. The notation is equivalent. We put $w^* = \mathrm{GCD}(v_0, v_1, \dots)$.

Property 4.4. ([19], Property 5.1)

(i) *Polygon $\mathcal{N}(f,\varphi_\ell)$ consists one segment S_ℓ with inclination v_ℓ/v_0 which touches the horizontal axis,*
(ii) $\deg_Y \mathrm{in}(f_{\varphi_\ell}, S_\ell) = \mathrm{GCD}(v_0, \dots, v_{\ell-1})/w^*$.
(iii) *Every root of* $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$ *has the multiplicity* $\mathrm{GCD}(v_0, \dots, v_\ell)/w^*$,
(iv) $t(f_{\varphi_\ell}, S_\ell) = \frac{\mathrm{GCD}(v_0,\dots,v_{\ell-1})}{\mathrm{GCD}(v_0,\dots,v_\ell)}$.

In addition to Property 4.4 we will need more precise information about the initial form $\mathrm{in}(f_{\varphi_\ell}, S_\ell)$. Let $w_\ell = \mathrm{GCD}(v_0, \dots, v_\ell)$, $u_\ell = w_{\ell-1}/w_\ell$, $\theta_\ell = v_\ell/v_0$.

Property 4.5. *With the previous notation there exist $c \neq 0$ and $\zeta \geq 0$ such that*

$$\mathrm{in}(f_{\varphi_\ell}, S_\ell) = cX^\zeta (Y^{u_\ell} - a_\ell^{u_\ell} X^{\theta_\ell u_\ell})^{w_\ell/w^*}\ .$$

Proof. See (e.g. [20], Lemma 6.1).

Let us return to tracks. Since $\mathrm{GCD}(v_0, \dots, v_{\ell_{\mathbf{h}}})/w^* = e_{\mathbf{h}} = 1$ then it follows from Property 4.4 (iii) that every root of the corresponding initial form is a single root. Therefore a track $\varphi_{\ell_{\mathbf{h}}} = y_{\mathbf{h}} \in T_*(f,X)$ cannot be extended in the sense of

Definition 4.1. Hence $T_*(f,X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}})$. Our effort was to obtain the equality $T_*(F_t,X) = T_*(f,X)$. However, we finished with the following two cases:

(I) $T_*(F_t,X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}})$,
(II) $T_*(F_t,X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}-1})$.

In both cases we want for $\ell < \ell_{\mathbf{h}}$ to have $\hat{\mu}(F_t,\varphi_\ell) = \hat{\mu}(f,\varphi_\ell)$. When $\ell = \ell_{\mathbf{h}}$ we want $\hat{\mu}(F_t,\varphi_{\ell_{\mathbf{h}}}) = \hat{\mu}(f,\varphi_{\ell_{\mathbf{h}}}) - \frac{1}{N_{\mathbf{h}-1}}$ in the first case and $\hat{\mu}(f,\varphi_{\ell_{\mathbf{h}}}) = \frac{1}{N_{\mathbf{h}-1}}$ in the second case. Since $\#\mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}}) = N_{\mathbf{h}-1}$ this will give $\mu(F_t) = \mu(f) - 1$ in both cases.

In order to describe the diagrams $\Delta f(X,\varphi_\ell + Y)$ and $\Delta F_t(X,\varphi_\ell + Y)$ we need the shapes of the diagrams $\Delta f(X,\varphi_\ell + Y)$ for $\ell = 1,\ldots,\ell_{\mathbf{h}}$ and $\Delta f_k(X,\varphi_\ell + Y)$ for $k = 1,\ldots,\mathbf{h}$ and $\ell = 1,\ldots,\ell_{\mathbf{h}}$. To this end let us recall facts from [13]. The contact exponent between the branch f and an arbitrary Puiseux series $z \in \mathbb{C}\{X\}^*$ is defined as

$$o_f(z) = \max\{\mathrm{ord}(z - \tau^0 * y), \ldots, \mathrm{ord}(z - \tau^{N-1} * y)\}\ . \tag{22}$$

Below, we describe the shapes of the diagrams by using the so-called Teissier notation. For $A, B \subset \mathbb{R}^2_+$ $A + B = \{a + b :\ a \in A,\ b \in B\}$, $\left\{\frac{a}{b}\right\}$ = the convex hull of $\{(a,0),(0,b)\} + \mathbb{R}^2_+$. Moreover $\{\frac{1}{\infty}\} = (1,0) + \mathbb{R}^2_+$ and $\{\frac{\infty}{1}\} = (0,1) + \mathbb{R}^2_+$ ($\left\{\frac{0}{0}\right\}$ is the identity). By convention the sum over the empty set equals $\{\frac{0}{0}\}$.

Property 4.6. (Properties 3.1 and 3.2 in [13])
Let $(b_0, b_1, \ldots, b_{\mathbf{h}})$ be the characteristic sequence of the branch.

(I) *If there exists the smallest integer k such that $o_f(z) \le b_k/b_0$ then*

$$\Delta f(X, z + Y) = \sum_{j=1}^{k-1} \left\{\frac{(b_j/b_0)(e_{j-1} - e_j)}{e_{j-1} - e_j}\right\} + \left\{\frac{o_f(z)\, e_{k-1}}{e_{k-1}}\right\}\ .$$

(II) *If $b_{\mathbf{h}}/b_0 < o_f(z)$ then*

$$\Delta f(X, z + Y) = \sum_{k=1}^{\mathbf{h}} \left\{\frac{(b_k/b_0)(e_{k-1} - e_k)}{e_{k-1} - e_k}\right\} + \left\{\frac{o_f(z)}{1}\right\}\ .$$

Corollary 4.7. (for f and φ_ℓ) We have $o_f(\varphi_\ell) = v_\ell/v_0$. Therefore:

(I) if there exists the smallest integer k such that $\ell \le \ell_k$ then

$$\Delta f(X, \varphi_\ell + Y) = \sum_{j=1}^{k-1} \left\{\frac{(b_j/b_0)(e_{j-1} - e_j)}{e_{j-1} - e_j}\right\} + \left\{\frac{(v_\ell/v_0)\, e_{k-1}}{e_{k-1}}\right\}\ ,$$

(II) if $\ell_{\mathbf{h}} < \ell$ then

$$\Delta f(X, \varphi_\ell + Y) = \sum_{k=1}^{\mathbf{h}} \left\{\frac{(b_k/b_0)(e_{k-1} - e_k)}{e_{k-1} - e_k}\right\} + \left\{\frac{v_\ell/v_0}{1}\right\}\ .$$

We can also describe $\Delta f_k(X, \varphi_\ell + Y)$ $(k = 1, \dots, \mathbf{h})$. The characteristic sequence of f_k has the form: $(b_0/e_{k-1}, b_1/e_{k-1}, \dots, b_{k-1}/e_{k-1})$ with the first sequence of divisors: $(e_0/e_{k-1}, e_1/e_{k-1}, \dots, e_{k-1}/e_{k-1})$. Let us observe that

$$o_{f_k}(\varphi_\ell) = \begin{cases} v_\ell/v_0 & \text{for } \ell < \ell_k \\ +\infty & \text{for } \ell = \ell_k \\ b_k/b_0 & \text{for } \ell_k < \ell \end{cases} . \tag{23}$$

Corollary 4.8. (for f_k and φ_ℓ)

(I) If $\ell < \ell_k$ then there exists the smallest integer $j \in \{1, \dots, k\}$ such that $\ell \le \ell_j$. Then

$$\Delta f_k(X, \varphi_\ell + Y) = \sum_{i=1}^{j-1} \left\{ \frac{(b_i/b_0)(e_{i-1}/e_{k-1} - e_i/e_{k-1})}{e_{i-1}/e_{k-1} - e_i/e_{k-1}} \right\} + \left\{ \frac{(v_\ell/v_0)\, e_{j-1}/e_{k-1}}{e_{j-1}/e_{k-1}} \right\} .$$

(II) If $\ell = \ell_k$ then

$$\Delta f_k(X, \varphi_{\ell_k} + Y) = \sum_{j=1}^{k-1} \left\{ \frac{(b_j/b_0)(e_{j-1}/e_{k-1} - e_j/e_{k-1})}{e_{j-1}/e_{k-1} - e_j/e_{k-1}} \right\} + \left\{ \frac{\infty}{1} \right\} .$$

(III) If $\ell_k < \ell$ then

$$\Delta f_k(X, \varphi_\ell + Y) = \sum_{j=1}^{k-1} \left\{ \frac{(b_j/b_0)(e_{j-1}/e_{k-1} - e_j/e_{k-1})}{e_{j-1}/e_{k-1} - e_j/e_{k-1}} \right\} + \left\{ \frac{b_k/b_0}{1} \right\} .$$

Below we apply the semigroup technique from [8]. Now, our aim is to construct $\omega_0, \dots, \omega_{\mathbf{h}}$ $(\mathbf{h} \ge 1)$ by using the longest track $y_{\mathbf{h}} = \varphi_{\ell_{\mathbf{h}}}$ in $T_*(f, X)$. We will apply the semigroup generators $\bar{b}_0, \bar{b}_1, \dots, \bar{b}_{\mathbf{h}}$ which satisfy relations $\bar{b}_0 = b_0$, $\bar{b}_1 = b_1$, $\bar{b}_{k+1} = n_k \bar{b}_k + b_{k+1} - b_k$ for $k = 1, \dots, \mathbf{h} - 1$. It follows from the above relation that $n_k \bar{b}_k < \bar{b}_{k+1}$ for $k = 1, \dots, \mathbf{h} - 1$ (God given inequality).

The following proposition follows from Corollaries 4.7 and 4.8.

Proposition 4.9.

(i) *For $k = 1, \dots, \mathbf{h} - 1$ the diagram $\Delta f_k(X, y_{\mathbf{h}} + Y)$ has the vertex on the horizontal axis with abscissa $\bar{b}_k/b_0$.*

(ii) *The diagram $\Delta f_{\mathbf{h}}(X, y_{\mathbf{h}} + Y)$ does not touch the horizontal axis and its lower vertex (with ordinate one) has the abscissa $(n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - b_{\mathbf{h}-1})/b_0$.*

(iii) *The last segment $S_{\mathbf{h}}$ of the diagram $\Delta f(X, y_{\mathbf{h}} + Y)$ has the inclination $|S_{\mathbf{h}}|_{\mathbf{H}}/|S_{\mathbf{h}}|_{\mathbf{V}} = b_{\mathbf{h}}/b_0$ and touches the horizontal axis at the point with abscissa $n_{\mathbf{h}}\bar{b}_{\mathbf{h}}/b_0$. The length of vertical projection is $|S_{\mathbf{h}}|_{\mathbf{V}} = n_{\mathbf{h}}$.*

(iv) *The straight line $\pi_{\mathbf{h}-1}$ determined by the penultimate segment of the diagram $\Delta f(X, \varphi_{\mathbf{h}} + Y)$ (the line and the segment have the inclination $b_{\mathbf{h}-1}/b_0$) crosses the horizontal axis at the point with abscissa $n_{\mathbf{h}}n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1}/b_0$.*

Let us notice that all the series $f_1(X, \varphi_{\mathbf{h}} + Y)$, $\dots$, $f_{\mathbf{h}}(X, \varphi_{\mathbf{h}} + Y)$, $f(X, \varphi_{\mathbf{h}} + Y)$ are in the ring $\mathbb{C}\{X^{1/N_{\mathbf{h}-1}}, Y\}$ where $N_{\mathbf{h}-1} = n_1 \dots n_{\mathbf{h}-1}$. Hence, all the points corresponding to nonzero coefficients have the form

$$\left(\frac{i}{n_1 \dots n_{\mathbf{h}-1}}, \beta\right) \tag{24}$$

for nonegative integers i, β. Now, let us consider nonegative integer numbers $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$. Let us denote

$$H = X^{\omega_0} f_1^{\omega_1} \dots f_{\mathbf{h}}^{\omega_{\mathbf{h}}} \,. \tag{25}$$

The polynomial H depends on $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$ what is not explicitly written.

Lemma 4.10. *Let $B(\alpha, \beta)$ be a point of the form* (24) *lying over the straight line $\pi_{\mathbf{h}-1}$ or on this line in the belt $0 \le \beta < n_{\mathbf{h}}$. Then the numbers $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$ may be chosen with condition $0 \le \omega_k < n_k$ $(k = 1, \dots, \mathbf{h})$ and such that the lowest vertex of the diagram $\Delta H(X, \varphi_{\mathbf{h}} + Y)$ equals B.*

Proof. From the fact that the diagram of the product equals the sum of diagrams of factors follows that the lowest vertex of the diagram $\Delta H(X, \varphi_{\mathbf{h}} + Y)$ is a linear combination of the lowest vertices of the diagrams $\Delta X = \{\frac{\infty}{1}\}$, $\Delta f_1(X, \varphi_{\mathbf{h}} + Y)$, $\dots$, $\Delta f_{\mathbf{h}}(X, \varphi_{\mathbf{h}} + Y)$ with coefficients $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}}$, respectively. From Proposition 4.9 (i) and (ii) it follows that the abscissa of the lowest vertex of the diagram $\Delta H(X, \varphi_{\mathbf{h}} + Y)$ equals

$$\omega_0 + \omega_1 \frac{\bar{b}_1}{b_0} + \dots + \omega_{\mathbf{h}-1} \frac{\bar{b}_{\mathbf{h}-1}}{b_0} + \omega_{\mathbf{h}} \left(\frac{n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - b_{\mathbf{h}-1}}{b_0}\right) \,. \tag{26}$$

The ordinate equals $\omega_{\mathbf{h}}$ hence we put $\omega_{\mathbf{h}} = \beta$. We want to choose $\omega_0, \omega_1, \dots, \omega_{\mathbf{h}-1}$ in order to have

$$\omega_0 + \omega_1 \frac{\bar{b}_1}{b_0} + \dots + \omega_{\mathbf{h}-1} \frac{\bar{b}_{\mathbf{h}-1}}{b_0} + \beta \left(\frac{n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - b_{\mathbf{h}-1}}{b_0}\right) = \alpha \,. \tag{27}$$

Then

$$\omega_0 \bar{b}_0 + \omega_1 \bar{b}_1 + \dots + \omega_{\mathbf{h}-1} \bar{b}_{\mathbf{h}-1} = \alpha b_0 - \beta(n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - b_{\mathbf{h}-1}) \,. \tag{28}$$

Notice that αb_0 is an integer divisible by $n_{\mathbf{h}}$. The value of the right side is fixed. There are unknowns $\omega_0, \dots, \omega_{\mathbf{h}-1}$ on the left side. We can apply the semigroup theory (e.g. [8]). Let us recall the notion of the conductor

$$c_k = (n_1 - 1)\bar{b}_1 + \dots + (n_k - 1)\bar{b}_k - \bar{b}_0 + e_k, \quad (k = 1, \dots, \mathbf{h}) \tag{29}$$

with the property that for every integer $c \ge c_k$ such that $c \equiv 0 (\operatorname{mod} e_k)$ there exists the unique sequence $\omega_0, \omega_1, \dots, \omega_k$ such that $\omega_0 \ge 0$, $0 \le \omega_1 < n_1$, $\dots$, $0 \le \omega_k < n_k$ satisfying $c = \omega_0 \bar{b}_0 + \omega_1 \bar{b}_1 + \dots + \omega_k \bar{b}_k$. Hence, it suffices to show that the right side R of (28) is greater than or equal to $c_{\mathbf{h}-1}$. Let us notice that the right side is divisible by $e_{\mathbf{h}-1} = n_{\mathbf{h}}$. It follows from the inequality $\beta \le n_{\mathbf{h}} - 1$ that

$$R = \alpha b_0 - \beta n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + \beta b_{\mathbf{h}-1} \ge \alpha b_0 - (n_{\mathbf{h}} - 1) n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + \beta b_{\mathbf{h}-1} \,. \tag{30}$$

Therefore

$$R \ge (\alpha n_{\mathbf{h}} - n_{\mathbf{h}} n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + \beta b_{\mathbf{h}-1}) + n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} \,. \tag{31}$$

The number in parantheses in nonnegative. It follows from the fact that the chosen point B of the form (24) lies over the straight line $\pi_{\mathbf{h}-1}$ or on this line (Proposition 4.9 (iv)). Hence

$$R \geq n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} \ . \tag{32}$$

In order to show that $R \geq c_{\mathbf{h}-1}$ we study the difference $R - c_{\mathbf{h}-1}$. The first $\mathbf{h}-1$ components in the formula on $c_{\mathbf{h}-1}$ are written below in the opposite order:

$$\begin{aligned} R - c_{\mathbf{h}-1} \geq & n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - (n_{\mathbf{h}-1}-1)\bar{b}_{\mathbf{h}-1} - (n_{\mathbf{h}-2}-1)\bar{b}_{\mathbf{h}-2} - \cdots - (n_1-1)\bar{b}_1 + \bar{b}_0 - e_{\mathbf{h}-1} \\ \geq & n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} - n_{\mathbf{h}-1}\bar{b}_{\mathbf{h}-1} + (\bar{b}_{\mathbf{h}-1} - n_{\mathbf{h}-2}\bar{b}_{\mathbf{h}-2}) + \cdots + (\bar{b}_2 - n_1\bar{b}_1) + \bar{b}_1 + \bar{b}_0 - e_{\mathbf{h}-1} \ . \end{aligned}$$

Since the numbers in parantheses are positive (God given inequality) we obtain

$$R - c_{\mathbf{h}-1} > \bar{b}_1 + \bar{b}_0 - e_{\mathbf{h}-1} \geq 0 \tag{33}$$

which finish the proof of the lemma. □

Main construction

Lemma 2 gives us some freedom to chose B. However, during the construction of the deformation $F_t = f + tH$ the point B is unique (in the fixed coordinate system). Every characteristic exponent may be written in the form

$$\frac{b_k}{b_0} = \frac{m_k}{n_1 \ldots n_k} \ , \quad \mathrm{GCD}(n_k, m_k) = 1 \ , \quad k = 1, \ldots, \mathbf{h} \ . \tag{34}$$

The pairs (n_1, m_1), …, $(n_{\mathbf{h}}, m_{\mathbf{h}})$ are called the characteristic Puiseux pairs. Applying the Euclid algorithm to the last characteristic pair we choose the unique integers i, j such that

$$\begin{cases} m_{\mathbf{h}} j - n_{\mathbf{h}} i = 1 \\ 0 < i < m_{\mathbf{h}} \\ 0 < j < n_{\mathbf{h}} \end{cases} . \tag{35}$$

Then we put

$$\tilde{\alpha} := \frac{\bar{b}_{\mathbf{h}} - b_{\mathbf{h}} + i}{N_{\mathbf{h}-1}} \ , \qquad \tilde{\beta} := n_{\mathbf{h}} - j \ . \tag{36}$$

We choose by Lemma 4.10 $\omega_0, \omega_1, \ldots, \omega_{\mathbf{h}} = \tilde{\beta}$ such that the lower vertex of the diagram $\Delta H(X, y_{\mathbf{h}} + Y)$ (25) equals $B(\tilde{\alpha}, \tilde{\beta})$. Recall that

$$0 \leq \omega_1 < n_1 \ , \ \ldots \ , \ 0 \leq \omega_{\mathbf{h}} < n_{\mathbf{h}} \ . \tag{37}$$

Now, we want to finish the proof. In the begining of this section we discussed two cases that allows to compare $T_*(f, X)$ and $T_*(F_t, X)$. Without loss of generality we assume that $b_0 = v_0$.

Proposition 4.11. (first case) *If one of the following conditions holds:*

(a) $\ell_{\mathbf{h}} = 1$,
(b) $\ell_{\mathbf{h}} \geq 2$ *and* $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \tilde{\beta}) > 1$,
(c) $\ell_{\mathbf{h}} \geq 2$ *and* $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \tilde{\beta}) = 1$ *but* $n_{\mathbf{h}} \geq 3$

then

(i) $T_*(F_t, X) = T_*(f, X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}})$.
(ii) *If* $\ell < \ell_{\mathbf{h}}$ *then* $\hat{\mu}(F_t, \varphi_\ell) = \hat{\mu}(f, \varphi_\ell)$.
(iii) $\hat{\mu}(F_t, \varphi_{\ell_{\mathbf{h}}}) = \hat{\mu}(f, \varphi_{\ell_{\mathbf{h}}}) - \frac{1}{N_{\mathbf{h}-1}}$.

Proposition 4.12. (second case)
If $\ell_{\mathbf{h}} \geq 2$ *and* $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \beta) = 1$ *and* $n_h = 2$ *then*

(i) $T_*(F_t, X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}-1})$.
(ii) *If* $\ell < \ell_{\mathbf{h}}$ *then* $\hat{\mu}(F_t, \varphi_\ell) = \hat{\mu}(f, \varphi_\ell)$.
(iii) $\hat{\mu}(f, \varphi_{\ell_{\mathbf{h}}}) = \frac{1}{N_{\mathbf{h}-1}}$.

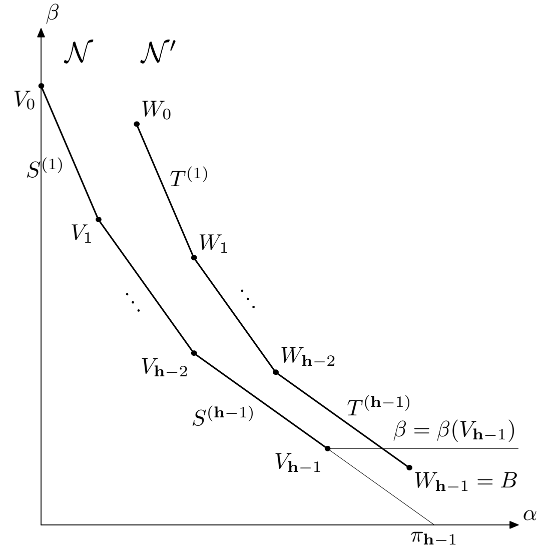
To finish the proof it suffices to verify propositions. Before this we should study relation of the Newton polygons of the diagrams $\Delta f(X, y_{\mathbf{h}}+Y)$ and $\Delta H(X, y_{\mathbf{h}}+Y)$. It follows from Corollary 4.7 that the Newton polygon of the first diagram has $\mathbf{h}$ segments $S^{(1)}, \ldots, S^{(\mathbf{h})}$ with respective inclinations $\frac{b_1}{b_0}, \ldots, \frac{b_{\mathbf{h}}}{b_0}$. Let $V_0, V_1, \ldots V_{\mathbf{h}}$ be succesive vertices of the first diagram (ordered from up to down). We have $S_k = \overline{V_{k-1}V_k}$ $(k = 1, \ldots, \mathbf{h})$; we use bar to denote segment. From Corollary 4.8 and from (25) we conclude that the Newton polygon of the second diagram has $\mathbf{h}-1$ segments $T^{(1)}, \ldots, T^{(\mathbf{h}-1)}$ with respective inclinations $\frac{b_1}{b_0}, \ldots, \frac{b_{\mathbf{h}-1}}{b_0}$. Let $W_0, W_1, \ldots W_{\mathbf{h}-1}$ be succesive vertices of the second diagram. We have $T_k = \overline{V_{k-1}V_k}$ $(k = 1, \ldots, \mathbf{h}-1)$. Recall that $W_{\mathbf{h}-1} = B(\tilde{\alpha}, \tilde{\beta})$ from the construction in Lemma 4.10 For a point (vertex) V we will write $\alpha(V)$ (resp. $\beta(V)$) to denote its abscissa (resp. ordinate).

Proposition 4.13.

(I) *For two above Newton polygons we consider the sets of first* $\mathbf{h}-1$ *segments* $\mathcal{N} = \{S^{(1)}, \ldots, S^{(\mathbf{h}-1)}\}$ *and* $\mathcal{N}' = \{T^{(1)}, \ldots, T^{(\mathbf{h}-1)}\}$. *We claim that* $\mathcal{N}'$ *lies over* $\mathcal{N}$ *in the weak sense: only the last segment of* $\mathcal{N}'$ *and the last segment of* $\mathcal{N}$ *may lay on the same straigh line.*
(II) *The inclination of straight line determined by* $V_{\mathbf{h}-1}$ *and* $W_{\mathbf{h}-1}$ *equals*

$$\frac{\alpha(W_{\mathbf{h}-1}) - \alpha(V_{\mathbf{h}-1})}{\beta(V_{\mathbf{h}-1}) - \beta(W_{\mathbf{h}-1})} = \frac{b_{\mathbf{h}}}{b_0} - \frac{1}{b_0(n_{\mathbf{h}} - \tilde{\beta})}.$$

Proof. (I). The vertices $V_{\mathbf{h}-2}$ and $V_{\mathbf{h}-1}$ determines the straight line $\pi_{\mathbf{h}-1}$. From the construction of $W_{\mathbf{h}-1} = B$ we have $\beta(W_{\mathbf{h}-1}) < \beta(V_{\mathbf{h}-1}) = n_{\mathbf{h}}$ and $W_{\mathbf{h}-1}$ lies over the line $\pi_{\mathbf{h}-1}$ or on this line.



Taking into consideration a geometrical argument to finish the proof if suffices to show that

$$|T^{(k)}| \leq e_{k-1} - e_k = |S^{(k)}| \text{ for } k = 1, \dots, \mathbf{h} - 1 \,. \tag{38}$$

By Corollaries 4.8 and 4.7 to each diagram $\Delta f_1(X, y_{\mathbf{h}} + Y), \dots, \Delta f_{\mathbf{h}}(X, y_{\mathbf{h}} + Y)$, $\Delta f(X, y_{\mathbf{h}} + Y)$ we assign the succesive inclinations that appear in their Newton polygons. We write ∞ if a diagram does not touch the horizontal axis. We write the multiplicities in the meaning of Theorem 2.1 (i) under the values.

$$
\begin{array}{ccccccc}
\Delta f_1(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{1} & & & & & \\
\Delta f_2(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{(n_1-1)} & \underbrace{(b_2/b_0)}_{1} & & & & \\
\Delta f_3(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{(n_1-1)n_2} & \underbrace{(b_2/b_0)}_{(n_2-1)} & \underbrace{(b_3/b_0)}_{1} & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & \\
\Delta f_{\mathbf{h}}(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{(n_1-1)n_2\dots n_{\mathbf{h}}} & \underbrace{(b_2/b_0)}_{(n_2-1)n_3\dots n_{\mathbf{h}}} & \underbrace{(b_3/b_0)}_{(n_3-1)n_4\dots n_{\mathbf{h}}} & \cdots & \underbrace{(b_{\mathbf{h}-1}/b_0)}_{(n_{\mathbf{h}-1}-1)} & \underbrace{\infty}_{1} \\
\Delta f(X, y_{\mathbf{h}} + Y) & \underbrace{(b_1/b_0)}_{e_0-e_1} & \underbrace{(b_2/b_0)}_{e_1-e_2} & \underbrace{(b_3/b_0)}_{e_2-e_3} & \cdots & \underbrace{(b_{\mathbf{h}-1}/b_0)}_{e_{\mathbf{h}-2}-e_{\mathbf{h}-1}} & \underbrace{(b_{\mathbf{h}}/b_0)}_{e_{\mathbf{h}-1}=n_{\mathbf{h}}}
\end{array}
$$

Applying (37) we can estimate

$$\begin{aligned}|T^{(1)}|_{\mathbf{V}} &= 1\cdot\omega_1 + (n_1-1)\cdot\omega_2 + (n_1-1)n_2\cdot\omega_3 + \cdots + (n_1-1)n_2\ldots n_{\mathbf{h}-1}\cdot\omega_{\mathbf{h}} \\ &\le (n_1-1) + (n_1-1)(n_2-1) + (n_1-1)n_2(n_3-1) + \cdots + (n_1-1)n_2\ldots n_{\mathbf{h}-1}(n_{\mathbf{h}}-1) \\ &= (n_1-1)n_2\ldots n_{\mathbf{h}} = e_0 - e_1 = |S^{(1)}|_{\mathbf{V}}\end{aligned}$$

We reason analogously for other segments and we finish the proof of (I).

Now, we prove (II). From Proposition 4.9 (iii) we obtain the equation of the straight line $\pi_{\mathbf{h}}$ that contains the segment $S_{\mathbf{h}} = \overline{V_{\mathbf{h}-1}V_{\mathbf{h}}}$:

$$\alpha b_0 + \beta b_{\mathbf{h}} = \bar{b}_{\mathbf{h}} n_{\mathbf{h}}\ . \tag{39}$$

From (35) and (36) we obtain that the coordinates of the point $W_{\mathbf{h}-1} = B$ satisfy:

$$\tilde{\alpha} b_0 + \tilde{\beta} b_{\mathbf{h}} = \bar{b}_{\mathbf{h}} n_{\mathbf{h}} - 1\ . \tag{40}$$

Substituting $\beta = n_{\mathbf{h}}$ to (39) we obtain

$$\alpha(V_{\mathbf{h}-1}) = \frac{n_{\mathbf{h}}(\bar{b}_{\mathbf{h}} - b_{\mathbf{h}})}{b_0}\ . \tag{41}$$

By (40) we have

$$\frac{\alpha(W_{\mathbf{h}-1}) - \alpha(V_{\mathbf{h}-1})}{\beta(V_{\mathbf{h}-1}) - \beta(W_{\mathbf{h}-1})} = \frac{\tilde{\alpha} b_0 - n_{\mathbf{h}}\bar{b}_{\mathbf{h}} + n_{\mathbf{h}} b_{\mathbf{h}}}{(n_{\mathbf{h}} - \tilde{\beta})b_0} = \frac{b_{\mathbf{h}}}{b_0} - \frac{1}{b_0(n_{\mathbf{h}} - \tilde{\beta})}\ . \tag{42}$$

This finishes the proof. □

In the next proposition we study relations between the diagrams $\Delta f(X, \varphi_\ell + Y)$ and $\Delta H(X, \varphi_\ell + Y)$ for $\ell < \ell_{\mathbf{h}}$. By using the Teissier notation of the diagram

$$\Delta = \sum_{i=1}^{n} \left\{\frac{a_i}{b_i}\right\}\ , a_i, b_i > 0,\ \text{at least one of } a_i, b_i \text{ is finite}\ , \quad i = 1, \ldots, n\ , \tag{43}$$

we can assign inclinations directly to the diagram. To $\{\frac{a_i}{b_i}\}$ we assign $\frac{a_i}{b_i}$ with convetion $\frac{a_i}{\infty} = 0$ and $\frac{\infty}{b_i} = \infty$ $(i = 1, \ldots, n)$. For a rational $\theta > 0$ we define the transformation $[\Delta]_\theta$ of the diagram Δ which replace the components with inclination strictly geater then θ by the respective components with inclination θ. We write

$$[\Delta]_\theta = \sum_{\frac{a_i}{b_i} \le \theta} \left\{\frac{a_i}{b_i}\right\} + \sum_{\frac{a_i}{b_i} > \theta} \left\{\frac{\theta b_i}{b_i}\right\}\ . \tag{44}$$

Clearly

$$\text{if } \Delta \subset \Delta' \text{ then } [\Delta]_\theta \subset [\Delta']_\theta\ . \tag{45}$$

Recall that $\theta_\ell = v_\ell / v_0$ (1). Clearly, $\theta_\ell = o_f(\varphi_\ell)$ (22).

Proposition 4.14. *Let $\ell < \ell_{\mathbf{h}}$. Then*

(a) $\Delta f(X, \varphi_\ell + Y) = [\Delta f(X, y_{\mathbf{h}} + Y)]_{\theta_\ell}$,
(b) $\Delta H(X, \varphi_\ell + Y) \subset [\Delta H(X, y_{\mathbf{h}} + Y)]_{\theta_\ell}$.

Proof. We use methods of Lemma 7.1 from [20]. □

We can consider the diagrams generated by points $V_1, \ldots, V_n$ in $\mathbb{R}^2_+$. By $\Delta\{V_1, \ldots, V_n\}$ we mean the convex hull of the union $V_1 + \mathbb{R}^2_+ \cup \cdots \cup V_n + \mathbb{R}^2_+$. Recall that $F_t = f + tH$.

Proposition 4.15. *Let $\ell < \ell_{\mathbf{h}}$. Then*

(i) $\Delta f(X, \varphi_\ell + Y) = \Delta F_t(X, \varphi_\ell + Y)$,
(ii) $\mathrm{cycle}(\varphi_\ell) \subset T_*(F_t, X)$.
(iii) $\hat{\mu}(f, \varphi_\ell) = \hat{\mu}(F_t, \varphi_\ell)$.

Proof. (i) The line ρ_ℓ with inclination θ_ℓ supporting $\Delta f(X, y_{\mathbf{h}} + Y)$ crosses the horizontal axis at the point A_ℓ. We have

$$[\Delta f(X, y_{\mathbf{h}} + Y)]_{\theta_\ell} = \Delta\{V_0, \ldots, V_{k-1}, A_\ell\} \tag{46}$$

with the smallest k such that $\theta_\ell \leq \frac{b_k}{b_0}$. Analogously, the line ρ'_ℓ with the same inclination supporting $\Delta H(X, y_{\mathbf{h}} + Y)$ meets the horizontal axis at B_ℓ. We have

$$[\Delta H(X, y_{\mathbf{h}} + Y)]_{\theta_\ell} = \Delta\{W_0, \ldots, W_{k-1}, B_\ell\} \tag{47}$$

where k is the smallest with $\theta_\ell \leq \frac{b_k}{b_0}$. Both parts of Proposition 4.13 give

$$\Delta\{W_0, \ldots, W_{k-1}, B_\ell\} \subset \Delta\{V_0, \ldots, V_{k-1}, A_\ell\} . \tag{48}$$

Hence $[\Delta H(X, y_{\mathbf{h}} + Y)]_{\theta_\ell} \subset [\Delta f(X, y_{\mathbf{h}} + Y)]_{\theta_\ell}$. From Proposition 4.14 we obtain $\Delta H(X, \varphi_\ell + Y) \subset \Delta f(X, \varphi_\ell + Y)$. This gives (i) for sufficiently small $t \neq 0$. Both parts (ii) and (iii) follow from (i). □

Below, we proof a useful lemma. Let us recall a classical fact.

Property 4.16. *Nonzero polynomials $f, g \in \mathbb{C}[X]$ have a common root if and only if there exist nonzero polynomials $a, b \in \mathbb{C}[X]$, $\deg a < \deg g$, $\deg b < \deg f$ such that $af - bg = 0$ in $\mathbb{C}[X]$.*

Lemma 4.17. *Let $f, g \in \mathbb{C}[X]$ be polynomials without common roots. Then for small $t \neq 0$ $f + tg \in \mathbb{C}[X]$ has only single roots.*

Proof. Let $h = fg' - f'g \in \mathbb{C}[X]$. From property we conclude that h is nonzero polynomial. Let

$$Z = \left\{ -\frac{f(c)}{g(c)} : h(c) = 0 \text{ and } g(c) \neq 0 \right\} . \tag{49}$$

Clearly, Z is finite (may be empty). We will show that for

$$t \in \mathbb{C} \setminus (Z \cup \{0\}) \tag{50}$$

the polynomial F_t has only single roots. For the contrary let us assume that $F_t(c) = F_t'(c) = 0$. Hence

$$\left\{ \begin{array}{rl} f(c) + t\,g(c) &= 0 \\ f'(c) + t\,g'(c) &= 0 \end{array} \right. . \tag{51}$$

Since the system has nonzero solution $(1, t)$, the determinant must be zero. Hence $h(c) = 0$. It must be $g(c) \neq 0$. From the first equation we obtain $t = -\frac{f(c)}{g(c)}$ which contadicts (50). □

Remark 4.18. (e.g.[20]) It is covenient to apply the initial form defined by the pair of positive weights (a, b). For $f = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbb{C}\{X^*, Y\}$ we put $\mathrm{ord}_{(a,b)} f = \min\{a\alpha + b\beta \,:\, c_{\alpha,\beta} \neq 0\}$, $\mathrm{in}_{(a,b)} f = \sum c_{\alpha\beta} X^\alpha Y^\beta$ where (α, β) correspond to nonzero coefficients and $a\alpha + b\beta = \mathrm{ord}_{\mathbf{v}} f$. We put $\mathrm{ord}_{\mathbf{v}} 0 = \infty$ and $\mathrm{in}_{\mathbf{v}} 0 = 0$. For $f, g \in \mathbb{C}\{X^*, Y\}$ we have $\mathrm{ord}_{\mathbf{v}}(fg) = \mathrm{ord}_{\mathbf{v}} f + \mathrm{ord}_{\mathbf{v}} g$ and $\mathrm{in}_{\mathbf{v}}(fg) = (\mathrm{in}_{\mathbf{v}} f)(\mathrm{in}_{\mathbf{v}} g)$.

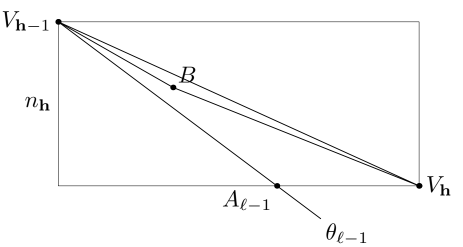
Verification of Propositions 4.11 and 4.12
Below, we will check both propositions.

Proof. Case (I) (a). The equality $\ell_{\mathbf{h}} = 1$ means that the series y has one characteristic pair and that the first exponent $\frac{v_1}{v_0} = \frac{b_1}{b_0} = \frac{m_1}{n_1}$ is characteristic. In this case $\Delta f(X, Y)$ has one segment S which joins $(0, n_1)$ and $(m_1, 0)$. By Properties 4.4 and 4.5 we have (up to a nonzero constant)

$$\mathrm{in}(f, S) = Y^{n_1} - a_1^{n_1} X^{m_1} \ .$$

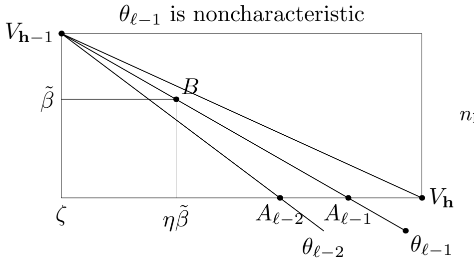
We put $F_t = f + tX^{\tilde\alpha} Y^{\tilde\beta}$ and we reason as in the Bodin's case. We have $T_*(f, X) = T_*(F_f, X) = \{0\}$ and $\hat\mu(F_t, 0) = \hat\mu(f, 0) - 1$.

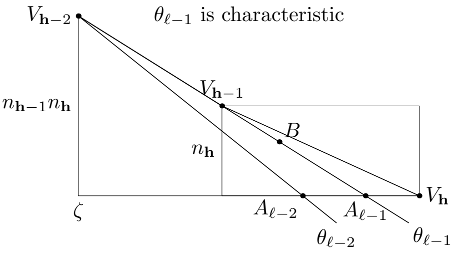
Case (I) (b). Now, we assume that $\ell_{\mathbf{h}} \geq 2$ and $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \beta) > 1$. For simplicity we write $\ell = \ell_{\mathbf{h}}$. We have $\deg \varphi_\ell = \theta_{\ell-1}$. Since $|\mathcal{N}(F_t, \varphi_\ell)| > 1$ then $\varphi_\ell \in T_*(F_t, X)$.



Hence $\mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}}) \subset T_*(F_t, X)$. We obtain the opposite inclusion from the equality $(f, X)_0 = (F_t, X)_0$ (which follows from Proposition 4.13) and by counting solutions of $F_t = 0$. Part (ii) follows from Proposition 4.15 (iii). As the case of Bodin we check that $\hat\mu(F_t, \varphi_\ell) = \hat\mu(f, \varphi_\ell) - \frac{1}{N_{\mathbf{h}-1}}$.

Case (I) (c) and Case (II). Let us assume that $\ell_{\mathbf{h}} \geq 2$ and $(b_{\mathbf{h}} - v_{\ell_{\mathbf{h}}-1})(n_{\mathbf{h}} - \beta) = 1$. As earlier we write $\ell = \ell_{\mathbf{h}}$. By using notation of Proposition 4.13 and from the proof of Proposition 4.15 the segments $V_{\mathbf{h}-1}A_{\ell-1}$ and $W_{\mathbf{h}-1}B_{\ell-1}$ lay on the same straigh line with the inclination $\theta_{\ell-1}$ ($W_{\mathbf{h}-1} = B$ and $A_{\ell-1} = B_{\ell-1}$).





We study $F_t(X, \varphi_{\ell-1} + Y)$. We have $\deg \varphi_{\ell-1} = \theta_{\ell-2}$ (we put $\theta_0 = 0$). Applying Property 4.5 we compute

$$\mathrm{in}_{(1,\theta_{\ell-1})} F_t(X, \varphi_{\ell-1} + Y) = \mathrm{in}_{(1,\theta_{\ell-1})} f(X, \varphi_{\ell-1} + Y) + t\,\mathrm{in}_{(1,\theta_{\ell-1})} H(X, \varphi_{\ell-1} + Y)\ .$$

Let us denote this form by I. For nonzero c, d and nonnegative ζ, η we have

$$\begin{aligned} I &= cX^{\zeta}\left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{n_{\mathbf{h}}} + t\,d^{\tilde{\beta}} X^{\eta\tilde{\beta}}\left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{\tilde{\beta}} \\ &= cX^{\zeta}\left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{\tilde{\beta}}\left[\left(Y^{u_{\ell-1}} - a_{\ell-1}^{u_{\ell-1}} X^{\theta_{\ell-1}u_{\ell-1}}\right)^{n_{\mathbf{h}}-\tilde{\beta}} + \frac{td^{\tilde{\beta}}}{c} X^{\zeta-\eta\tilde{\beta}}\right]. \end{aligned}$$

The right factor is nodegenerate by Lemma 4.17. If $\tilde{\beta} > 1$ (case (I) c) then $a_{\ell-1}X^{\theta_{\ell-1}}$ is a multiple root of $\mathrm{in}_{(1,\theta_{\ell-1})}F_t$. Hence

$$\varphi_{\ell} = \varphi_{\ell-1} + a_{\ell-1}X^{\theta_{\ell-1}} \in T_*(F_t, X)\ . \tag{52}$$

To obtain (ii) we reason as in the previous case. We have

$$\begin{aligned} \hat{\mu}(f, \varphi_{\ell}) &= 2\,\mathrm{Area}(V_{\mathbf{h}-1}A_{\ell-1}V_{\mathbf{h}}) - |A_{\ell-1}V_{\mathbf{h}}|\ , \\ \hat{\mu}(F_t, \varphi_{\ell}) &= 2\,\mathrm{Area}(BA_{\ell-1}V_{\mathbf{h}}) - |A_{\ell-1}V_{\mathbf{h}}| \end{aligned}$$

where $|\ldots|$ stands for the length of a segment. Clearly $\hat{\mu}(F_t, \varphi_{\ell}) = \hat{\mu}(f, \varphi_{\ell}) - \frac{1}{N_{\mathbf{h}}}$.

When $\tilde{\beta} = 1$ (equivalently $n_{\mathbf{h}} = 2$, case (II)) the form I is nondegenerate. Hence $T_*(f, X) = \mathrm{cycle}(\varphi_1) \cup \cdots \cup \mathrm{cycle}(\varphi_{\ell_{\mathbf{h}}-1})$. We obtain $\hat{\mu}(f, \varphi_{\ell}) = \frac{1}{N_{\mathbf{h}}}$. This finishes the proof of Propositions 4.11 and 4.12 and the proof of Theorem 1.1 □

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 121–136
DOI: https://doi.org/10.18778/8331-092-3.10

REAL NULLSTELLENSATZ AND SUMS OF SQUARES

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Abstract. In this paper we highlight the foundational principles of sums of squares in the study of Real Algebraic Geometry. To this aim the article is designed as mainly a self-contained presentation of a variation of the standard proof of Real Nullstellensatz, the only relevant omission being the (long) proof of the Tarski-Seidenberg theorem. On the way we see how the theory follows closely developments in algebra and model theory due to Artin and Schreier. This allows us to present on the way Artin's solution to Hilbert's 17th Problem: whether positive polynomials are sums of squares. These notes are intended to be accessible to math students of any level.

1. Introduction

Any sum of squares of real numbers is equal zero if and only if the numbers are zero themselves; this is not true anymore over the algebraic closure of the real field. These fundamental facts underlie a host of subtle differences of Algebraic Geometry over the Real and the Complex numbers. The first and foremost difference is the Nullstellensatz, a theorem which describes the relation between algebraic objects and their vanishing sets. The complex Nullstellensatz asks the defining ideal of a set to be radical, whereas the Real Nullstellensatz demands more: for the ideal to be real, that is to have the property that if a sum of squares is an element of this ideal, then all summands are elements of the ideal also.

This may come as surprise, but the Real Nullstellensatz was unknown until the paper [Risler, 1970] of Jean-Jacques Risler in 1970. By all means, the sums of squares were already a very prominent element in the study of Algebraic Geometry over the reals. In 1900 among the famous problems of David Hilbert was the following, the 17th Problem: is any nonnegative polynomial a sum of squares? This

2010 *Mathematics Subject Classification.* 13J30 (12D15, 14P05).
Key words and phrases. Real Nullstellensatz, sums of squares, real field, real closed field, Artin-Schreier, Hilbert's 17th problem, positive polynomials, real radical.

question lies naturally in Hilbert's general predisposition to formalize mathematics, since being a sum of squares is an algebraic certificate for nonnegativity. It was already discovered by Hilbert that one cannot demand every positive polynomial to be a sum of squares of polynomials. Nevertheless, one had to wait for Emil Artin to present in 1927 a positive solution for rational functions, [Artin, 1927]. This comes therefore as no surprise that elements of Artin-Schreier Theory are useful in the proof of Real Nullstellensatz. Thus we will introduce some elements of the theory and use the opportunity to present a full proof of Artin's solution to Hilbert's 17th Problem.

Research on Null-, Nichtnull- and Positivstellensätze, and sums of squares continues, nowadays motivated by pursuit of efficient optimization algorithms. For a panorama of modern developments one can consult [Marshall, 2008], [Scheiderer, 2009] or [Lasserre, 2015]. As a sidenote, one would like to remark that by [Delzell, 1984] a nonnegative polynomial is even a sum of squares of regulous functions, i.e. rational functions extending continuously to their indeterminacy loci, which currently are quite intensively studied, compare [Fichou et al., 2016]. The aforementioned fact can be seen as basis of Nullstellensatz for regulous functions which again demands the defining ideals to be simply radical, as it was all the time in the complex case.

This note was designed foremost as a self-contained presentation of a variation of the standard proof of Real Nullstellensatz, we will omit only the (long but elementary) proof of the Tarski-Seidenberg theorem. These notes are intended to be accessible to math students of any level. Notes are organized as follows: presentation of the Real Nulsellensatz is given in Section 2 followed by explanation of notation and notions, as well as essential properties and proofs of intermediate results in Sections 3, 4 and 5. In Section 6 one finds the presentation and Artin's solution of Hilbert's 17th Problem and the paper ends with presentation of proof of Real Nullstellensatz over real closed fields in Section 7. On first lecture it is advised for a novice reader to prove Propositions and Properties left without proof.

2. Real Nullstellensatz

Every real algebraic set in $\mathbb{R}^n$ is defined to be the vanishing set of an ideal $I \triangleleft \mathbb{R}[X_1, \dots, X_n]$, i.e. it is a set of the form

$$V(I) = \{x \in \mathbb{R}^n : \ \forall_{f \in I} f(x) = 0\}.$$

Note that every polynomial ideal I is finitely generated by, say, $f_1, \dots, f_k$, hence any real algebraic set can be given by one equation $f_1^2 + \cdots + f_k^2 = 0$. We say an ideal I is real if from $\sum a_j^2 \in I$ follows all $a_j \in I$, see Section 3.

On the other hand, for a set $V \subset \mathbb{R}^n$ denote the defining ideal

$$\mathcal{I}(V) = \{f \in \mathbb{R}[X] : \ \forall_{x \in V} f(x) = 0\}$$

i.e. $\mathcal{I}(V)$ is the largest ideal in $\mathbb{R}[X_1, \dots, X_n]$ such that all its elements vanish on V. Obviously, always $I \subset \mathcal{I}(V(I))$.

Real Nulstellensatz ties the geometric meaning of ideals with the algebraic meaning of sets in the real euclidean space in the following way:

Theorem 1 (Real Nullstellensatz)**.** *Let $I \triangleleft \mathbb{R}[X_1, \dots, X_n]$.*

$$I = \mathcal{I}(V(I)) \iff I \text{ is real}$$

Proof of Real Nullstellensatz is given in the last Section. Reader is advised to start with the proof and go back to relevant sections when needed.

3. Basic algebra

Throughout this section let R be a commutative ring (with unity) and $I \triangleleft R$ an ideal.

Definition 3.1. *I is real if*

$$a_1^2 + \cdots + a_k^2 \in I \Rightarrow a_1, \dots, a_k \in I$$

for any $a_1, \dots, a_k \in R$.

Property 3.2. *(1) If an ideal is prime, then it is radical.*
(2) If an ideal is real, then it is radical.

Property 3.3. *I is prime iff the quotient ring R/I is an integral domain i.e. has no zero divisors.*

Property 3.4. *(1) Field R embeds naturally into $R[X_1, \dots, X_n]/I$ if $I \neq R[X_1, \dots, X_n]$.*
(2) Integral domain R embeds naturally into its field of fractions $\mathrm{Quot}(R)$*.*

Definition 3.5. *I is primary if*

$$ab \in I \Rightarrow a \in I \textit{ or } b^m \in I \text{ for some } m \in \mathbb{N}.$$

Definition 3.6. *We say that the commutative ring is noetherian if every ascending chain of ideals stabilizes.*

The above is equivalent to saying that every ideal is finitely generated. Note that every field is noetherian, because it contains only two ideals (0) and (1).

Theorem 3.7 (Hilbert's basis theorem)**.** *If R is a noetherian ring, then the ring of polynomials $R[X_1, \dots, X_n]$ is also noetherian.*

Theorem 3.8 (Noether-Lasker Theorem)**.** *Assume ring is noetherian. Every ideal is an intersection of finitely many primary ideals.*

Proof. We divide the proof into two steps.

- *Every ideal is a finite intersection of irreducible ideals.*

We say that an ideal I is irreducible if for any two ideals J, K if $I = J \cap K$, then $I = J$ or $I = K$. The proof is standard for noetherian rings:

Let A be the set of all ideals which are not a finite intersection of irreducible ideals. Take $I \in A$. If I cannot be expressed as an intersection of two ideals different from I, then I is irreducible. Therefore $I \notin A$. Hence $I = J_1 \cap K_1$. Obviously, either $J_1 \in A$ or $K_1 \in A$. Set $I_1 = J_1$ if $J_1 \in A$ or $I_1 = K_1$ otherwise. Proceed inductively, given $I_k \in A$ we have $I_k = J_k \cap K_k$ and $I_k \neq J_k$, $I_k \neq K_k$. Put

$$I_{k+1} = \begin{cases} J_k & \text{if } J_k \in A \\ K_k & \text{otherwise} \end{cases}$$

We get an ascending sequence

$$I \subset I_1 \subset \dots$$

of ideals. Since R is noetherian, we get $I_k = I_N$ for all $k \geq N$ and some $N \in \mathbb{N}$. But then $I_N = I_{N+1}$ contrary to assumption. Therefore $A = \emptyset$. This ends the proof.

- *Every irreducible ideal is primary*

Take an irreducible ideal I and take $ab \in I$. We will use quotients of ideals to prove that $a \in I$ or $b^m \in I$.

Define $J_k = I : (b^k) = \{c \in R : cb^k \in I\}$. We have that J_k are ideals and

$$I = J_0 \subset J_1 \subset J_2 \subset \dots$$

Since R is noetherian, the sequence stabilizes. Let J_N be such that $J_k = J_N$ for all $k \geq N$.

Put $J = J_N$ and $K = I + (b^N)$. Then obviously $I \subset J \cap K$. Moreover, if $c \in J \cap K$, then

$$c = i + fb^N,\ i \in I \tag{1}$$

and

$$b^N c \in I.$$

Multiplying both sides of (1) above by b^N we get

$$cb^N - i = fb^{2N}.$$

Hence $fb^{2N} \in I$. Therefore, $f \in J_{2N} = J_N$. Hence $fb^N \in I$ and from the form (1) we see $c \in I$. Therefore, $I = J \cap K$.

Since I is irreducible, we get either $I = K = I + (b^m)$ and $b^m \in I$ or $I = J_N$. In the latter case we have $I = J_N \supset J_1 \supset J_0 = I$, hence $J_1 = I$. Since $ab \in I$, hence $a \in I : (b) = I$. ☐

Corollary 3.9 (Prime decomposition of a radical). *Assume ring is noetherian. Every radical ideal is a finite intersection of minimal prime ideals.*

Here a prime ideal p is minimal with respect to I if $I \subset p$ and for any p' prime: $I \subset p' \subset p \Rightarrow p' = p$.

Proof. Three easy steps.

• *The radical of primary ideal is prime*

Let I be primary and $\sqrt{I} = \{a \in R : a^m \in I \text{ for some } m\}$ be its radical. Take $ab \in \sqrt{I}$. Then $(ab)^m \in I$. Since I is primary, we get $a^m \in I$ or $b^{km} \in I$. From definition of radical, either $a \in \sqrt{I}$ or $b \in \sqrt{I}$.

• Since $I = p_1 \cap \dots \cap p_k$ with p_i primary ideals due to Noether-Lasker Theorem and I is radical, then

$$I = \sqrt{I} = \sqrt{p_1 \cap \dots \cap p_k} = \sqrt{p_1} \cap \dots \cap \sqrt{p_k},$$

where every $\sqrt{p_i}$ is prime.

• *The prime ideals in decomposition can be taken as minimal.*

We have $I = p_1 \cap \dots \cap p_k$ with all p_i prime. Fix $p_i =: p$. Consider any chain $(P_\alpha)_\alpha$ with respect to inclusion of prime ideals P_α such that $p \supset P_\alpha \supset I$ and $P_\alpha \subset P_\beta$ for $\alpha \geq \beta$. Then $P := \cap_\alpha P_\alpha$ is a prime ideal. Indeed, let $ab \in P$. Then $ab \in P_\alpha$ for every α. Assume $a, b \notin P$, then $a, b \notin P_\alpha$ for some α (α can be chosen in common for a, b because of inclusions). But this is contrary to assumption that P_α is prime. Hence every chain has a lower bound. Therefore by Kuratowski-Zorn Lemma[1] there exists a minimal element P_i. The prime ideal P_i is a minimal prime containing I by its definition.

One has $I = p_1 \cap \dots \cap p_i \cap \dots \cap p_m = p_1 \cap \dots \cap P_i \cap \dots \cap p_m$. Apply above reasoning to every ideal p_i in the representation. □

Proposition 3.10. *Assume ring is noetherian. All minimal prime ideals containing a real ideal are real.*

Proof. Let I be a real ideal. Since real ideal is radical, from Corollary 3.9 we can write $I = p_1 \cap \dots \cap p_r$ with p_i minimal prime ideals containing I. Assume p_1 is not real. Then we can take $a_1^2 + \dots + a_k^2 \in p_1$ such that $a_1 \notin p_1$. Since p_l are minimal, we can choose $b_l \in p_l \setminus p_1$ for $l = 2, \dots, r$. Put $b = \Pi_{l=2,\dots,r} b_l$. We have $b \notin p_1$ by definition of b, because p_1 is prime. Then

$$(a_1 b)^2 + \dots + (a_k b)^2 = (a_1^2 + \dots + a_k^2) b^2 \in p_1 \cap \bigcap_{l=2,\dots,r} p_l = I$$

and since I is real, we have $a_1 b \in I \subset p_1$. Since p_1 is prime, we get $a_1 \in p_1$ or $b \in p_1$. This gives a contradiction. Hence $a_1, \dots, a_k \in p_1$ and p_1 is real. □

Knowing there exists prime decomposition of radical ideals, we can reformulate Proposition 3.10 in a following way.

Corollary 3.11 (Real prime decomposition of real ideal)**.** *Assume ring is noetherian. Every real ideal is a finite intersection of minimal real prime ideals.*

Now, the following paragraph is not necessary for proof of RN, but is basic and of interest in view of Artin-Lang homomorphism theorem.

---

[1]Kuratowski-Zorn Lemma: If every chain in a partially ordered set is bounded from below, then there exists a minimal element in the set.

Proposition 3.12. *Let R be a commutative ring. An R-algebra A is finitely generated iff it is isomorphic to a quotient ring $R[X]/I$ for some polynomial ring over R and an ideal $I \lhd R[X]$.*

Proof. Suppose A is finitely generated as an R-algebra, this means there exist polynomials $f_1, \ldots, f_k \in R[X_1, \ldots, X_n]$ such that $A = R[f_1, \ldots, f_k]$. Then put $\Phi : R[X_1, \ldots, X_k] \to A$ as $\Phi(f) = f(f_1, \ldots, f_k)$. Without doubt Φ is a surjective homomorphism. Take $I := \ker\Phi$. Then $R[X_1, \ldots, X_k]/I$ is isomorphic to A.

Now suppose that $R[X_1, \ldots, X_k]/I$ is isomorphic to A. Since the natural homomorphism $\Phi : R[X_1, \ldots, X_k] \ni f \to f + I \in A$ is surjective and $\Phi(f) = f(\Phi(X_1), \ldots, \Phi(X_k))$, we get $A = \Phi(R[X_1, \ldots, X_k]) = R[\Phi(X_1), \ldots, \Phi(X_k)]$. □

4. Elements of Artin-Schreier Theory

One property that separates complex and real numbers is zeros of sums of squares.

Definition 4.1. *A field R is real if*

$$a_1^2 + \cdots + a_k^2 = 0 \Rightarrow a_1, \ldots, a_k = 0$$

(or satisfies any of the equivalent conditions of Theorem 4.7).

You can see that complex numbers cannot be a real field since $i^2 + 1^2 = 0$. The Artin-Schreier Theory deals with this in a model-theoretic way.

Another thing that sets apart real and complex numbers is the ordering.

Definition 4.2. *Let R be a ring. We say that $\leq$ is a total (linear) ordering of R if it is an ordering*

(i) $a \leq a$
(ii) $(a \leq b \wedge b \leq c) \Rightarrow a \leq c$ TRANSITIVE
(iii) $(a \leq b \wedge b \leq a) \Rightarrow a = b$ ANTISYMMETRIC

which is total (linear)

(iv) $a \leq b \vee b \leq a$

and consistent with addition and multiplication

(v) $a \leq b \Rightarrow (\forall_c \quad a + c \leq b + c)$
(vi) $(0 \leq a \wedge 0 \leq b) \Rightarrow 0 \leq ab$

We write $a < b$ when $a \leq b$ and $a \neq b$.

Property 4.3. *If ring R is ordered, then*

(1) $0 \leq a^2$*, in particular* $0 < 1$
(2) $0 \leq a \Rightarrow -a \leq 0$

Moreover, if R is a field, then

(3) $0 < a < b \Rightarrow 0 < \frac{1}{b} < \frac{1}{a}$
(4) $0 < ab \iff 0 < \frac{a}{b} \wedge b \neq 0$

Corollary 4.4. *If the ring R is ordered, then $\mathbb{N} \subset R$.*

If a field R is ordered, then $\mathbb{Q} \subset R$. In particular, $\mathrm{char} R = 0$.

Denote by R^2 all squares of elements of R.

Let us denote by $\sum R^2$ all finite sums of squares of elements of R.

If ring R is ordered, then $0 \leq a$ for all $a \in \sum R^2$. Not all rings can be ordered: note that for complex numbers -1 is a square, so ordering would imply all complex numbers to be zero.

Let us introduce a set defining an ordering.

Definition 4.5. *We say $P \subset R$ is a proper cone, if*

(a) $\sum R^2 \subset P$
(b) $P + P \subset P$, $P \cdot P \subset P$ CLOSED UNDER ADDITION AND MULTIPLICATION
(c) $-1 \notin P$ PROPER
(d) $-P \cap P = \{0\}$ ANTISYMMETRIC

A proper cone P is said to be a positive cone if

(e) $P \cup -P = R$ TOTAL

Naturally, $-P := \{a \in R : -a \in P\}$. Note that if $\sum R^2$ is a positive cone, then it is the unique positive cone of R.

Property 4.6. *There is a one-to-one correspondence between total orderings of R and positive cones of R. The correspondence is given by*

$$a \leq b \iff b - a \in P.$$

First Artin-Schreier Theorem gives characterization of ordered fields as real fields.

Theorem 4.7 (Artin-Schreier Theorem for real fields)**.** *Let R be a field. Following conditions are equivalent*

(1) R *is real i.e.* $a_1^2 + \cdots + a_k^2 = 0 \Rightarrow a_1, \ldots, a_k = 0$
(2) -1 *is not a sum of squares in* R
(3) R *can be ordered*
(4) R *contains a positive cone*

Proof. (1) $\iff$ (2) If $-1 \in \sum R^2$, then $-1 = a_1^2 + \cdots + a_k^2$. Hence $0 = 1^2 + a_1^2 + \cdots + a_k^2$ and R is not real. If $\sum_{j=1,\ldots,k} a_j^2 = 0$ and $a_1 \neq 0$, then $\sum_{j \neq 1} \left(\frac{a_j}{a_1}\right)^2 = -1$.

(3) $\iff$ (4) By Property 4.6.

(3) $\Rightarrow$ (2) Assume R is ordered. If $-1 = \sum_{j=1,\ldots,k} a_j^2$, then $0 \leq -1$. Hence $0 < 1 + (-1) = 0$ which gives a contradiction.

(2) $\Rightarrow$ (3) We will show the following.

• *For R real there exists a maximal proper cone in R. A maximal proper cone is a positive cone.*

The set $\sum R^2$ is a proper cone by assumption that -1 is not a sum of squares. Consider any chain (P_α) of proper cones. Then $P := \bigcup(P_\alpha)$ is a proper cone. Indeed, it is obvious that P satisfies points (a)-(c) of the definition. To prove (d) it suffices to note that $P_\alpha \cap -P_\beta = \{0\}$ for all α, β. Hence $0 \in P \cap -P \subset \{0\}$. Therefore, every chain is bounded from above and by Kuratowski-Zorn Lemma there exists a maximal proper cone $P_\leq$ in R.

Assume $P_\leq$ is not a positive cone. Then for $c \notin P_\leq \cup -P_\leq$ we have that c is not a sum of squares and $P_c := P_\leq + cP_\leq$ is the smallest proper cone containing $P_\leq \cup \{c\}$. Since $P_\leq$ is maximal, we get $P_\leq = P_c$. Hence $c \in P_\leq$. Contradiction.

Therefore, every real field contains a positive cone, and equivalently it can be ordered. $\square$

Proposition 4.8. *Let R be a ring and $I \lhd R$ a prime ideal. Field of fractions $\mathrm{Quot}(R/I)$ is real iff I is real.*

Proof. Note that $(a+I)/(b+I) = 0$ in $\mathrm{Quot}(R/I)$ iff $a \in I$ and $b \notin I$. In particular

$$\sum_{i=1,\ldots,k} \left(\frac{f_i + I}{g_i + I}\right)^2 = 0 \iff \sum_{i=1,\ldots,k} \left(\frac{f_i g_1 \cdots g_k}{g_i}\right)^2 \in I.$$

So if we assume I is real, then for $\sum_{i=1,\ldots,k} \left(\frac{f_i+I}{g_i+I}\right)^2 = 0$ we get $\frac{f_i g_1 \cdots g_k}{g_i} \in I$ for every i. Therefore $f_i/g_i = 0$ for every i. On the other hand, if $\mathrm{Quot}(R/I)$ is real and we take $f_1^2 + \cdots + f_k^2 \in I$, then $(f_1 + I)^2 + \cdots + (f_k + I)^2 = 0$ and it follows $f_i + I = 0$ for all i. Therefore, $f_i \in I$. $\square$

Definition 4.9. *A field R is algebraically closed if any univariate polynomial over R has a root in R.*

Theorem 4.10. *For any field R if a field C is an algebraic extension of R and every polynomial $R[t]$ has a root in C, then C is algebraically closed.*

This characterization of extensions is classic for field theory, for proof you can look up [Isaacs, 1980].

Definition 4.11. *A real field R is real closed if its algebraic extension $R[\sqrt{-1}] = R[X]/(X^2+1)$ is proper and algebraically closed (or when R satisfies any of the equivalent conditions of Theorem 4.13)*

Note $R(a) = R[a]$ for algebraic extension of field R.

Remark 4.12. The field $\mathbb{R}$ is a real closed field.

Theorem 4.13 (Artin-Schreier Theorem for real closed fields)**.** *Let R be a field. Following conditions are equivalent:*

(1) R is real closed i.e. its algebraic extension $R[\sqrt{-1}]$ is proper and algebraically closed.
(2) R is real and has no (proper) algebraic extension which is real
(3) the positive cone of R is the squares R^2 and any odd-degree polynomial has a root in R

Proof. In the proof we will use a following remark

- *For a field $R \neq R[\sqrt{-1}]$ we have*

$$\left(R = R^2 \cup -R^2 \ \wedge \ R^2 = \sum R^2\right) \iff R[\sqrt{-1}] = \left(R(\sqrt{-1})\right)^2$$

Indeed, assume $R = R^2 \cup -R^2$ and $R^2 = \sum R^2$. Take any $a + \sqrt{-1}b$ with $a, b \in R$. The discriminant of $f = 4X^2 - 4aX - b^2$ is $(4a)^2 + (4b)^2 \in R^2$, hence a root c of f lies in R. Since $R = R^2 \cup -R^2$, we get $c = \alpha^2$ or $c = (\sqrt{-1}\alpha)^2$. Put $x = \alpha$ and $y = \frac{b}{2\alpha}$ in first case, or $x = \sqrt{-1}\alpha$ and $y = \frac{b}{\sqrt{-1}\alpha}$ otherwise. Then $x + \sqrt{-1}y \in R[\sqrt{-1}]$ and $(x + \sqrt{-1}y)^2 = a + \sqrt{-1}b$.

Assume $R[\sqrt{-1}] = (R[\sqrt{-1}])^2$. Take $a \in R$. There is $b + \sqrt{-1}c$, $b, c \in R$, such that $a = (b + \sqrt{-1}c)^2 = b^2 - c^2 + 2\sqrt{-1}bc$. Hence $b = 0$ or $c = 0$ and $a = -c^2$ or $a = b^2$ respectively. This proves $R = R^2 \cup -R^2$. To prove $R^2 = \sum R^2$ it suffices to show $a^2 + b^2$ is a square for some $a, b \in R$. Take $c, d \in R$ such that $a + \sqrt{-1}b = (c + \sqrt{-1}d)^2$. Then $a = c^2 - d^2$, $b = 2cd$ and $a^2 + b^2 = (c^2 + d^2)^2$. This ends proof of the remark.

(1)$\Rightarrow$(2) Since $R[\sqrt{-1}]$ is a proper algebraic closure of R, in particular we have $\sqrt{-1} \notin R$ and $R[\sqrt{-1}] = (R[\sqrt{-1}])^2$. Hence $R = R^2 \cup -R^2$, $R^2 = \sum R^2$ and $R^2 \cap -R^2 = \{0\}$. Therefore R has a positive cone, hence is real.

Any proper algebraic extension of R contains an element $a + \sqrt{-1}b \in R[\sqrt{-1}] \setminus R$. Since $b \neq 0$ we have $R[a + \sqrt{-1}b]$ equals

$$R[X] \ / \ (x^2 - 2ax + a^2 + b^2),$$

thus $a - \sqrt{-1}b \in R[a + \sqrt{-1}b]$. Hence $(a + \sqrt{-1}b - (a - \sqrt{-1}b))/2b = \sqrt{-1}$ and $R[a + \sqrt{-1}b] = R[\sqrt{-1}]$. Hence any proper algebraic extension of R is algebraically closed. Algebraically closed field is never real.

(2)$\Rightarrow$(3) Suppose $a \in R \setminus R^2$. Then $R[\sqrt{a}]$ is an algebraic extension of R, by assumption it is not real. Hence

$$-1 = \sum_j (b_j + c_j\sqrt{a})^2 = \sum b_j^2 + a\sum c_j^2 + \sqrt{a}\sum 2b_jc_j.$$

Therefore $\sum 2b_jc_j = 0$ and $a = -(1^2 + \sum b_j^2)/\sum c_j^2$. Hence $a \leq 0$. Thus every positive element is a square.

Now we need to show every odd-degree polynomial has a root in R. Any polynomial of degree 1 is linear and has a root in R. Assume all odd-degree polynomials of degree $< d$ have a root in R. Let $f \in R[X]$ be of odd degree d and suppose f does not have a root in R. If f was reducible, then one of the factors would be an odd-degree polynomial of degree lower than f, hence f would have a root in R. Therefore f is irreducible over R. Then $R[f] = R[X]/(f)$ is an algebraic extension of R. By assumption the field of fractions is not real. Therefore there exist $g_1, \ldots, g_k$ of degrees $< d$ such that

$$-1 = \sum (g_j + (f))^2 = \sum g_j^2 + (f).$$

Note that $\deg \left(\sum g_j^2\right) \leq 2d - 2$ and is even (because the leading coefficient is a sum of squares in the real field R, thus it does not vanish). Hence $-1 = \sum g_j^2 + fh$ for some h of odd degree $\leq d-2$. By inductive assumption, h has a root a in R. We get $-1 = \sum g_j^2(a) + f(a)h(a) = \sum (g_j(a))^2$, so $-1 \in \sum R^2$. Contradiction.

(3)⇒(1) Under assumption (3) we have $-1 \notin R^2$, hence $R[\sqrt{-1}] \neq R$.

We will show any polynomial over R of degree $d = 2^m n$, n odd, has a root in $R[\sqrt{-1}]$ by induction on m. When $m = 0$ we get the claim from assumption (3). Assume for any $m' < m$ the assumption holds. Consider polynomial f of degree $d = 2^m n$. Let $a_1, \ldots, a_d$ be roots of f in the algebraic closure of R. For $N \in \mathbb{N}$ put

$$g_N(X) = \Pi_{i<j}(X - a_i - a_j - N a_i a_j).$$

The polynomial g_N is of degree $d(d-1)/2 = 2^{m-1}(2^m n - 1)$ and it is symmetric in a_j, the roots of f. From fundamental theorem of symmetric polynomials, see for instance [Macdonald, 1979], we get that coefficients of g_N can be expressed in terms of coefficients of f, hence $g_N \in R[X]$. From inductive assumption every g_N has a root in $R[\sqrt{-1}]$. Hence there exist i, j and $N, N' \in \mathbb{N}$, $c, c' \in R[\sqrt{-1}]$ such that $a_i + a_j + N a_i a_j = c = c' + (N - N')a_i a_j$. Therefore $a_i a_j$ and $a_i + a_j$ are elements of $R[\sqrt{-1}]$.

We have $(X - a_i)(X - a_j) = X^2 - (a_i + a_j)X + a_i a_j$ is a quadratic polynomial over $R[\sqrt{-1}]$ with roots a_i, a_j and its discriminant is $(a_i + a_j)^2 - 4a_i a_j = (a_i - a_j)^2$. Since $R = R^2 \cup -R^2$, then $R[\sqrt{-1}] = (R[\sqrt{-1}])^2$. Hence exists $c \in R[\sqrt{-1}]$ such that $c^2 = (a_i - a_j)^2$. Therefore from formulæ for solving quadratic equations we get a_i or $a_j \in R[\sqrt{-1}]$ and f has a root in $R[\sqrt{-1}]$. This ends the inductive proof. □

Definition 4.14. *We say that a real field $\overline{R}$ is an extension of an ordered ring R if R embeds into $\overline{R}$ with its ring operations and ordering.*

Theorem 4.15. *Every real field has a (unique) minimal extension to a real closed field.*

Proof. Note that algebraically closed field is not a real field, because -1 is a square.

Take a real field R with ordering $\leq$ and its algebraic closure C. Consider any chain $(R_\alpha, \leq_\alpha)$ of algebraic extensions of R (contained in C) with consistent orderings. The field $\bigcup R_\alpha$ is an algebraic extension of R (because it is contained in the algebraic closure). Moreover, if $a_1^2 + \cdots + a_k^2 = 0$ for $a_1, \ldots, a_k \in \bigcup R_\alpha$, we get $a_1, \ldots, a_k \in R_\alpha$ for some α. Since R_α is real, then $a_1 = \cdots = a_k = 0$ and $\bigcup R_\alpha$ is real. Hence by Kuratowski-Zorn Lemma there exists a maximal real field $\overline{R} \subset C$ that is an algebraic extension of R with consistent ordering. The only algebraic extension of $\overline{R}$ is C and C is not real. Hence $\overline{R}$ is a real closed field. Obviously, if $R \subset R' \subset \overline{R}$ and R' is real closed, then $R' = \overline{R}$. □

Uniqueness in the theorem is up to an order-preserving isomorphism. For instance, one can define infinitely many orderings in $\mathbb{R}(t)$ and some of them have non-isomorphic extensions if we ask the isomorphism to respect the order.

5. Tarski's Transfer Principle

Definition 5.1. *We say that a formula is a boolean combination in variables $X_1, \ldots, X_n$ over an ordered ring R if it is a (syntax correct) finite combination of formulas of the form $f(X_1, \ldots, X_n) \geq 0$ with $f \in R[X_1, \ldots, X_n]$ and the logic operators $\vee, \wedge$ and $\neg$.*

Note that a polynomial is a finite (syntax correct) combination of elements of the field, variables $X_1, \ldots, X_n$, addition and multiplication.

Definition 5.2. *A first order formula over an ordered ring is a (syntax correct) finite combination of $\wedge, \vee, \neg$, boolean combinations over the ordered field and quantifiers $\forall, \exists$. The variables which are not under range of any of the quantifiers are called free variables (and the formula is in fact a sentential function in the free variables).*

The two definitions above are far from precise, for more exact formulation see [Robinson, 1963, Chapter VIII].

For instance $\Phi(X, Y) :\ X^2 + 2Y^2 \leq 0 \Rightarrow Y = 0$ is a boolean combination with free variables X, Y. Then $\Phi_1(Y) :\ \exists_x\ \Phi(x, Y)$ is a first order formula with free variable Y and $\Phi_2 :\ \forall_y\ \Phi_1(x, y)$ is also a first order formula without free variables, Φ_2 is a true statement. The formula $\psi(X) :\ \exists_y \sum_{j=1}^{\infty} X^j < y$ is not a first order formula.

We treat a first order formula Φ over R as a formula over an extension R_1 of R by taking the range in the quantifiers as R_1.

Remark 5.3. Formulas without free variables are either true or false.

We will now state and leave without proof the Tarski's Quantifier Elimination Theorem known in real algebraic geometry as Tarski-Seidenberg Theorem, see [Bochnak et al., 1998], [Tarski, 1951] or [Robinson, 1963] for different presentations of its proof.

Theorem 5.4 (Tarski-Seidenberg Theorem)**.** *Let R be an ordered ring. Let $b(X_0, X_1, \ldots, X_n)$ be a boolean combination. There exists a boolean combination $B(X_1, \ldots, X_n)$ such that for any real closed field R_1 extending R we have*

$$\{x \in R_1^n : \exists_{x_0 \in R_1} \ b(x_0, x)\} = \{x \in R_1^n : B(x)\}$$

i.e. the projection of a semialgebraic set is semialgebraic.

This is equivalent to the following

Theorem 5.5 (Quantifier Elimination)**.** *Let R be an ordered ring. For every first order formula $\Phi(X)$ over R there exists a boolean combination $B(X)$ over R such that for any real closed field R_1 extending R we have*

$$\forall_{x \in R_1} \big(\Phi(x) \iff B(x)\big).$$

It is important to note that quantifier elimination holds in the class of algebraically closed fields for constructible sets (see discussion of Lefschetz Principle and Minor Lefschetz Principle in [Seidenberg, 1958] or [Eklof, 1973]).

Now we can prove Tarski's transfer principle

Theorem 5.6 (Tarski's Transfer Principle)**.** *Let R be an ordered ring. Let R_1, R_2 be real closed extensions of R and $B(X_1, \ldots, X_n)$ a boolean combination over R. Then*

$$\exists_{x \in R_1^n} \ B(x) \iff \exists_{x \in R_2^n} \ B(x)$$

Tarski's Transfer Principle can be equivalently stated as follows: theory of real closed fields is model-complete.

Proof. Note that since R is ordered, it is an integral domain and by Proposition 4.8 and Theorem 4.15, there exist real closed extensions of R.

Take $B(X_1, \ldots, X_n)$ a boolean combination over R. By Tarski-Seidenberg Theorem and finite induction we can eliminate the quantifier in the formula $\exists_{x_1,\ldots,x_n} B(x_1, \ldots, x_n)$ i.e. there exists a boolean combination $\tilde{B}$ such that for any real closed extension R_1 of R we have

$$\exists_{x \in R_1^n} \ B(x) \iff \forall_{y \in R_1} \exists_{x \in R_1^n} \ B(x) \iff \forall_{y \in R_1} \ \tilde{B} \iff \tilde{B}.$$

The formula $\tilde{B}$ does not have free variables, therefore it is either true or false. Due to Tarski's Quantifier Elimination it has uniform logical value over all real closed fields extending R, in particular over R_1 and R_2. $\square$

6. Artin's solution of Hilbert's 17th Problem

Following theorems are not necessary for the proof of RN, but of interest partly because Artin-Schreier Theory was developed to answer the following question:

Hilbert's 17th Problem Is every positive polynomial a sum of squares of rational functions?

In fact, the problem dates back to Minkowsky. Moreover, Hilbert considered mainly polynomials with rational coefficients. Hilbert already proved that there exist polynomials positive on $\mathbb{R}^n$ such that they are not sums of squares of polynomials. On the other hand, all nonnegative polynomials of degree d in n variables are sums of squares of polynomials if and only if $d \leq 2$ or $n = 1$ or $d = 4$ and $n = 2$ (see for instance [Bochnak et al., 1998, Section 6.3]).

Theorem 6.1 (Solution to Hilbert's 17th Problem)**.** *Let R be a real closed field and Q its subfield with the positive cone $P = Q \cap R^2$. Take $f \in Q[X_1, \dots, X_n]$ which is nonnegative i.e.*

$$\forall_{x \in R^n}\ f(x) \geq 0.$$

Then

$$f \in \sum P \cdot (Q(X))^2$$

i.e. $f(X) = \sum a_j q_j^2(X)$ *with* $a_j \in P$ *and* $q_j \in (Q(X))^2$.

Proof. Take $f \in Q[X_1, \dots, X_n]$ nonnegative and suppose $f \notin \sum P \cdot (Q(X))^2$. Hence either $-f \in \sum P \cdot (Q(X))^2$ or not. In both cases, we can extend the proper cone $\sum P \cdot (Q(X))^2$ to a positive cone P' of $Q(X_1, \dots, X_n)$ such that $-f \in P'$.

Write $f = \sum a_\alpha X^\alpha$ with $a_\alpha \in Q$. Consider the first order variable-free formula Φ with coefficients in Q of the form

$$\Phi : \quad \exists_{x_1, \dots, x_n} \sum a_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n} < 0.$$

Note that Φ is equivalent to

$$\exists_{x_1, \dots, x_n} f(x_1, \dots, x_n) < 0$$

From the choice of ordering of $Q(X)$, the statement Φ is true over the real closure of $Q(X)$. By Tarski's Transfer Principle, Φ is also true over R. Therefore, there exists $x \in R^n$ such that $f(x) < 0$ which is against nonnegativity of f. □

In particular the above theorem is the desired solution to Hilbert's problem: every nonnegative real polynomial is a sum of squares of real rational functions ($R = Q = \mathbb{R}$). Moreover, every polynomial with rational coefficients is a sum of squares of functions in $\mathbb{Q}(X)$ ($R = \mathbb{R}, Q = \mathbb{Q}$).

In the original solution of Hilbert's problem by Artin an important tool was:

Theorem 6.2 (Artin-Lang Homomorphism Theorem)**.** *Let $R \subset R_1$ be real closed fields and A a finitely generated R-algebra. If there is a homomorphism $\phi_1 : A \to R_1$, then there exists a homomorphism $\phi : A \to R$.*

Proof. We may assume $A = R[X_1, \dots, X_n]/I$ by Proposition 3.12. Take a homomorphism $\phi_1 : A \to R_1$ and put $y = (\phi_1(X_1), \dots, \phi_1(X_n)) \in R_1^n$. Since

$R[X_1, \dots, X_n]$ is noetherian, consider finitely many generators $f_1, \dots, f_k$ of I. For any polynomial $f = \sum a_\alpha X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ we have

$$\phi(f+I) = \phi\left(\sum a_\alpha (X_1+I)^{\alpha_1} \cdots (X_n+I)^{\alpha_n}\right) = \\ = \sum a_\alpha \phi(X_1+I)^{\alpha_1} \cdots \phi(X_n+I)^{\alpha_n} = f(y).$$

Therefore $f_1(y) = \cdots = f_k(y) = 0$. By Tarski's Transfer Principle we get there exists $x \in R$ such that $f_1(x) = \cdots = f_k(x) = 0$. Now we see the homomorphism $\phi : A \to R$ given by assignment $X_i \to x_i$ is well-defined. □

7. Proof of Real Nullstellensatz

In this section we prove Real Nullstellensatz. Careful reader may note that in previous sections we worked with real fields and in particular $\mathbb{R}$ is real. Hence we will prove the following more general statement to be true:

Let R be a real closed field and $I \lhd R[X_1, \dots, X_n]$. We have

$$I = \mathcal{I}(V(I)) \iff I \text{ is real}$$

7.1. **Proof of RN in easy direction.** Note that for any arbitrary set $V \subset R^n$, the ideal $\mathcal{I}(V)$ is real. Assume $I = \mathcal{I}(V)$. Take $a_1^2 + \cdots + a_k^2 \in I$. Hence $a_1^2(x) + \cdots + a_k^2(x) = 0$ at every point $x \in V$. Therefore, $a_1 = \cdots = a_k \equiv 0$ on V. Hence $a_1, \dots, a_k \in \mathcal{I}(V) = I$ and I is real. This holds in particular when $I = \mathcal{I}(V(I))$.

7.2. **Proof of RN for prime ideals.** Take a prime real ideal $I \subsetneq R[X]$. To prove RN it suffices to show that $I \supset \mathcal{I}(V(I))$.

Take $f \notin I$ and denote $g_1, \dots, g_k$ the generators of I. Due to Proposition 4.8 and Theorem 4.15 we can take R_1, the real closure of the real field $\mathrm{Quot}(R/I)$. Naturally R embeds into R_1, see Property 3.4, and one can check the natural embedding preserves the order. Note that 0 in R_1 is the image of I.

Consider elements $y_1 = X_1 + I, \dots, y_n = X_n + I$ of R_1 and the boolean combination

$$B(Y_1, \dots, Y_n): \; g_1(Y) = \cdots = g_k(Y) = 0 \wedge f(Y) \neq 0$$

defined over R.

Since f is polynomial i.e. $f = \sum a_\alpha X^\alpha$ a finite sum, we get

$$f(y) = \sum a_\alpha y^\alpha = \sum a_\alpha (X_1+I)^{\alpha_1} \cdots (X_n+I)^{\alpha_n} = \left(\sum a_\alpha X^\alpha\right) + I = f + I$$

Hence $f(y) = f + I \neq I = 0$ since $f \notin I$.

Analogously we show $g_1(y) = \cdots = g_k(y) = 0$.

The fields R_1 and R are both real closed fields over R. Therefore, from Tarski's Transfer Principle we get

$$\exists_{y \in R_1^n} \; B(y) \;\Rightarrow\; \exists_{x \in R^n} \; B(x).$$

Since the left-hand is true, there exists $x \in R^n$ such that $g_1(x) = \cdots = g_k(x) = 0$ and $f(x) \neq 0$. Hence $f(x) \neq 0$ for $x \in V(I)$. Therefore, $f \notin \mathcal{I}(V(I))$ and this ends the proof.

7.3. **Proof of RN for any ideals.** We will show that if RN is true for prime ideals, then it is true for any ideal.

Assume the left implication of RN holds for real prime ideals. Take any real ideal I. Hence I is radical and from prime decomposition of Corollary 3.9 we have

$$I = \bigcap_{i=1,\ldots,r} p_i \tag{2}$$

where p_i are minimal prime ideals and from Proposition 3.11 follows that the ideals p_i are real.

Hence p_l in equality (2) are real. Since we have RN is true for prime ideals and defining ideal of union of sets is equal to intersection of defining ideals of the sets, we get

$$\begin{aligned}\mathcal{I}(V(I)) = \mathcal{I}\left(V(\bigcap_{l=1\ldots,r} p_l)\right) = \mathcal{I}\left(\bigcup_{l=1\ldots,r} V(p_l)\right) = \\ = \bigcap_{l=1\ldots,r} \mathcal{I}(V(p_l)) = \bigcap_{l=1\ldots,r} p_l = I.\end{aligned}$$

This ends the proof. □

Acknowledgements

This note is based on a part of the course Tópicos de Singularidades – Real Algebraic Geometry for postgraduate students at Universidade Federal de Ceará. The author would like to thank the participating students André Costa, Emanoel Ferreira, Felipe Fernandes and Sergio Alvarez for their diligence and valuable feedback.

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 137–146
DOI: https://doi.org/10.18778/8331-092-3.11

SOME NOTES ON THE LÊ NUMBERS IN THE FAMILY OF LINE SINGULARITIES

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Abstract. In this paper we introduce the jumps of the Lê numbers of non-isolated singularity f in the family of line deformations. Moreover, we prove the existence of a deformation of a non-degenerate singularity f such that the first Lê number is constant and the zeroth Lê number jumps down to zero. We also give estimations of the Lê numbers when the critical locus is one-dimensional. These give a version of the celebrated theorem of A. G. Kouchnirenko in this case.

1. Introduction

The most important topological invariant associated with a complex analytic function f with an isolated singularity at 0, is its Milnor number at 0. It is well known that this invariant is upper-semicontinuous in the family of singularities. Therefore it allows to define the jump of the Milnor number as the minimum non-zero difference $\mu(f) - \mu((f_t))$, where (f_t) is a deformation of f. S. Guzein-Zade [6] and A. Bodin [1] began the research devoted to this notion. In the papers [2,7,8,17] authors computed the jump of the Milnor number in the different classes deformations.

If f has a non-isolated singularity at 0, the Milnor number can not be defined. But there exist some numbers called Lê numbers, which play a similar role to the Milnor number in the isolated case. These numbers were defined by D. Massey (see [13–15]). Roughly speaking they describe a handle decomposition of the Milnor fibre (see [15, Theorem 3.3]). We recall that families with constant Lê numbers satisfy remarkable properties. For example, in [14], Massey proved that under appropriate conditions the diffeomorphism type of the Milnor fibrations associated

2010 *Mathematics Subject Classification.* 32S25, 14J17, 14J70.
Key words and phrases. Jump of Lê numbers, Non-isolated hypersurface singularity, Lê numbers, Newton diagram, Modified Newton numbers, Iomdine-Lê-Massey formula.

with the members of such family is constant. In [5], J. Fernández de Bobadilla showed that in the special case of families of 1-dimensional singularities, the constancy of Lê numbers implies the topological triviality of the family at least if $n \geq 5$.

Analogously as the Milnor number, the tuple of the Lê numbers has upper-semicontinuity property in the lexicographical order. Therefore, it is possible to distinguish two types of jumps. The first is the jump up of the tuple of the Lê numbers and the second is the jump up of the Lê number $\lambda^d_{f,z}(0)$, where d is a dimension of the critical locus.

In general, the Lê numbers are not topological invariants. However, it turns out that in the family of aligned singularities they are topological invariants (see [15, Corollary 7.8]). In the paper we focus our attention on the class of line singularities (see definition 5.1). It is the simplest class of aligned singularities. In the paper we consider deformations mainly in this class. Our main theorem (Theorem 5.3) guarantees the existence of a deformation (f_t) of a non-degenerate singularity $f = f_0$ with $\lambda^0_{f_0,z}(0) > 0$, such that $\lambda^0_{(f_t),z}(0) = 0$ and $\lambda^1_{(f_t),z}(0) = \lambda^1_{f,z}(0)$. In terms of a handle decomposition of the Milnor fibre it means that handles of the highest dimension disappear and others remains unchanged (see Remark 5.4).

Using Theorem 5.3 we introduce the minimal jump of the tuple of Lê numbers. In this class we can interpret the jump of the tuple of Lê numbers as a measure of "nearness" of the cycles (see Remark 5.8). Moreover, we show the interesting fact that there exists f such that the minimal jump of $\lambda^1_{f,z}(0)$ is greater then one (see Proposition 5.11). What is surprising, in the class of line singularities $\lambda^0_{f,z}(0) \neq 1$ (see Proposition 5.9). From this fact and Example 5.10 it follows that the "minimal jump" of $\lambda^0_{f,z}(0)$ is greater then one.

In the last section we give estimations of Lê numbers in terms of the Newton diagram when the critical locus is one-dimensional (see Theorem 6.1). This is a generalization of the Kouchnirenko theorem in this case.

2. Preliminary

Lê numbers are intersection multiplicity of certain analytic cycles — so-called *Lê cycles* — with certain affine subspaces. The Lê cycles are defined using the notion of gap sheaf. In this section, we briefly recall these definitions which are essential for the paper. We follow the presentation given by Massey in [13–15].

2.1. **Gap sheaves.** Let $(X, \mathscr{O}_X)$ be a complex analytic space, $W \subseteq X$ be an analytic subset of X, and $\mathscr{I}$ be a coherent sheaf of ideals in $\mathscr{O}_X$. As usual, we denote by $V(\mathscr{I})$ the analytic space defined by the vanishing of $\mathscr{I}$. At each point $x \in V(\mathscr{I})$, we want to consider scheme-theoretically those components of $V(\mathscr{I})$ which are not contained in W. For this purpose, we look at a minimal primary decomposition of the stalk $\mathscr{I}_x$ of $\mathscr{I}$ in the local ring $\mathscr{O}_{X,x}$, and we consider the ideal $\mathscr{I}_x \neg W$ in $\mathscr{O}_{X,x}$ consisting of the intersection of those (possibly embedded) primary components Q of $\mathscr{I}_x$ such that $V(Q) \not\subseteq W$. This definition does not depend on the choice of the minimal primary decomposition of $\mathscr{I}_x$. Now, if we perform the operation described above at the point x simultaneously at all points of $V(\mathscr{I})$, then we obtain a coherent sheaf of ideals called a *gap sheaf* denoted by $\mathscr{I}\neg W$. Hereafter, we shall denote by $V(\mathscr{I})\neg W$ the scheme (i.e., the complex analytic space) $V(\mathscr{I}\neg W)$ defined by the vanishing of the gap sheaf $\mathscr{I}\neg W$.

2.2. **Lê cycles and Lê numbers.** Let $n \geq 2$. Consider an analytic function $f\colon (U,0) \to (\mathbb{C},0)$, where U is an open neighbourhood of 0 in $\mathbb{C}^n$, and fix a system of linear coordinates $z = (z_1, \dots, z_n)$ for $\mathbb{C}^n$. Let Σf be the critical locus of f. For $0 \leq k \leq n-1$, the kth (relative) *polar variety* of f with respect to the coordinates z is the scheme

$$\Gamma^k_{f,z} := V\left(\frac{\partial f}{\partial z_{k+1}}, \dots, \frac{\partial f}{\partial z_n}\right)\neg \Sigma f.$$

The analytic cycle

$$[\Lambda^k_{f,z}] := \left[\Gamma^{k+1}_{f,z} \cap V\left(\frac{\partial f}{\partial z_{k+1}}\right)\right] - \left[\Gamma^k_{f,z}\right]$$

is called the kth *Lê cycle* of f with respect to the coordinates z. (We always use brackets $[\cdot]$ to denote analytic cycles.) The kth *Lê number* $\lambda^k_{f,z}(0)$ of f at $0 \in \mathbb{C}^n$ with respect to the coordinates z is defined to be the intersection number

$$\lambda^k_{f,z}(0) := \big([\Lambda^k_{f,z}] \cdot [V(z_1, \dots, z_k)]\big)_0 \tag{2.1}$$

provided that this intersection is 0-dimensional or empty at 0; otherwise, we say that $\lambda^k_{f,z}(0)$ is *undefined*.[1] For $k = 0$, the relation (2.1) means

$$\lambda^0_{f,z}(0) = \big([\Lambda^0_{f,z}] \cdot U\big)_0 = \left[\Gamma^1_{f,z} \cap V\left(\frac{\partial f}{\partial z_1}\right)\right]_0.$$

For any $\dim_0 \Sigma f < k \leq n-1$, the Lê number $\lambda^k_{f,z}(0)$ is always defined and equal to zero. For this reason, we usually only consider the Lê numbers

$$\lambda^{\dim_0 \Sigma f}_{f,z}(0), \dots, \lambda^0_{f,z}(0),$$

and we denote this tuple by $\lambda_{f,z}(0)$. Note that if 0 is an *isolated* singularity of f, then $\lambda^0_{f,z}(0)$ (which is the only possible non-zero Lê number) is equal to the Milnor number $\mu_f(0)$ of f at 0.

Now, we introduce the cycle of the critical locus (see [15, Proposition 1.15]). Let $d = \dim_0 \Sigma f$. We define

$$[\Sigma f] = \sum_{i=0}^{d} \lambda^i_{f,z}(0) |[\Lambda^i_{f,z}]|. \tag{2.2}$$

[1] As usual, $[V(z_1, \dots, z_k)]$ denotes the analytic cycle associated to the analytic space defined by $z_1 = \cdots = z_k = 0$. The notation $\big([\Lambda^k_{f,z}] \cdot [V(z_1, \dots, z_k)]\big)_0$ stands for the intersection multiplicity at 0 of the analytic cycles $[\Lambda^k_{f,z}]$ and $[V(z_1, \dots, z_k)]$.

3. Lê numbers of a deformation

Let $f\colon (U,0)\to(\mathbb{C},0)$ be an analytic function , where U is an open neighbourhood of 0 in $\mathbb{C}^n$, and fix a system of linear coordinates $z=(z_1,\dots,z_n)$ for $\mathbb{C}^n$.

A deformation of f is an analytic function

$$F\colon (D\times U, D\times\{0\})\to(\mathbb{C},0),$$

where D is an open neighbourhood of the origin in $\mathbb{C}$, such that $F(0,z)=f(z)$ for any $z\in\mathbb{C}^n$. We will shortly write $f_t(z):=F(t,z)$, $(f_t):=F$.

Assume that $d=\dim_0\Sigma f\geq 1$ and the Lê numbers $\lambda^k_{f_t,z}(0)$ are defined for all $k\leq d$ and all t sufficiently small.

Theorem 3.1. *(Uniform Iomdine-Lê-Massey formula, [15, Theorem 4.15]) For sufficiently large integer j and any sufficiently small complex number t, we have the following properties:*

1. $\Sigma(f_t+z_1^j)=\Sigma f_t\cap V(z_1)$ *in a neighbourhood of the origin;*
2. $\dim_0\Sigma(f+z_1^j)=d-1$*;*
3. *the Lê numbers $\lambda^k_{f_t+z_1^j,\tilde{z}}(0)$ exist for all $0\leq k\leq d-1$ and*

$$\lambda^0_{f_t+z_1^j,\tilde{z}}(0)=\lambda^0_{f_t,z}(0)+(j-1)\lambda^1_{f_t,z}(0); \tag{3.1}$$

$$\lambda^k_{f_t+z_1^j,\tilde{z}}(0)=(j-1)\lambda^{k+1}_{f_t,z}(0) \quad \textit{for} \quad 1\leq k\leq d-1; \tag{3.2}$$

where $\lambda^k_{f_t+z_1^j,\tilde{z}}(0)$ is the kth Lê number of $f_t+z_1^j$ at 0 with respect to the rotated coordinates $\tilde{z}=(z_2,\dots,z_n,z_1)$.

Now, we define the Lê numbers of a deformation F. For this reason we will prove the following.

Proposition 3.2. *The numbers $\lambda^k_{f_t,z}(0)$, $k\leq d$ are independent of small $t\neq 0$.*

Proof. By Uniform Iomdine-Lê-Massey formula inductively we get that for $0\ll j_1\ll\cdots\ll j_d$ and small t,

$$f_{t,d}:=f_t+z_1^{j_1}+\cdots+z_d^{j_d}$$

has an isolated singularity at the origin. By upper-semicontinuity of Milnor number we have the number $\mu(f_{t,d})$ is constant for small $t\neq 0$. By (3.1) we obtain that the number

$$\lambda^1_{f_{t,d-1}}(0)=\mu(f_{t,d+1})-\mu(f_{t,d}) \tag{3.3}$$

is also constant for small $t\neq 0$. Now, by (3.3) and (3.1) we get that

$$\lambda^0_{f_{t,d-1}}(0)=\mu(f_{t,d})-(j_d-1)\lambda^1_{f_{t,d-1}}(0)$$

is also constant for small $t\neq 0$. In similar way, by induction and using (3.1) and (3.2) we finally get the assertion. □

Definition 3.3. By the Lê numbers of a deformation (f_t) we mean

$$\lambda^k_{(f_t),z}(0) := \lambda^k_{f_t,z}(0), \quad k \leq d,$$

for sufficiently small $t \neq 0$.

By Proposition 3.2 this definition is correct.

Like the Milnor number is upper-semicontinuous, the Lê numbers have also this property treated as tuple (see [15]). Precisely, we have the following.

Theorem 3.4. *(Upper-semicontinuity of Lê numbers, [15, Corollary 4.16]) The tuple of Lê numbers*

$$\left(\lambda^d_{f_t,z}(0), \ldots, \lambda^0_{f_t,z}(0)\right)$$

is lexicographically upper-semicontinuous in the t variable, i.e. for all sufficiently small $t \neq 0$, either

$$\lambda^d_{f,z}(0) > \lambda^d_{f_t,z}(0)$$

or

$$\lambda^d_{f,z}(0) = \lambda^d_{f_t,z}(0) \quad \text{and} \quad \lambda^{d-1}_{f,z}(0) > \lambda^{d-1}_{f_t,z}(0)$$

or

$$\vdots$$

or

$$\lambda^d_{f,z}(0) = \lambda^d_{f_t,z}(0), \ldots, \lambda^1_{f,z}(0) = \lambda^1_{f_t,z}(0) \quad \text{and} \quad \lambda^0_{f,z}(0) \geq \lambda^0_{f_t,z}(0).$$

In other words $\lambda_{(f_t),z}(0) \prec \lambda_{f,z}(0)$, where $\prec$ is the lexicographical order.

4. Jump of Lê numbers

Let $F = (f_t)$ be a deformation of f such that $\dim_0 \Sigma f_t = \dim_0 \Sigma f$ for sufficiently small t. By the above semicontinuity, we can consider the jump of Lê numbers of a deformation F in the lexicographical order.

Definition 4.1. By the *jump* $\delta_{F,z}(0)$ of a deformation F we mean

$$\lambda_{f,z}(0) - \lambda_{F,z}(0).$$

By the Theorem 3.4 and the fact that we can always deform f to be smooth, we have

$$\mathbf{0} \prec \delta_{F,z}(0) \prec \lambda_{f,z}(0).$$

Example 4.2. Let $f(x,y,z) = y^2 + z^3$. Then $\Sigma f = \{y = z = 0\}$. It easy to check that $\lambda_{f,z}(0) = (2,0)$. Taking the following sequence of deformations $f^k_t = f + tx^k z^2$, we obtain $\lambda_{f^k_t,z}(0) = (1, 3k-1)$. This shows that $\delta_{f^k_t,z}(0) = (1, 1-3k)$ can be arbitrary small.

5. Main theorem

Let $n \geq 2$ and $f\colon (U, 0) \to (\mathbb{C}, 0)$ be an analytic function , where U is an open neighbourhood of 0 in $\mathbb{C}^n$.

Definition 5.1. We say that f is a *line singularity* if Σf is Oz_1 i.e. $\Sigma f = \{z \in \mathbb{C}^n \colon z_2 = \cdots = z_n = 0\}$ and $f|_{V(z_1)}$ has an isolated singularity at the origin.

Let f be a line singularity and let $F = (f_t)$ be its deformation.

Definition 5.2. We say that (f_t) is a *family of line singularities (F is a line deformation of f)* if Σf_t is z_1-axis and $f_t|_{V(z_1)}$ has an isolated singularity at the origin for each t near $0 \in \mathbb{C}$.

Observe that in the Example 4.2, $\lambda^0_{f,z}(0) = 0$. In the case $\lambda^0_{f,z}(0) > 0$, we give the proof of the following theorem in the class of non-degenerate line singularities (see Appendix A). We believe that it is also true for all line singularities.

Theorem 5.3. *Let $f\colon (U, 0) \to (\mathbb{C}, 0)$ be a non-degenerate line singularity, where U is an open neighbourhood of* 0 *in $\mathbb{C}^n$. Assume that $z = (z_1, \ldots, z_n)$ be prepolar coordinates for f i.e. $f|_{z_1=0}$ has an isolated singularity at* 0*. If $\lambda^0_{f,z}(0) > 0$, then there exists a line deformation (f_t) such that $\lambda^0_{(f_t),z}(0) = 0$ and $\lambda^1_{(f_t),z}(0) = \lambda^1_{f,z}(0)$.*

Proof. Take
$$f_t(z_1, \ldots, z_n) = f(z_1 + t, z_2, \ldots, z_n).$$
Since $z = (z_1, \ldots, z_n)$ are prepolar coordinates for f, then $f|_{z_1=0}$ has an isolated singularity at 0. Since $(f_t)|_{z_1=0}$ is a deformation of $f|_{z_1=0}$, then $(f_t)|_{z_1=0}$ has an isolated singularity at 0. Therefore by [15, Remark 1.9] $(z_1, \ldots, z_n)$ are prepolar coordinates for (f_t) and $\lambda^0_{(f_t),z}(0)$, $\lambda^1_{(f_t),z}(0)$ exist. Since f and (f_t) are the line singularities, by [10, 11, 15] we have
$$\lambda^1_{f,z}(0) = \mu(f|_{z_1=\varepsilon}) = \mu(f|_{z_1=\varepsilon+t}) = \mu((f_t)|_{z_1=\varepsilon}) = \lambda^1_{(f_t),z}(0).$$
We will show $\lambda^0_{(f_t),z}(0) = 0$. Since f is non-degenerate (f_t) is also non-degenerate. Moreover
$$\Gamma((f_t)) = \Gamma((f_t)|_{z_1=0}).$$
To prove it we identify the monomials of (f_t) with associated points of $\operatorname{supp}(f_t)$. The monomials, which are vertices of $\Gamma((f_t))$ do not depend on variable z_1. Indeed, suppose to the contrary that a monomial $z_1^{\alpha_1} z_2^{\beta_2} \ldots z_n^{\beta_n}$ is a vertex of $\Gamma((f_t))$. Hence by the form of (f_t) monomial $z_2^{\beta_2} \ldots z_n^{\beta_n}$ is a point of $\operatorname{supp}(f_t)$. Take the hyperplane supporting $\Gamma_+((f_t))$ in $z_1^{\alpha_1} z_2^{\beta_2} \ldots z_n^{\beta_n}$. Then every point of $\operatorname{supp} f_t$ lies on this hyperplane or above. But the point $z_2^{\beta_2} \ldots z_n^{\beta_n}$ lies below it. This gives the contradiction. Therefore by [4] we have
$$\lambda^0_{(f_t),z}(0) = \lambda^0_{(f_t)|_{z_1=0},z}(0) = 0.$$
The last equality follows from the definition of Lê numbers and the fact that $((f_t)|_{z_1=0})'_{z_1} \equiv 0$. This gives the assertion. $\square$

Remark 5.4. Roughly speaking, the deformation in the main theorem "straightens" the line singularity along its critical locus.

Example 5.5. Let $f(x,y,z) = y^2 + z^3 + x^2z^2$. Then $\Sigma f = \{y = z = 0\}$. It is easy to check that $\lambda_{f,z}(0) = (1,5)$. Take the line deformation $f_t = f + tz^2$. We have $\lambda_{(f_t),z}(0) = (1,0)$. Hence $\delta_{(f_t),z}(0) = (0,5)$.

Let f be a line, non-degenerate singularity such that $\lambda^0_{f,z}(0) > 0$. By Theorem 5.3 we can correctly define the minimal jump of f as follows.

Definition 5.6. By the *minimal jump* $\delta_{f,z}(0)$ of a singularity f we mean

$$\min\{\delta_{F,z}(0)\colon F \text{ is a deformation of } f, \delta_{F,z}(0) \succ 0\},$$

where the above minimum is taken in the lexicographical order.

Definition 5.7. By the *minimal jump in the class of line deformation* $\delta^l_{f,z}(0)$ of a singularity f we mean

$$\min\{\delta_{F,z}(0)\colon F \text{ is a line deformation of } f, \delta_{F,z}(0) \succ 0\}.$$

Remark 5.8. By (2.2), when$f$ and (f_t) are line singularities we have

$$[\Sigma f] = \lambda^1_{f,z}(0)[Oz_1] + \lambda^0_{f,z}(0)[0],$$
$$[\Sigma f_t] = \lambda^1_{(f_t),z}(0)[Oz_1] + \lambda^0_{(f_t),z}(0)[0].$$

In this case one can interpret $\delta_{(f_t),z}(0)$ as a "nearness" of the above cycles.

Proposition 5.9. *Let f be a line singularity. Then*

$$\lambda^0_{f,z}(0) \neq 1.$$

Proof. Suppose to the contrary that $\lambda^0_{f,z}(0) = 1$. It means by definition that

$$\left([\Gamma^1_{f,z}] \cdot \left[V\left(\frac{\partial f}{\partial z_1}\right)\right]\right)_0 = 1. \tag{5.1}$$

Let $[\Gamma^1_{f,z}] = \sum_{i=1}^k a_i[\Upsilon^i]$, where Υ^i are irreducible components of $\Gamma^1_{f,z}$. By (5.1) we have

$$\sum_{i=1}^k a_i\left([\Upsilon^i] \cdot \left[V\left(\frac{\partial f}{\partial z_1}\right)\right]\right)_0 = 1.$$

Therefore $k = 1$, $\Gamma^1_{f,z}$ is irreducible. Let $\varphi\colon (\mathbb{C},0) \to (\mathbb{C}^n,0)$ be a parametrization of Υ^1. Hence

$$\operatorname{ord}\left(\frac{\partial f}{\partial z_1} \circ \varphi\right) = 1.$$

This implies that $\operatorname{ord} f'_{z_1} = 1$. Hence, for some i we have

$$f(z_1,\ldots,z_n) = az_1z_i + \ldots$$

$a \neq 0$. Then $f'_{z_i}(t,0,\ldots,0) \neq 0$. This and the assumption Σf is z_1-axis gives the contradiction. $\square$

Example 5.10. Let $f(x,y,z) = y^2 + z^3 + xz^2$. Then $\Sigma f = \{y = z = 0\}$. It is easy to check that $\lambda_{f,z}(0) = (1,2)$. Take deformations $f_t = f + tz^2$. Then $\Sigma f_t = \Sigma f$ and $\lambda_{(f_t),z}(0) = (1,0)$. By Proposition 5.9 $\delta^l_{f,z}(0) = (0,2)$.

Proposition 5.11. *There exists a singularity $f\colon (\mathbb{C}^3,0) \to (\mathbb{C},0)$ such that*

$$\min\{\lambda^1_{f,z}(0) - \lambda^1_{F,z}(0) > 0\colon F \textit{ is a line deformation of } f\} > 1.$$

Proof. Take

$$f(x,y,z) = y^4 + z^4 + y^2z^2.$$

We check that f is a line singularity and for sufficiently small $\varepsilon \neq 0$ [15, Remark 1.19]

$$\lambda^1_{f,z}(0) = \mu_0(f|_{x=\varepsilon}) = 9.$$

Let $F = (f_t)$ be a line deformation of f. By [15, Remark 1.19] and [2, Theorem 3.1] we have

$$\lambda^1_{(f_t),z}(0) = \mu_0((f_t)|_{x=\varepsilon}) \leq 7.$$

This ends the proof. □

6. Estimation of Lê numbers

Let $f\colon (U,0) \to (\mathbb{C},0)$ be a singularity, where U is an open neighbourhood of 0 in $\mathbb{C}^n$. Suppose that $z = (z_1,\dots,z_n)$ is prepolar coordinates for f and $\dim_0 \Sigma f = 1$.

Theorem 6.1.

$$\lambda_{f,z}(0) \succ (\widetilde{\nu}_1(f_1), (-1)^n + \nu_0(f_1) + \widetilde{\nu}_1(f_1)),$$
$$\lambda^1_{f,z}(0) \geq \widetilde{\nu}_1(f_1),$$

where $f_1 = f + z_1^\alpha$, α is sufficiently big and $\nu_0(f_1)$, $\widetilde{\nu}_1(f_1)$ are modified Newton numbers (see [4]). The equalities hold, if f is non-degenerate.

Proof. If f is non-degenerate, then the assertion follows from [4, Theorem 4.1]. Assume now that f is degenerate. Since the non-degeneracy is open condition (see [16, Appendix]) there exists a non-degenerate deformation (f_t) of f with the same Newton diagram. Since the modified Newton numbers depend only on the Newton diagram, modified Newton numbers of f and (f_t) are the same. Since $z = (z_1,\dots,z_n)$ is prepolar coordinates for f it is also prepolar for (f_t). By [15, Theorem 1.28] the Lê numbers of (f_t) exist. Hence by [4, Theorem 4.1] we have

$$\lambda_{(f_t),z}(0) = (\widetilde{\nu}_1(f_1), (-1)^n + \nu_0(f_1) + \widetilde{\nu}_1(f_1)),$$
$$\lambda^1_{(f_t),z}(0) = \widetilde{\nu}_1(f_1).$$

On the other hand, by the upper-semicontinuity of Lê numbers we get

$$\lambda_{f,z}(0) \succ \lambda_{(f_t),z}(0),$$
$$\lambda^1_{f,z}(0) \geq \lambda^1_{(f_t),z}(0).$$

Summing up, we get the assertion. □

Appendix A. Newton diagram

Here, the reference is Kouchnirenko [9].

Let $z := (z_1, \ldots, z_n)$ be a system of coordinates for $\mathbb{C}^n$, let U be an open neighbourhood of the origin in $\mathbb{C}^n$, and let

$$f\colon (U,0) \to (\mathbb{C},0), \quad z \mapsto f(z) = \sum_\alpha c_\alpha z^\alpha,$$

be an analytic function, where $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $c_\alpha \in \mathbb{C}$, and z^α is a notation for the monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

The *Newton polyhedron* $\Gamma_+(f)$ of f (at the origin and with respect to the coordinates $z = (z_1, \ldots, z_n)$) is the convex hull in $\mathbb{R}_+^n$ of the set

$$\bigcup_{c_\alpha \neq 0} (\alpha + \mathbb{R}_+^n).$$

For any $v \in \mathbb{R}_+^n \setminus \{0\}$, put

$$\begin{aligned}
&\ell(v, \Gamma_+(f)) := \min\{\langle v, \alpha\rangle \,;\, \alpha \in \Gamma_+(f)\},\\
&\Delta(v, \Gamma_+(f)) := \{\alpha \in \Gamma_+(f) \,;\, \langle v, \alpha\rangle = \ell(v, \Gamma_+(f))\},
\end{aligned}$$

where $\langle \cdot\,, \cdot\rangle$ denotes the standard inner product in $\mathbb{R}^n$. A subset $\Delta \subseteq \Gamma_+(f)$ is called a *face* of $\Gamma_+(f)$ if there exists $v \in \mathbb{R}_+^n \setminus \{0\}$ such that $\Delta = \Delta(v, \Gamma_+(f))$. The *dimension* of a face Δ of $\Gamma_+(f)$ is the minimum of the dimensions of the affine subspaces of $\mathbb{R}^n$ containing Δ. The *Newton diagram* (also called *Newton boundary*) of f is the union of the compact faces of $\Gamma_+(f)$. It is denoted by $\Gamma(f)$. We say that f is *convenient* if the intersection of $\Gamma(f)$ with each coordinate axis of $\mathbb{R}_+^n$ is non-empty (i.e., for any $1 \leq i \leq n$, the monomial $z_i^{\alpha_i}$, $\alpha_i \geq 1$, appears in the expression $\sum_\alpha c_\alpha z^\alpha$ with a non-zero coefficient).

For any face $\Delta \subseteq \Gamma(f)$, define the *face function* f_Δ by

$$f_\Delta(z) := \sum_{\alpha \in \Delta} c_\alpha z^\alpha.$$

We say that f is *Newton non-degenerate* (in short, non-degenerate) on the face Δ if the equations

$$\frac{\partial f_\Delta}{\partial z_1}(z) = \cdots = \frac{\partial f_\Delta}{\partial z_n}(z) = 0$$

have no common solution on $(\mathbb{C} \setminus \{0\})^n$. We say that f is *(Newton) non-degenerate* if it is non-degenerate on every face Δ of $\Gamma(f)$.

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 147–162
DOI: https://doi.org/10.18778/8331-092-3.12

LECTURES ON POLYNOMIAL EQUATIONS: MAX NOETHER'S FUNDAMENTAL THEOREM, THE JACOBI FORMULA AND BÉZOUT'S THEOREM

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In memory of Jacek Chądzyński

Streszczenie. Using some commutative algebra we prove Max Noether's Theorem, the Jacobi Formula and Bézout's Theorem for systems of polynomial equations defining transversal hypersurfaces without common points at infinity.

The classical theorems on polynomial equations: Max Noether's Fundamental Theorem, The Jacobi Formula and Bézout's Theorem were presented in nineteenth-century literature (see for example [La] and [Ne]) for polynomial equations with indeterminate coefficients. In this article we give the present-day version of these theorems. To prove Max Noether's Fundamental Theorem which is basic for our approach we use Hilbert's Nullstellensatz and the Cohen-Macauley property of parameters. An elementary proof of the Cohen-Macauley property is given in [Pł].

1. Introduction

Let K be an algebraically closed field (of arbitrary characteristic). For any polynomial $P = P(X) \in K[X]$ in n variables $X = (X_1, \ldots, X_n)$ we denote by $\deg P$ the total degree of P and by P^+ the principal part of P, i.e. the sum of all monomials of degree $\deg P$ appearing in P. By convention $\deg 0 = -\infty$, $0^+ = 0$.

2010 *Mathematics Subject Classification.* Primary 12XXX, Secondary 14H20.
Key words and phrases. polynomial equations; Cohen–Macaulay Property; Hilbert's Nullstellensatz.
The first version of this article was published in the Proceedings of XXXI Workshop on Analytic and Algebraic Complex Geometry.

Definition 1. *Let $F_i \in K[X]$, $1 \leqslant i \leqslant n$ be nonconstant polynomials in n variables $X = (X_1, \dots, X_n)$. The system of polynomial equations $F_1(X) = \dots = F_n(X) = 0$ is* general *if the following conditions hold*

(1) *the system of polynomial equations $F_1(X) = \dots = F_n(X) = 0$ has no solutions at infinity i.e. the system of homogeneous equations $F_1^+(X) = \dots = F_n^+(X) = 0$ has in K^n only the zero-solution $X = 0$;*
(2) *all solutions in K^n of the system $F_1(X) = \dots = F_n(X) = 0$ are simple i.e. the jacobian* $\det\left(\frac{\partial F_i}{\partial X_j}\right)$ *does not vanish on the solutions of this system.*

Let us consider some examples:

(1) The system of linear equations $a_{i1}X_1 + \dots + a_{in}X_n - b_i = 0$, $1 \leqslant i \leqslant n$ is general if and only if $\det(a_{ij}) \neq 0$.
(2) If $F_i = X_i^{d_i} + c_{i1}X_i^{d_i - 1} + \dots + c_{id_i} \in K[X_i]$, $1 \leqslant i \leqslant n$, are one-variable polynomials of degree $d_i > 0$ with simple roots then the system $F_1(X_1) = \dots = F_n(X_n) = 0$ is general.
(3) Let $s_i(X)$, $1 \leqslant i \leqslant n$ be symmetric polynomials defined by identity

$$(T - X_1) \cdots (T - X_n) = T^n + s_1(X)T^{n-1} + \dots + s_n(X)$$

i.e.

$$s_1(X) = -(X_1 + \dots + X_n), \cdots, \quad s_n(X) = (-1)^n X_1 \cdots X_n.$$

Let $D(s_1, \dots, s_n)$ be the discriminant of the polynomial $T^n + s_1T^{n-1} + \dots + s_n$ with general coefficients $s_1, \dots, s_n$. Recall that

$$D(s_1(X), \dots, s_n(X)) = \left(\det\left(\frac{\partial s_i(X)}{\partial X_j}\right)\right)^2 = \prod_{i=1, i>j}^{n} (x_i - x_j)^2$$

(see pages 150-151 of [Pe]).

It is easy to see that the system of polynomial equations $s_1(X) - a_1 = \dots = s_n(X) - a_n = 0$, where $a_i \in K$, is general if and only if $D(a_1, \dots, a_n) \neq 0$.

In the sequel we put $F = (F_1, \dots, F_n) \in K[X]^n$, $\operatorname{Jac} F = \det\left(\frac{\partial F_i(X)}{\partial X_j}\right)$ and $V(F) = \{x = (x_1, \dots, x_n) \in K^n \ : \ F_1(x) = \dots = F_n(x) = 0\}$. The system of polynomial equations $F_1(X) = \dots = F_n(X) = 0$ will be denoted $F = 0$.

Now we may formulate the three classical theorems mentioned in the title of these lectures.

Theorem 1 (Max Noether's Fundamental Theorem)**.** *Let $F = 0$ be a general system of polynomial equations. If a polynomial G vanishes on the set $V(F)$ then there exists polynomials $A_1, \dots, A_n \in K[X]$ such that*

$$G = \sum_{i=1}^{n} A_i F_i \quad \textit{and} \ \deg A_i F_i \leqslant \deg G \quad \textit{for } i \in \{1, \dots, n\}.$$

We will give the proof of Theorem 1 in Section 3 of these notes. Note that with the notations of Theorem 1 we have $\deg G = \max_{i=1}^{n}(\deg A_i F_i)$ since the inequality $\deg G \leqslant \max_{i=1}^{n}(\deg A_i F_i)$ is obvious. The following property is an immediate consequence of Max Noether's Theorem.

Corollary 1. *The solutions of the general system of polynomial equations* $F_1(X) = \cdots = F_n(X) = 0$ *do not lie on a hypersurface of degree strictly less than* $\min_{i=1}^{n}(\deg F_i)$. *Moreover the system* $F_1(X) = \cdots = F_n(X) = 0$ *has at least one solution in* K^n.

Proof. If the solutions of the system $F_1(X) = \cdots = F_n(X) = 0$ lie on the hypersurface $G(X) = 0$ then $\deg G = \max_{i=1}^{n}(\deg A_i F_i) \geqslant \min_{i=1}^{n}(\deg F_i)$. This proves the first assertion. To check the second assertion suppose that the system $F_1(X) = \cdots = F_n(X) = 0$ has no solutions in K^n. Taking $G = 1$ we get $\deg G \geqslant \min_{i=1}^{n}(\deg F_i) > 0$ by the first part of the corollary. Contradiction.

Using Max Noether's Fundamental Theorem we prove in Section 4

Theorem 2 (The Jacobi Formula)**.** *Let* $F = 0$ *be a general system of polynomial equations. Then the set* $V = V(F)$ *is finite and for every polynomial* $H \in K[X]$ *of degree* $\deg H < \sum_{i=1}^{n}(\deg F_i - 1)$ *one has*

$$\sum_{x \in V(F)} \frac{H(x)}{\operatorname{Jac} F(x)} = 0.$$

Note that if $n = 1$ then the Jacobi Formula follows easily from the Lagrange Interpolation Theorem: let $F(X) = (X - x_1) \cdots (X - x_d) \in K[X]$ be a univariate polynomial of degree $d > 1$ such that $x_i \neq x_j$ for $i \neq j$. Then

$$H(X) = \sum_{i=1}^{d} \frac{H(x_i)}{F'(x_i)} (X - x_1) \cdots (\widehat{X - x_i}) \cdots (X - x_d)$$

provided that $H(X)$ is a polynomial of degree strictly less than d.

The assumption on the degree of H cannot be weakened. If $\operatorname{char} K = 0$ then $H = \operatorname{Jac} F$ is of degree $\sum_{i=1}^{n}(\deg F_i - 1)$ and $\sum_{x \in V(F)} \frac{H(x)}{\operatorname{Jac} F(x)} = \sharp V(F) \neq 0$.

Corollary 2 (The Cayley-Bacharach Theorem). *If a polynomial H of degree strictly less than $\sum_{i=1}^{n}(\deg F_i - 1)$ vanishes on all points of $V = V(F)$ but one then it necessarily vanishes on V.*

The oldest result on general systems of polynomial equations is due to Étienne Bézout (Théorie générale des équations algébriques, Paris, 1770).

Theorem 3 (Bézout's Theorem). *Let $F = 0$ be a general system of polynomial equations. Then it has exactly $\prod_{i=1}^{n} \deg F_i$ solutions.*

We give the proof of Theorem 3 in Section 3. To prove Béout's Theorem we will use Max Noether's Fundamental Theorem and the Poincaré series (see Section 5).

2. Homogeneous systems of parameters

Let $\phi = (\phi_1, \ldots, \phi_n)$ be a sequence of homogeneous polynomials $\phi_i \in K[X]$, $X = (X_1, \ldots, X_n)$. Using Hilbert's Nullstellensatz we check

Lemma 1. *Let K be an algebraically closed field. Then the following conditions are equivalent:*

(1) *the system of homogeneous equations $\phi_1(X) = \cdots = \phi_n(X) = 0$ has in K^n only the zero-solution $X = 0$.*
(2) *there is an integer $N > 0$ such that all monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, $\alpha_1 + \cdots + \alpha_n = N$ belong to the ideal $I(\phi) = (\phi_1, \ldots, \phi_n)K[X]$ generated by $\phi_1, \ldots, \phi_n$ in $K[X]$.*

Now let K be an arbitrary field.

Definition 2. *The sequence of homogeneous forms $\phi = (\phi_1, \ldots, \phi_n) \in K[X]^n$ is a homogeneous system of parameters (h.s.o.p.) if the ideal generated by $\phi_1, \ldots, \phi_n$ in $K[X]$ contains all monomials of sufficiently high degree i.e. if it satisfies the second condition of the above lemma.*

The following result on h.s.o.p. is basic for us. For the proof see [St] (page 37, *The Cohen-Macauley property*).

Theorem 4. *If $\phi = (\phi_1, \ldots, \phi_n) \in K[X]^n$ is a h.s.o.p. then for every k, $0 < k < n$ and for every homogeneous polynomial ψ such that $\psi\phi_{k+1} \in (\phi_1, \ldots, \phi_k)K[X]$ we have $\psi \in (\phi_1, \ldots, \phi_k)K[X]$.*

3. Proof of Max Noether's Fundamental Theorem

Let $F_1, \dots, F_n \in K[X]$ be polynomials (we do not assume that the system $F_1(X) = \dots = F_n(X) = 0$ is general!) in n variables $X = (X_1, \dots, X_n)$ with coefficients in an algebraically closed field K. Let $G \in K[X]$. We say that the sequence $G, F_1, \dots, F_n$ satisfies Noether's conditions at $x \in K^n$ if there exists a polynomial $D_x = D_x(X) \in K[X]$ such that $D_x(x) \neq 0$ and $D_x G$ is in the ideal $(F_1, \dots, F_n)K[X]$.

Lemma 2. *Let $G, F_1, \dots, F_n \in K[X]$ be polynomials such that for every $x \in K^n$ the sequence $G, F_1, \dots, F_n$ satisfies Noether's conditions at x. Then $G \in (F_1, \dots, F_n)K[X]$.*

Proof. The system of polynomial equations $D_x(X) = 0$, $x \in K^n$ has no solutions in K^n. Therefore by Hilbert's Nullstellensatz there exists a family of polynomials $M_x(X)$, $x \in K^n$ such that $\sharp\{x \in K^n \ : \ M_x(X) \neq 0\} < +\infty$ and $\displaystyle\sum_{x \in K^n} M_x D_x = 1$ in $K[X]$. Then we get $G = \left(\displaystyle\sum_{x \in K^n} M_x D_x\right) G = \displaystyle\sum_{x \in K^n} M_x (D_x G) \in (F_1, \dots, F_n)K[X]$.

Remark 1. *If $x \notin V(F_1, \dots, F_n)$ then for any polynomial G the sequence $G, F_1, \dots, F_n$ satisfies Noether's conditions at x. It suffices to take $D_x = F_i$ where F_i is such that $F_i(x) \neq 0$.*

Lemma 3. *Let $F_1, \dots, F_n \in K[X]$ be polynomials such that $F_1(x) = \dots = F_n(x) = 0$ and $\det\left(\dfrac{\partial F_i}{\partial X_j}(x)\right) \neq 0$ at a point $x = (x_1, \dots, x_n) \in K^n$. Then there is a polynomial $D_x(X) \in K[X]$ such that $(X_i - x_i)D_x \in (F_1, \dots, F_n)K[X]$ for $i \in \{1, \dots, n\}$ and $D_x(x) \neq 0$.*

Proof. Write $F_i(X) = (X_1 - x_1)D_{i1}(X) + \dots + (X_n - x_n)D_{in}(X)$ in $K[X]$ for $i \in \{1, \dots, n\}$. Differentiating and putting $X = x$ we get $D_{ij}(x) = \dfrac{\partial F_i}{\partial X_j}(x)$. Let $D_x(X) := \det(D_{ij}(X))$. Then $D_x(x) \neq 0$ and by Cramer's Rule $(X_i - x_i)D_x(X) \in (F_1, \dots, F_n)K[X]$.

Proposition 1. *Let $F_1, \dots, F_n \in K[X]$ be polynomials such that for every $x \in V(F_1, \dots, F_n)$ one has $\det\left(\dfrac{\partial F_i}{\partial X_j}(x)\right) \neq 0$. Let $G \in K[X]$ be a polynomial such that $G(x) = 0$ for all $x \in V(F_1, \dots, F_n)$. Then $G \in (F_1, \dots, F_n)K[X]$.*

Proof. Let $x = (x_1, \dots, x_n) \in K^n$. If $x \in V(F_1, \dots, F_n)$ then $G(X) = \displaystyle\sum_{i=1}^{n} (X_i - x_i)G_i(X)$. By Lemma 3 there is a polynomial $D_x(X) \in K[X]$ such that $(X_i - x_i)D_x(X) \in (F_1, \dots, F_n)K[X]$. Thus $D_x G \in (F_1, \dots, F_n)K[X]$. By Lemma 2 and Remark 1 we get $G \in (F_1, \dots, F_n)K[X]$.

What remains to be proved in Noether's Theorem is the bound on the degrees.

Proposition 2. *Let $F_1,\dots,F_n \in K[X]$ be nonconstant polynomials such that the homogeneous forms $F_i^+ \in K[X]$, $i \in \{1,\dots,n\}$, form a h.s.o.p. Then for every $G \in (F_1,\dots,F_n)K[X]$ there exists $A_1,\dots,A_n \in K[X]$ such that $G = \sum_{i=1}^{n} A_iF_i$ and $\deg(A_iF_i) \leqslant \deg(G)$ for $i \in \{1,\dots,n\}$.*

Proof. Let X_0 be a new variable and let $\tilde{G}(X_0,X)$, $\tilde{F}_i(X_0,X)$, $i \in \{1,\dots,n\}$, be the homogenization of $G(X)$ and $F_i(X)$ for $i \in \{1,\dots,n\}$. Recall that $\tilde{G}(X_0,X) = X_0^{\deg G} G\left(\dfrac{X_1}{X_0},\dots,\dfrac{X_n}{X_0}\right)$. Since $G \in (F_1,\dots,F_n)K[X]$ we get $X_0^N\tilde{G} \in (\tilde{F}_1,\dots,\tilde{F}_n)K[X_0,X]$ for an integer $N > 0$. It is easy to see that $X_0^N, \tilde{F}_1,\dots,\tilde{F}_n$ form a h.s.o.p. in $K[X_0,X]$. By Theorem 4 X_0^N is not a zero-divisor mod $(\tilde{F}_1,\dots,\tilde{F}_n)$ and we may write $\tilde{G} = \sum_{i=1}^{n} \psi_i\tilde{F}_i$ where ψ_i are homogeneous polynomials such that $\psi_i\tilde{F}_i$ is either 0 or of degree $\deg\tilde{G}$. Let $A_i(X) = \psi_i(1,X)$ for $i \in \{1,\dots,n\}$. Putting $X_0 = 1$ in the identity $\tilde{G} = \sum_{i=1}^{n} \psi_i\tilde{F}_i$ we get $G = \sum_{i=1}^{n} A_iF_i$ and $\deg(A_iF_i) \leqslant \deg G$ for $i \in \{1,\dots,n\}$.

Remark 2. *With the assumptions of Proposition 2 one has $\max_{i=1}^{n}(\deg A_iF_i) = \deg G$ and $G^+ = \sum_{i\in I} A_i^+F_i^+$ where $I = \{i : \deg(A_iF_i) = \deg(G)\}$. In particular $G^+ \in (F_1^+,\dots,F_n^+)$.*

Proof of Max Noether's Fundamental Theorem. Max Noether's Theorem follows immediately from Proposition 1 and Proposition 2.

4. Proof of the Jacobi formula

Lemma 4. *Let $F = (F_1,\dots,F_n) \in K[X]^n$ be polynomials with coefficients in a field K. Then the set $W = \{x \in K^n : F(x) = 0 \text{ and } \operatorname{Jac} F(x) \neq 0\}$ is finite.*

Proof. By Lemma 3 for every $x \in W$ there is a polynomial $D_x = D_x(X)$ such that $D_x(x) \neq 0$ and

$$(X_i - x_i)D_x \in (F_1,\dots,F_n) \text{ for } i = 1,\dots,n.$$

Let us put $U_x = \{\tilde{x} \in K^n : D_x(\tilde{x}) \neq 0\}$ for every $x \in W$. Then $U_x \subseteq K^n$ is a Zariski open subset of K^n and $W \cap U_x = \{x\}$. Since $K[X]$ is a noetherian ring there exists a finite sequence $x_1,\dots,x_s \in W$ such that $\bigcup_{x\in W} U_x = \bigcup_{i=1}^{s} U_{x_i}$. Obviously $W = \{x_1,\dots,x_s\}$.

Now, let $F = (F_1, \dots, F_n) \in K[X]^n$ be a sequence of polynomials such that the set $V = V(F)$ is finite. If $R, S \in K[X]$ and $S(x) \neq 0$ for all $x \in V$ then we define the *trace* of $\dfrac{R}{S}$ with respect to F by putting $\operatorname{Tr}_F\left(\dfrac{R}{S}\right) := \displaystyle\sum_{x\in V} \frac{R(x)}{S(x)}$.

If the system of polynomial equations $F = 0$ has only simple solutions then $\operatorname{Tr}_F\left(\dfrac{H}{\operatorname{Jac} F}\right) = \displaystyle\sum_{x\in V} \frac{H(x)}{\operatorname{Jac} F(x)}$ is well-defined.

Lemma 5. *Let $F = (F_1, \dots, F_n) \in K[X]^n$ and $G = (G_1, \dots, G_n) \in K[X]^n$ be such that the systems of polynomial equations $F = 0$ and $G = 0$ have only simple zeroes. Suppose that $G_i = \sum_{j=1}^{n} A_{ij} F_j$ in $K[X]$. Let $A = \det(A_{ij})$. Then $\operatorname{Tr}_F\left(\dfrac{H}{\operatorname{Jac} F}\right) = \operatorname{Tr}_G\left(\dfrac{AH}{\operatorname{Jac} G}\right)$.*

Proof. Differentiating the identities

$$G_i = \sum_{j=1}^{n} A_{ij} F_j \tag{1}$$

we get

$$\operatorname{Jac} G \equiv A \operatorname{Jac} F \ (\mathrm{mod}\ (F_1, \dots, F_n)K[X]). \tag{2}$$

From (1) and (2) we get that for all $x \in K^n$, $F(x) = 0$ if and only if $G(x) = 0$ and $A(x) \neq 0$. Indeed, if $F(x) = 0$ then $G(x) = 0$ by (1) and $\operatorname{Jac} G(x) = A(x)\operatorname{Jac} F(x)$ by (2). Thus $\operatorname{Jac} G(x) \neq 0$ by the hypothesis that all the zeroes of the system $G = 0$ are simple, consequently we get $A(x) \neq 0$.

On the other hand suppose that $G(x) = 0$ and $A(x) \neq 0$. Then from (1) we get $0 = \sum_{j=1}^{n} A_{ij}(x) F_j(x)$ for $i \in \{1, \dots, n\}$ and $F_j(x) = 0$ by Cramer's Rule. Summing up we have $V(F) = V(G) \setminus V(A)$ and $\operatorname{Jac} G = A \operatorname{Jac} F$ on $V(F)$.

Now, we get

$$\operatorname{Tr}_F\left(\frac{H}{\operatorname{Jac} F}\right) = \sum_{x\in V(F)} \frac{H(x)}{\operatorname{Jac} F(x)} =$$

$$= \sum_{x\in V(G)\setminus V(A)} \frac{A(x)H(x)}{\operatorname{Jac} G(x)} = \sum_{x\in V(G)} \frac{A(x)H(x)}{\operatorname{Jac} G(x)} = \operatorname{Tr}_G\left(\frac{AH}{\operatorname{Jac} G}\right).$$

Lemma 6. *If $G = (G_1, \dots, G_n) \in K[X]^n$ where $G_i = G_i(X_i) \in K[X_i]$, $i \in \{1, \dots, n\}$, are nonconstant polynomials with simple zeroes then for every polynomial $H \in K[X]$, $\deg H < \sum_{i=1}^{n} (\deg G_i - 1)$ one has $\operatorname{Tr}_G\left(\dfrac{H}{\operatorname{Jac} G}\right) = 0$.*

Proof. By linearity of the trace we may assume that $H = X_1^{a_1} \cdots X_n^{a_n}$. It is easy to see that $\operatorname{Tr}_G \left(\frac{H}{\operatorname{Jac} G}\right) = \operatorname{Tr}_{G_1}\left(\frac{X_1^{a_1}}{G_1'}\right) \cdots \operatorname{Tr}_{G_n}\left(\frac{X_n^{a_n}}{G_n'}\right)$. If $\deg H = \sum_{i=1}^n a_i < \sum_{i=1}^n (\deg G_i - 1)$ then $a_i < \deg G_i - 1$ for some $i \in \{1, \ldots, n\}$ and $\operatorname{Tr}_{G_i}\left(\frac{X_i^{a_i}}{G_i'}\right) = 0$. Consequently $\operatorname{Tr}_G\left(\frac{H}{\operatorname{Jac} G}\right) = 0$ and we are done.

Proof of the Jacobi Formula. Let $F = 0$ be a general system of polynomial equations. Then the set $V = V(F)$ is finite by Lemma 4 (and non-empty by Corollary 1). Let $\Pi_i : K^n \longrightarrow K$ be the projection given by $\Pi_i(x_i, \ldots, x_n) = x_i$ and put $G_i(X_i) = \prod_{x_i \in V_i} (X_i - x_i) \in K[X_i]$ where $V_i = \Pi_i(V(F))$. Then $G_i(X_i)$ is a polynomial with simple zeroes vanishing on V. By Max Noether's Fundamental Theorem we may write $G_i = A_{i1}F_1 + \cdots + A_{in}F_n \in K[X]$ with $\deg(A_{ij}F_j) \leqslant \deg G_i$ for $i \in \{1, \ldots, n\}$. Let $A = \det(A_{ij})$. For any permutation $(j_1, \ldots, j_n)$ of $(1, \ldots, n)$ we get $\deg(\pm A_{1j_1} \cdots A_{nj_n}) \leqslant (\deg G_1 - \deg F_{j_1}) + \cdots + (\deg G_n - \deg F_{j_n}) = \sum_{i=1}^n (\deg G_i - \deg F_i)$ and consequently $\deg A \leqslant \sum_{i=1}^n (\deg G_i - \deg F_i)$.

Let $H \in K[X]$ be a polynomial such that $\deg H < \sum_{i=1}^n (\deg F_i - 1)$. Therefore $\deg(AH) < \sum_{i=1}^n (\deg G_i - \deg F_i) + \sum_{i=1}^n (\deg F_i - 1) = \sum_{i=1}^n (\deg G_i - 1)$. Let $G = (G_1, \ldots, G_n)$. By Lemma 5 and Lemma 6 we get $\operatorname{Tr}_F\left(\frac{H}{\operatorname{Jac} F}\right) = \operatorname{Tr}_G\left(\frac{AH}{\operatorname{Jac} G}\right) = 0$.

5. Poincaré series

Let K be an arbitrary field (not necessarily algebraically closed).

Let $\phi_1, \ldots, \phi_n \in K[X]$, $X = (X_1, \ldots, X_n)$ be a sequence of homogeneous forms of degrees $d_1, \ldots, d_n > 0$. For any integer $d \geqslant 0$ we denote by $K[X]_d$ the linear K-linear subspace of $K[X]$ generated by monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$, $\alpha_1 + \cdots + \alpha_n = d$. For any integer m, $1 \leqslant m \leqslant n$ we put $(\phi_1, \ldots, \phi_m)_d$ the K-linear subspace of $K[X]_d$ consisted of the sums $\alpha_1\phi_1 + \cdots + \alpha_m\phi_m$ where α_i are homogeneous polynomials such that $\alpha_i\phi_i$ is either 0 or of degree d. We put, by convention, $(\phi_1, \ldots, \phi_m)_d = (0)_d$ if $m = 0$.

Theorem 5. *Suppose that $\phi_1, \ldots, \phi_n$ is a sequence of homogeneous parameters in $K[X]$. Then for any integer m, $0 \leqslant m \leqslant n$ we have*

$$\sum_{d \geqslant 0} (\dim_K K[X]_d/(\phi_1, \ldots, \phi_m)_d)\, T^d = \frac{\prod_{i=1}^m (1 - T^{d_i})}{(1 - T)^n}.$$

Remark 3. *The formal power series which appears on the left side of the above identity is named the* Poincaré series *of the graded algebra* $K[X]/(\phi_1,\dots,\phi_m) \simeq \bigoplus_{d\geqslant 0} K[X]_d/(\phi_1,\dots,\phi_m)_d$.

To prove Theorem 5 we need two lemmas.

Lemma 7. $\sum_{d\geqslant 0}(\dim_K K[X]_d)\, T^d = \dfrac{1}{(1-T)^n}$.

Proof. Let $T_1,\dots,T_n$ be new variables. Then

$$\left(\sum_{\alpha_1\geqslant 0} T_1^{\alpha_1}\right)\cdots\left(\sum_{\alpha_n\geqslant 0} T_n^{\alpha_n}\right) = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbf{N}^n} T_1^{\alpha_1}\cdots T_n^{\alpha_n}.$$

Let T be a variable. Substituting $T_1=\dots=T_n=T$ we get

$$\left(\sum_{\alpha\geqslant 0} T^{\alpha}\right)^n = \sum_{(\alpha_1,\dots,\alpha_n)\in\mathbf{N}^n} T^{\alpha_1+\dots+\alpha_n} =$$

$$= \sum_{d\geqslant 0}\left(\sum_{\alpha_1+\dots+\alpha_n=d} 1\right) T^d = \sum_{d\geqslant 0}(\dim_K K[X]_d) T^d$$

and the Lemma follows since $\sum_{\alpha\geqslant 0} T^{\alpha} = \dfrac{1}{1-T}$ in $\mathbf{Z}[T]$.

Lemma 8.

(1) $\dim_K K[X]_d/(\phi_1,\dots,\phi_m)_d = \dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d$ for $d<d_m$.
(2) $\dim_K K[X]_d/(\phi_1,\dots,\phi_m)_d = \dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d -$
$- \dim_K K[X]_{d-d_m}/(\phi_1,\dots,\phi_{m-1})_{d-d_m}$ for $d\geqslant d_m$.

Proof. Property *1.* is obvious since $(\phi_1,\dots,\phi_m)_d = (\phi_1,\dots,\phi_{m-1})_d$ for $d<d_m$. Let U be a K-linear space of finite dimension. Then for any subspaces $W,V\subseteq U$ such that $W\subseteq V$ we have $\dim_K U/W = \dim_K U/V + \dim_K V/W$. Taking $U = K[X]_d$, $V=(\phi_1,\dots,\phi_m)_d$ and $V=(\phi_1,\dots,\phi_{m-1})_d$ we get

$$\text{(3)}\quad \begin{aligned}\dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d &= \dim_K K[X]_d/(\phi_1,\dots,\phi_m)_d\\ &+ \dim_K(\phi_1,\dots,\phi_m)_d/(\phi_1,\dots,\phi_{m-1})_d.\end{aligned}$$

By Theorem 4 ϕ_m is not a zero-divisor mod $(\phi_1,\dots,\phi_{m-1})$. Consequently the mapping $A\longrightarrow A\phi_m$ where $A\in K[X]_{d-d_m}$ induces an isomorphism of spaces $(\phi_1,\dots,\phi_m)_d/(\phi_1,\dots,\phi_{m-1})_d$ and $K[X]_{d-d_m}/(\phi_1,\dots,\phi_{m-1})_{d-d_m}$ and we get

$$\text{(4)}\quad \dim_K(\phi_1,\dots,\phi_m)_d/(\phi_1,\dots,\phi_{m-1})_d = \dim_K K[X]_{d-d_m}/(\phi_1,\dots,\phi_{m-1})_{d-d_m}.$$

From (3) and (4) we obtain Property *2.* of Lemma.

Now we can give

Proof of Theorem 5.

If $m=0$ then the formula follows from Lemma 7. Suppose that $m>0$ and that Theorem 5 holds for $m-1$. So we have

$$\sum_{d\geqslant 0}(\dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d)\,T^d=\frac{(1-T^{d_1})\cdots(1-T^{d_m-1})}{(1-T)^n}.$$

Using Lemma 8 we get

$$\begin{aligned}\sum_{d\geqslant 0}&(\dim_K K[X]_d/(\phi_1,\dots,\phi_m)_d)\,T^d=\\&=\sum_{d\geqslant 0}(\dim_K K[X]_d/(\phi_1,\dots,\phi_{m-1})_d)\,T^d-\\&\qquad-\sum_{d\geqslant d_m}(\dim_K K[X]_{d-d_m}/(\phi_1,\dots,\phi_{m-1})_{d-d_m})\,T^d=\\&=\frac{(1-T^{d_1})\cdots(1-T^{d_{m-1}})}{(1-T)^n}-\frac{(1-T^{d_1})\cdots(1-T^{d_{m-1}})}{(1-T)^n}T^{d_m}=\\&=\frac{(1-T^{d_1})\cdots(1-T^{d_m})}{(1-T)^n}.\end{aligned}$$

Corollary 3. *If $\phi_1,\dots,\phi_n$ is a system of homogeneous parameters in $K[X]$ with $\deg\phi_i=d_i$, then*

$$\dim_K K[X]/(\phi_1,\dots,\phi_n)=d_1\cdots d_n.$$

Proof. If $m=n$ then by Theorem 5 we get

$$\begin{aligned}&\sum_{d\geqslant 0}(\dim_K (K[X]_d/(\phi_1,\dots,\phi_n)_d)\,T^d=\\&\qquad=(1+T+\dots+T^{d_1-1})\cdots(1+T+\dots+T^{d_n-1}).\end{aligned}\tag{5}$$

Therefore $\dim_K(K[X]_d/(\phi_1,\dots,\phi_n)_d)=0$ for $d>\sum_{i=1}^{n}(d_i-1)$. Substituting $T=1$ in (5) we get

$$\sum_{d\geqslant 0}\dim_K K[X]_d/(\phi_1,\dots,\phi_n)_d=d_1\cdots d_n.$$

It suffices to observe that $K[X]/(\phi_1,\dots,\phi_n)$ and $\oplus_{d\geqslant 0}K[X]_d/(\phi_1,\dots,\phi_n)_d$ are K-isomorphic.

Let $d_1, \dots, d_n > 0$ be a sequence of positive integers. For any $d \geqslant 0$ we put

$$(6)\quad \nu_d(d_1, \dots, d_n) = \sharp \left\{ (\alpha_1, \dots, \alpha_n) : 0 \leqslant \alpha_i < d_i \text{ for } i = 1, \dots, n,\ \sum_{i=1}^{n} \alpha_i = d \right\} .$$

Lemma 9. *To abbreviate the notation we put $\nu_d = \nu_d(d_1, \dots, d_n)$.*

(i) $(1 + T + \dots T^{d_1-1}) \dots (1 + T + \dots T^{d_n-1}) = \sum_{d \geqslant 0} \nu_d$,

(ii) $\sum_{d \geqslant 0} \nu_d = d_1 \dots d_n$,

(iii) *Let* $\delta = \sum_{i=1}^{n} (d_i - 1)$. *Then* $\nu_d = \nu_{\delta - d}$ *for* $0 \leqslant d \leqslant \delta$.

Proof. Property (i) is obvious. Putting $T = 1$ we get (ii). The polynomial on the left side of (i) is recurrent, hence it follows (iii).

Proposition 3. $\dim_K K[X]_d/(\phi_1, \dots, \phi_n)_d = \nu_d(d_1, \dots, d_n)$.

Proof. Use formula (5) and Lemma 9 (i) .

6. Monomial bases

We keep the notation and assumptions of Section 5. In particular, K is an arbitrary field. Let $A = K[X]/I$ be an affine algebra of finite dimension $D = \dim_K A$. A monomial basis of A mod. the ideal I is a sequence of monomials $e_0, \dots, e_{D-1} \in K[X]$ such that the images of $e_0, \dots, e_{D-1}$ in A form a linear basis of A.

Proposition 4. *Let $F_1, \dots, F_n \in K[X]$ be nonconstant polynomials such that the homogeneous forms $F_1^+, \dots, F_n^+$ form h.s.o.p. Let $I(F) = (F_1, \dots, F_n)$ and $I(F^+) = (F_1^+, \dots, F_n^+)$. Then any monomial basis mod $I(F^+)$ is a monomial basis mod $I(F)$.*

Proof. Let $\epsilon_0, \epsilon_1, \dots, \epsilon_{D-1}$ be a monomial basis. We will check that $\epsilon_0, \epsilon_1, \dots, \epsilon_{D-1}$ is a linear basis mod $I(F)$. First, let us prove that $\epsilon_0, \epsilon_1, \dots, \epsilon_{D-1}$ are linearly independent mod $I(F)$. Suppose that there is a non-zero sequence $c_0, \dots, c_{D-1} \in K$ such that $c_0\epsilon_0 + \dots + c_{D-1}\epsilon_{D-1} \equiv 0$ mod $I(F)$. Let $I = \{i : c_i \neq 0\}$ and $I_0 = \{i \in I : \deg(\sum_j c_j \epsilon_j) = \deg \epsilon_i\}$. Then, by Remark 2 we get $\sum_{i \in I_0} c_i \epsilon_i \equiv 0$ (mod $I(F^+)$) which contradicts the linear independence of ϵ_i mod $I(F^+)$.

To check that every polynomial G is a linear combination of ϵ_i mod $I(F)$ we use induction on $\deg G$. Let $N > 0$ be an integer and suppose that every polynomial of degree strictly less than N is a linear combination of ϵ_i mod $I(F)$. Let G be a polynomial of degree N. It suffices to check that G^+ is a linear combination of

$\epsilon_0, \ldots, \epsilon_{D-1}$ mod $I(F)$. Since $\epsilon_0, \ldots, \epsilon_{D-1}$ form a linear basis mod $I(F^+)$ we may write

$$G^+ = \phi_1 F_1^+ + \cdots + \phi_n F_n^+ + \sum_i c_i \epsilon_i$$

where ϕ_i are homogeneous forms such that $\phi_i F_i^+$ is of degree $\deg G^+ = N$. Write $F_i = F_i^+ + R_i$, $1 \leqslant i \leqslant n$, where $\deg R_i < \deg F_i^+$. Then we get

$$G^+ = \phi_1(F_1 - R_1) + \cdots + \phi_n(F_n - R_n) + \sum_i c_i \epsilon_i \equiv$$
$$\equiv \phi_1(-R_1) + \cdots + \phi_n(-R_n) + \sum_i c_i \epsilon_i \bmod I(F)$$

where $\deg(-\phi_1 R_1 - \cdots - \phi_n R_n) < N$ and we are done.

Theorem 6. *If $F_1, \ldots, F_n$ are nonconstant polynomials, $d_1 = \deg F_1, \ldots, d_n = \deg F_n$ such that the forms $F_1^+, \ldots, F_n^+$ form a homogeneous system of parameters then*

$$\dim_K K[X]/I(F) = d_1 \ldots d_n \ .$$

Proof. Proposition (4) implies that $\dim_K K[X]/I(F) = \dim_K K[X]/I(F^+)$. Use Corollary 3.

Theorem 7. *With the assumptions of Theorem 6 there exists a monomial basis mod the ideal $I(F)$ such that*

$$\sharp\{i : \deg e_i = d\} = \nu_d(d_1, \ldots, d_n)$$

for any $d \geqslant 0$.

Proof. According to Proposition 4 it suffices to prove the theorem for ideal $I(F^+)$. Let $\epsilon_0, \epsilon_1, \ldots, \epsilon_{D-1}$ be a monomial basis mod $I(F^+)$. Fix an integer $d \geqslant 0$. Since $K[X]/I(F^+) = \bigoplus K[X]_d/I(F^+)_d$ the images of $\epsilon_0, \epsilon_1, \ldots, \epsilon_{D-1}$ of degree d form a basis of the space $K[X]_d/I(F^+)_d$ which is of dimension $\nu_d(d_1, \ldots, d_n)$ by Proposition 3.

7. Proof of Bézout's Theorem

We keep the notations of Introduction. We consider a general system of polynomial equations $F = 0$ and its set of solutions $V(F)$. We know that $V(F)$ is non-empty (see Corollary 1) and finite (see Lemma 4). Let us denote $I(F)$ the ideal generated by polynomials $F_1, \ldots, F_n$ in the ring $K[X]$. To prove Bézout's Theorem we need

Lemma 10.
$$\sharp V(F) = \dim_K K[X]/I(F).$$

Proof. Let us consider the K-algebra $K[V]$ of polynomial functions on the set $V = C(F)$. It is easy to see that the family $\{e_x : x \in V\}$ where $e_x(x) = 1$ and $e_x(x') = 0$ for $x' \in V \setminus \{x\}$ is a K-linear basis of $K[V]$. Thus $\dim_K K[V] = \sharp V$. On the other hand the K-linear homomorphism $\sigma : K[X] \to K[V]$ defined by $\sigma(P) = P_{|V}$, has by Proposition 1 the kernel $I(V)$. Thus $K[V]$ and $K[V]/I(F)$ are isomorphic and the lemma follows.

Proof of Theorem 3. By Lemma 10 and Theorem 6 we have

$$\sharp V(F) = \dim_K K[X]/I(F) = \prod_{i=1}^{n} \deg F_i.$$

The reader will find more about Bézout's Theorem in [LJ].

8. Application to real algebraic geometry

Let $F = (F_1, \ldots, F_n) \in \mathbb{R}[X]^n$ be nonconstant polynomials in n variables $X = (X_1, \ldots, X_n)$ of degrees $d_1, \ldots, d_n > 0$. Suppose that the system of polynomial equations $F = 0$ is general (see Definition 1). Let $V = V(F)$ be the set of all complex solutions of $F = 0$ and let $V_{\mathbb{R}} = V(F) \cap \mathbb{R}^n$. Let $J_F = \operatorname{Jac} F$. We define $\operatorname{ind} F = \sum_{a \in V_{\mathbb{R}}} \operatorname{sgn} J_F(a)$ (the index of vector field F). We define the Petrovskii number $\Pi(d_1, \ldots, d_n)$ by the formula

$$\Pi(d_1, \ldots, d_n) = \sharp \left\{ (\alpha_1, \ldots, \alpha_n) : 0 \leqslant \alpha_i < d_i, \sum_{i=1}^{n} \alpha_i = \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) \right\}.$$

Clearly, if $\sum_{i=1}^{n} (d_i - 1)$ is an odd number then $\Pi(d_1, \ldots, d_n) = 0$. Note also that $\Pi(d_1, d_2) = \min\{d_1, d_2\}$ if $d_1 + d_2 \equiv 0 \pmod 2$.

The following theorem may be considered as a real counterpart of Bézout's theorem.

Theorem 8 (Petrovskii-Oleinik Inequality)**.** *With the notation and assumptions introduced above*
$$|\operatorname{ind} F| \leqslant \Pi(d_1, \ldots, d_n).$$

The inequality figuring in Theorem 8 was proved by Arnold [A] and called by him the Petrovskii-Oleinik inequality. Khovanskii [Kh] proved an inequality of this type for the index of polynomial vector field in the open set defined by an equation $P > 0$.

Proof of the Petrovskii-Oleinik inequality.

1. Preliminaries

Let $V \subset \mathbb{C}^n$ be a finite subset of $\mathbb{C}^n$ such that if $a = (a_1, \ldots, a_n) \in V$ then $\overline{a} = (\overline{a_1}, \ldots, \overline{a_n}) \in V$. Let $\mathbb{R}[V]$ be the set of all functions $f : V \to \mathbb{C}$ such that $\overline{f(a)} = f(\overline{a})$ for $a \in V$. Then $\mathbb{R}[V]$ is an algebra over $\mathbb{R}$, $\dim_{\mathbb{R}} \mathbb{R}[V] = \sharp V$. Let $\phi \in \mathbb{R}[V]$ be a fixed function which is nowhere 0. We consider the bilinear form B_ϕ on $\mathbb{R}[V]$ defined by

$$B_\phi(f, g) = \sum_{a \in V} \phi(a) f(a) g(a).$$

Lemma 11. *The quadratic form $Q_\phi(f) = B_\phi(f, f)$ takes real values and is non-degenerate. The signature $\sigma(Q_\phi)$ of Q_ϕ is equal to*

$$\sum_{a \in V \cap \mathbb{R}} \operatorname{sgn} \phi(a).$$

Proof of Lemma 11. Let $V = \{a_1, \ldots, a_r, b_1, \ldots, b_s, \overline{b_1}, \ldots, \overline{b_s}\}$ where $\overline{a_i} = a_i$ for $i = 1, \ldots, r$, $\overline{b_j} \neq b_j$ for $j = 1, \ldots, s$ are pairwise different. We have

$$Q_\phi(f) = \sum_{i=1}^{r} \phi(a_i) f(a_i)^2 + 2 \sum_{j=1}^{s} \operatorname{Re} \{\phi(b_j) f(b_j)\}.$$

Let $Q_i(f) = \phi(a_i) f(a_i)^2$ $(i = 1, \ldots, r)$ and $R_j(f) = \phi(b_j) f(b_j)^2$ $(j = 1, \ldots, s)$. Then $\operatorname{rank} Q_i = 1$, $\sigma(Q_i) = \operatorname{sgn} \phi(a_i)$, $\operatorname{rank} R_j = 2$, $\sigma(R_j) = 0$. The subspaces corresponding to linear forms $f \to f(a_i)$ and $f \to f(b_j)$ are orthogonal with respect to the form B_ϕ. Therefore

$$\begin{aligned} \operatorname{rank} Q_\phi &= \operatorname{rank} Q_1 + \cdots + \operatorname{rank} Q_r + \operatorname{rank} R_1 + \cdots + \operatorname{rank} R_s \\ &= r + 2s = \sharp V = \dim_{\mathbb{R}} \mathbb{R}[V] \end{aligned}$$

and

$$\sigma(Q_\phi) = \sigma(Q_1) + \cdots + \sigma(Q_r) + \sigma(R_1) + \cdots + \sigma(R_s) = \sum_{i=1}^{r} \operatorname{sgn} \phi(a_i).$$

Lemma 12. *Let N be any linear subspace of $\mathbb{R}[V]$ on which Q_ϕ is identically equal to zero. Then $\sigma(Q_\phi) \leqslant \dim_{\mathbb{R}} \mathbb{R}[V] - 2 \dim_{\mathbb{R}} N$.*

Proof. The lemma follows from Witt's theorem (see [L]), p. 592, Corollary 10.4).

Let V be the set of all complex solutions of the general system of real equations $F_1 = 0, \ldots, F_n = 0$ of degrees $d_1, \ldots, d_n > 0$. Note that $\dim_{\mathbb{R}} \mathbb{R}[V] = \sharp V = \dim_{\mathbb{C}} \mathbb{C}[V] = d_1 \cdots d_n$ by Bézout's theorem. For any polynomial $H \in \mathbb{R}[X_1, \ldots, X_n]$

we define a function $[H]$ of $\mathbb{R}[V]$ by putting $[H](a) = H(a)$ for $a \in V$. Let us consider the subspace of $\mathbb{R}[V]$:

$$N = \left\{ [H] \in \mathbb{R}[V] : \deg H < \frac{1}{2} \sum_{i=1}^{n} (d_i - 1) \right\}.$$

If $[H] \in N$ then $\deg H^2 < \sum_{i=1}^{n} (d_i - 1)$ and by the Jacobi formula

$$\sum_{a \in V} \frac{H(a)^2}{\operatorname{Jac} F(a)} = 0.$$

Let $\phi = \dfrac{1}{\operatorname{Jac} F}$. Then the subspace N is contained in the cone $Q_\phi^{-1}(0)$. By Lemma 12 we get

$$|\operatorname{ind} F| = \left| \sum_{a \in V} \operatorname{sgn} \operatorname{Jac} F(a) \right| = |\sigma(Q_\phi)| \leqslant \dim_{\mathbb{R}} \mathbb{R}[V] - 2 \dim_{\mathbb{R}} N$$
$$< d_1 \cdots d_n - 2 \dim_{\mathbb{R}} N.$$

By Theorem 7 there exists a monomial basis $e_0, \ldots, e_n$ of $\mathbb{R}[V]$ such that

$$\sharp\{i : \deg e_i = d\} = \nu_d(d_1, \ldots, d_n) \quad \text{for } d \geqslant 0.$$

Let $\delta = \sum_{i=1}^{n} (d_i - 1)$. Then

$$\begin{aligned} \dim_{\mathbb{R}} N &= \textit{number of elements in monomial basis of degree } < \frac{1}{2}\delta \\ &= \textit{number of elements in monomial basis of degree } > \frac{1}{2}\delta \end{aligned}$$

by Lemma 9 (iii).

Therefore

$$\begin{aligned} 2 \dim_{\mathbb{R}} N &= \textit{number of elements in monomial basis of degree } \neq \frac{1}{2}\delta \\ &= d_1 \ldots d_n - \nu_{\frac{1}{2}\delta}(d_1, \ldots, d_n) \end{aligned}$$

and

$$|\operatorname{ind} F| = |\sigma(Q_\phi)| \leqslant \dim_{\mathbb{R}} \mathbb{R}[V] - 2 \dim_{\mathbb{R}} N = \nu_{\frac{1}{2}\delta}(d_1, \ldots, d_n) = \Pi(d_1, \ldots, d_n).$$

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Acknowledgments
This article was prepared for the lectures delivered at the Algebra Department of La Laguna University in June 2009. I thank Professor Evelia García Barroso for the invitation and great hospitality. I am also indebted to Professor Patrick Popescu-Pampu who read the article and made useful remarks.

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 163–173
DOI: https://doi.org/10.18778/8331-092-3.13

AN INVITATION TO THE POSITIVITY AND GEOMETRY OF ALGEBRAIC CYCLES

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1. Problems

The purpose of this work is an introduction and overview of geometric and numeric properties of algebraic cycles in smooth projective varieties. We recall or propose several problems, which we consider worth to study. We are mainly interested in, but do not restrict our story to, codimension 2 cycles in projective spaces. These are points in $\mathbb{P}^2$, curves in $\mathbb{P}^3$, surfaces in $\mathbb{P}^4$ and so on.

We focus on the *positivity aspects* of such cycles on the one hand, and on attached *asymptotic invariants* on the other hand.

Whereas positivity for divisors (i.e. codimension 1 cycles) is either well understood or there is at least a clear conjectural picture, the study of positivity of higher codimension cycles has been taken on seriously in this century and the theory is much less developed.

Historically, there has been a lot of interest in the geometry of space curves. A lot of research focused on the study of Hilbert schemes $H(d, g)$ of smooth, irreducible curves of degree d and genus g in $\mathbb{P}^N$. The Hilbert scheme perspective is naturally associated to degeneration techniques. Whereas these techniques are inevitable our objective is to inform about another approach motivated by recent results of Fulger, Lehmann and others.

One of the most fundamental questions asked, in the context of positivity, about divisors is whether a divisor is effective and if it is so, the next question is: what is the dimension of the linear system it lives in. Similar approach can be taken on studying cycles of higher codimension. We recall here the relevant invariant, which

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2010 *Mathematics Subject Classification.* 14C25, 14H50, 14M07.
Key words and phrases. ACM subvarieties, algebraic cycles, Hartshorne conjecture, mobility count, space curves, postulation problems for cycles.

is the *mobility count* as introduced in early works of Daniel Perrin [25] and which is a natural generalisation of the dimension of a linear system of divisors.

Let X be a smooth variety of dimension n and let $\alpha \in N_k(X)$ be an effective integral k-cycle (k is here the *dimension* of the support of α, thus $k = n-1$ if α is a divisor). The *mobility count* $\mathrm{mc}(\alpha)$ of α is the maximal number of general points in X that can be imposed on the class of α (i.e. there exists an effective cycle, numerically equivalent to α, passing through all these points). Note that for a divisor, the mobility count is essentially (up to some issues between the numerical and linear equivalence) the dimension of the linear system generated by this divisor.

It is natural to expect that if X has a simple, or well-known structure, the mobility count should be easily performed. Surprisingly, this is not the case! Already in the case of curves in $\mathbb{P}^3$ the picture is far from being complete. Indeed, given a positive number s, Perrin asked what is the minimal degree $d(s)$ of a curve $C \subset \mathbb{P}^3$, with $\mathrm{mc}(C) \geqslant s$. Interestingly, it is not known in general. Note that for divisors the same question is an elementary exercise. This motivates the first problem we propose to study.

Problem 1.1. *Find new, lower and upper, bounds on the numbers $d(s)$. Additionally, introduce and study numbers $d(s,m)$ corresponding to cycles which pass through s general points and have there multiplicity at least m (thus $d(s) = d(s,1)$ for all $s \geqslant 1$).*

We expect that this is a hard problem. Introducing the multiplicities is hard already in the setting of divisors. On the other hand, even partial results in this direction would mark a landscape, which seems unexplored to large extend so far.

The volume of a divisor can be thought of as an asymptotic version of the dimension of the linear system. An important advantage of the volume when compared with the dimension of the linear series is that the volume depends only on the numerical equivalence class of the underlying divisor, whereas the dimension depends in general on the linear equivalence class. Motivated by the concept of the volume of divisors Lehman [20] and Xiao [30] introduced the notion of the *mobility.* For cycles of dimension k the mobility is defined as follows

$$\mathrm{mob}(\alpha) = \limsup_{m \to \infty} \frac{\mathrm{mc}(m\alpha)}{m^{n/(n-k)}/n!}.$$

Thus if the codimension of α is 1, then $\mathrm{mob}(\alpha) = \mathrm{vol}(\alpha)$, i.e., we recover the volume of the divisor. For cycles of higher codimension mobility parallels essential properties of the volume. In particular, Lehman [20, Theorem 1.2] showed that the mobility is a continuous $\frac{n}{n-k}$-homogeneous function on the space of k-cycles $N_k(X)$.

Whereas, at least for big and nef divisors, the count of the volume reduces to the simple computation of the self-intersection number of the divisor, the picture is much more mysterious for cycles of higher codimension. Suffices it to say that already the number $\mathrm{mob}(\ell)$ for a line $\ell \subseteq \mathbb{P}^3$ is not known! The best estimate up to date $1 \leqslant \mathrm{mob}(\ell) \leqslant 3.54$ is due to Lehmann [20]. Establishing the exact value of $\mathrm{mob}(\ell)$ is related to the enumerative problem mentioned above: what is the minimal degree of a curve in $\mathbb{P}^3$ passing through s general points? It is expected that the answer is governed by the values of $(6s)^{\frac{2}{3}}$ but this is not known. Accordingly, it is conjectured that the actual value is $\mathrm{mob}(\ell) = 1$. All this motivates he second problem we put forward.

Problem 1.2. *Improve the upper bound on the number* $\mathrm{mob}(\ell)$*. Or even show* $\mathrm{mob}(\ell) = 1$*.*

In fact, it is expected that complete intersection curves are subject to the following conjectural statement.

Conjecture 1.3 (Lehmann)**.** *Let X be a projective variety of dimension n and let L be an ample line bundle on X. Then for all k in the range $0 < k < n$ there is*

$$\mathrm{mob}(L^{n-k}) = \mathrm{vol}(L).$$

It is natural to wonder what happens for cycles which are not complete intersections. Thus we will be also interested in the number $\mathrm{mob}(T)$ for T, a twisted cubic. This is the first interesting example of a non-complete intersection curve in $\mathbb{P}^3$. As in the case of Problem 1.1 for points of higher multiplicity, we were not able to trace down any results in this direction in the literature. Note, that almost certainly it is not $\mathrm{mob}(T) = 3\,\mathrm{mob}(\ell)$.

Both problems put forward so far concern postulation on codimension 2 cycles. However, such cycles can be used themselves to formulate postulation problems on divisors. The best known examples of such problems concern points in $\mathbb{P}^2$. We mention here two most prominent conjectures in the field. Nagata's Conjecture, going back to 1959, is analogous to Problem 1.1. It predicts that the minimal degree $n(s,m)$ of a plane curve passing through $s \geqslant 10$ general points with multiplicity at least m is subject to the following inequality

$$n(s,m) > m\sqrt{s}.$$

The Segre-Harbourne-Gimigliano-Hirschowitz Conjecture (SHGH for short) is somewhat more geometrical in nature. Since this is not our objective to study this conjecture, we provide a somewhat simplified formulation, basically due to Segre. Given s general points in $\mathbb{P}^2$, the linear system of curves passing through these points with fixed multiplicities is either non-special (i.e. its dimension is provided by a simple calculation of Hilbert polynomials) or the system contains a non-reduced base curve.

Passing to the postulation of higher dimensional cycles, it is natural to replace, in the first step, points by flats, i.e., linear subspaces in projective spaces. The landscape here is much less explored. It has been proved by Hartshorne and Hirschowitz in [15] that a general collection of lines in $\mathbb{P}^N$ imposes independent conditions on forms of any degree d. In other words, their result shows that finding the minimal degree of a hypersurfaces containing s general lines in $\mathbb{P}^N$ boils down to an easy calculation of Hilbert functions. Allowing only one fat line introduces a lot of complications, see our work with Bauer, Di Rocco, Schmitz and Szemberg [3] and the work of Aladpoosh [1]. To the best of our knowledge the postulation problem for planes in $\mathbb{P}^4$ is not solved. An obvious, new complication, arising here, is that planes in $\mathbb{P}^4$ are not disjoint. This gets even more involved for unions of codimension 2 flats in higher dimensional projective spaces as not only the flats intersect each other but their intersections interact with other intersections as well. This makes our next problem pretty challenging.

Problem 1.4. *Study the postulation problem for unions of general flats of codimension* 2 *in projective spaces.*

We expect that there is no simple analogy of the Hartshorne-Hirschowitz result and that some special linear systems can be identified in this way. Such systems might in turn prove quite useful, for example, in the area of birational geometry in the spirit of [7].

As already indicated in Problem 1.1 there is an additional difficulty when we consider postulation of multiple (fat) points. This is especially transparent in the case of Nagata's Conjecture. For reduced points, its prediction is trivial and can be proved by linear algebra. Introducing singularities changes the problem dramatically.

Bocci and Chiantini initiated in [5] a new line of investigation. They considered ideals of sets Z of *arbitrary* points in $\mathbb{P}^2$ and asked how the assumption that the difference between the minimal degree of a curve passing through Z and that passing doubly through Z is minimal (i.e. equal 1) influences the geometry of Z. They showed that the constrain is serious and the points in Z either form what is now known as a star configuration (see [10]) or they are all collinear. This result has been generalised for flats of codimension 2 by Janssen [18] and Haghighi, Zaman Fashami and Szemberg [8] under the additional assumption that the union of studied flats is an arithmetically Cohen-Macaulay variety. Already in $\mathbb{P}^3$ it is natural to study the same problem for connected curves. This is exactly the next and the last problem we want to spell out.

Problem 1.5. *Let $C \subseteq \mathbb{P}^3$ be a smooth (or just connected) curve such that the minimal degree of a generator of its ideal $I(C)$ is α and the minimal degree of the second symbolic power $I(C)^{(2)}$ is $\alpha + 1$. Show that then C is contained in a hyperplane.*

In a sense, this is the most challenging problem, because there is no analogy in the literature to build on. Note that proving the reverse implication in Problem 1.5 is elementary.

2. Significance

On smooth varieties (for example in projective spaces) codimension 1 subvarieties (divisors) are in one-to-one correspondence with sections of line bundles.

Higher codimension cycles can be obtained by intersecting divisors. It is natural to wonder if one obtains all cycles in this way. A classical result along these lines is due to Noether and Lefschetz.

Theorem 2.1 (Noether-Lefschetz)**.** *Let S be a general surface of degree $d \geqslant 4$ in $\mathbb{P}^3$. Then any curve C contained in S is a complete intersection, i.e., it is cut out by another surface $S' \subset \mathbb{P}^3$.*

This result, proved in the final form only in 1985 by Griffiths and Harris [11], stimulated a lot of research on higher codimension cycles. In particular, Griffiths and Harris raised the following interesting question.

Conjecture 2.2 (Degree Conjecture)**.** *Let $X \subset \mathbb{P}^4$ be a general threefold of degree $d \geqslant 6$ and let C be a curve contained in X. Is then the degree of C a multiple of d?*

This conjecture would easily follow if the following statement, analogous to Theorem 2.1, would hold: Any curve as in the Degree Conjecture is the intersection of X with some surface $S \subset \mathbb{P}^4$. Voisin showed in 1988 that this statement is false. Of course her result shows that an even more naive hope that C might be an intersection of three threefolds in $\mathbb{P}^4$ fails, but it was already known at the point when the conjecture was stated. Interestingly, the Degree Conjecture is still open. Some strong evidence supporting the Conjecture has been provided recently by Kollár. This developments and names appearing here prove that *studying codimension 2 cycles in algebraic varieties is an important and hard problem in contemporary algebraic geometry.*

We have just seen that one cannot expect in general that cycles of high codimension come up as intersections of cycles of lower codimension. There is another possibility to extend the relation between effective divisors and sections of line bundles. To this end one can study sections in higher rank locally free sheaves, i.e., vector bundles. For codimension 2 cycles it is natural to look at vector bundles of rank 2. Whereas this relation again fails in general, even for cycles in projective spaces, there is a very challenging conjecture due to Hartshorne [19, Conjecture 3.2.8].

Conjecture 2.3 (Hartshorne)**.** *Let $X \subset \mathbb{P}^N$ be a smooth, irreducible subvariety of codimension 2. For $N \geqslant 6$ it follows that X is a complete intersection.*

This conjecture has stimulated a lot of research on codimension 2 subvarieties. The proof with available methods seems out of reach. However one can hope that assuming the Conjecture (possibly in a slightly stronger form) one can obtain progress in Problem 1.4 precisely in the cases where iterated intersections between involved flats become messy. It is also possible that progress on Problem 1.4 can be obtained without assuming Hartshorne's conjecture, but studying instead, from the perspective of our problem, its consequences. By this we mean the general yoga that small codimension subvarieties in projective spaces behave cohomologically like complete intersections and it is the cohomology we are interested in.

In the formulation provided above, Conjecture 2.3 is equivalent to the following statement, see [14].

Conjecture 2.4 (Hartshorne's splitting conjecture). *Any rank* 2 *vector bundle on* $\mathbb{P}^N$ *with* $N \geqslant 6$ *splits as a direct sum of line bundles.*

It is well-known that this conjecture fails in lower dimensional projective spaces, perhaps the most prominent example being that of Horrocks-Mumford bundle [17]. In any case, it is clear that vector bundle techniques come naturally into the picture, when higher codimension subvarieties are considered. It is also worth to mention the following two facts. A well-known splitting criterion due to Horrocks asserts that a vector bundle E on the projective space $\mathbb{P}^N$ splits if and only if $H^i(\mathbb{P}^N, E \otimes \mathcal{O}_{\mathbb{P}^N}(d)) = 0$ for all d and all $0 < i < N$. An improvement, due to Evans and Griffiths, asserts that it suffices to check the splitting for $i < \min\{N, \mathrm{rk}(E)\}$. Hence for the rank 2 case (i.e. codimension 2 cycles) it suffices to check the vanishing $H^1(\mathbb{P}^N, E) = 0$ *for all* rank 2 vector bundles E.

Among codimension 2 subvarieties in projective spaces, curves in $\mathbb{P}^3$ play a special role. This is due to the fact that, by an elementary classical theorem, any smooth curve can be embedded into the projective space of dimension 3 (very much as any smooth variety of dimension n can be embedded into $\mathbb{P}^{2n+1}$). Thus one cannot expect that curves in $\mathbb{P}^3$ enjoy any special properties, contrary e.g. to surfaces in $\mathbb{P}^4$ as only special surfaces can be embedded into $\mathbb{P}^4$. On the other hand there are certain constraints relating the genus and the degree of space curves, worked out by Gruson and Peskine [13]. Further refinements in term of the ideal $I(C)$ defining C have been obtained by Gruson, Lazarsfeld and Peskine in [12]. The ideals with the simplest structure are those generated by a regular sequence. For codimension two subvarieties there is a striking result due to Gaeta [9] to the effect that any arithmetically Cohen-Macaulay (ACM) subvariety X is minimally linked to a complete intersection subvariety, see [24] for the modern treatment of liason theory and much more. The study of non-ACM subschemes of projective spaces is an area of active research to which we hope to attract even more attention, see e.g. [23] for a nice introduction to this circle of ideas. In particular, Problem 1.5 is stated without assuming that the considered curve is an ACM-subscheme. Note that this assumption was inevitable in the approach taken on by Janssen [18] and in the generalisations proved in [8]. The reason is the application of the Hilbert-Burch theorem, which gives a useful description of the defining ideal of an arithmetically Cohen-Macaulay subvariety of codimension 2 in a projective space (or more generally: in a smooth projective variety). Of course, dealing with a connected curve C provides much stronger tools, including the normal sheaf and the infinitesimal neighbourhoods of C. In this context we quote the following illuminating result due to Ran [26].

Theorem 2.5 (Ran). *Let X be a degree d, locally complete intersection, codimension* 2 *subvariety in* $\mathbb{P}^{n+2}$. *Let N_X be the normal sheaf of X and let* $\bigwedge^2 N_X = \mathcal{O}_X(a)$. *If*

either $a \geqslant d/n + n$*;*
or $d \leqslant n$*,*

then X *is a complete intersection.*

3. Possible path of research

As already stated in Section 1, even the conjectural picture of the positivity of codimension 2 subvarieties is far from being well understood. The most striking manifestation of how little is known is the problem to determine the mobility of the class of a line in $\mathbb{P}^3$. Therefor it seems reasonable to focus on the phenomenological part of the research, namely to explore connected codimension 2-cycles given by the intersections of hyperplane arrangements in projective spaces, e.g., defined as skeletons of Fermat arrangements [27] and thus collecting data before approaching the more general research problems. Such a strategy turned out to be quite successful in different areas of studies, for instance in questions concerning the existence of unexpected subvarieties and in the problems revolving around the containment conjectures. Skeletons of Fermat arrangements are an interesting testing ground for various hypotheses. In a sense, they resemble the so called star configurations, see [10], which are close to general arrangements on the one hand and special enough to exhibit interesting patterns on the other hand. Such patterns are particularly reach in algebraic objects connected to both classes varieties (i.e. skeletons of Fermat and star configurations), starting with their defining equations and various powers of them.

The proposed path of research consists of the following four questions:

(1) Find minimal degree $d(s)$ of a connected curve $C \subset \mathbb{P}^3$ passing though s general points.
(2) Improve an upper bound on the mobility of the class of a line in $\mathbb{P}^3$.
(3) Study postulation problems for unions of (general) flats of codimension 2 in $\mathbb{P}^N$.
(4) Explore the fattening effect for connected curves in $\mathbb{P}^3$.

These problems are clearly divided in two groups: one concerning curves and the other higher dimensional subvarieties. In both cases we propose to test working hypothesis with symbolic algebra programs. This approach is well established in this area, see for example the Crelle work of Holme and Schneider [16].

4. Related methods

The Castelnuovo-Mumford regularity (CM-regularity for short) is a fundamental invariant in commutative algebra and algebraic geometry. It has (informally) appeared in works of Guido Castelnuovo, long before it has been formally defined. Castelnuovo studied linear series on curves in $\mathbb{P}^3$ cut out by surfaces of fixed degree. In modern terms, he was interested in determining the dimension of the vector space of global sections $H^0(C, \mathcal{O}_C(d))$ for a curve C in $\mathbb{P}^3$. For d large enough, this dimension is, by the Riemann-Roch theorem, equal to $cd - g(C)$, where c is the degree of C and $g(C)$ is the genus. The CM-regularity of $I(C)$ makes sense of the phrase "large enough", turning it to an effective statement. Along these lines, there is the following interesting problem due to Eisenbud and Goto.

Conjecture 4.1 (Regularity Conjecture)**.** *For a non-reduced and connected in codimension* 1 *subscheme* $X \subset \mathbb{P}^N$*, there is*

$$\operatorname{reg}(I(X)) \leqslant \deg(X) - \operatorname{codim}(X) + 1.$$

This conjecture has been recently shown to fail spectacularly by McCullough and Peeva [22]. We suggest to explore methods from [22] in order to deal with problems listed in the previous section. In particular, it seems feasible to couple these new methods with those of Bertram, Ein and Lazarsfeld from [4]. One of results of that paper seems of direct interest.

Theorem 4.2 (Bertram-Ein-Lazarsfeld)**.** *Let X be a subvariety in projective space $\mathbb{P}^N$ defined scheme-theoretically by equations of degree $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_m$ (i.e. the equations in the minimal set of generators of the ideal $I(X)$ have these degrees). Then X is*

$$(d_1 + \ldots + d_e - e + 1) - \textit{regular},$$

where $e = \operatorname{codim}(X)$*.*

This result is of course in general far from the Regularity Conjecture but, on the other hand, it can be very useful towards solving Problem 1.5. Even if the Problem cannot be solved in the full generality, it could be more tractable, in particular with Theorem 4.2, for curves defined by quadratic equations.

In order to attack Problem 1.1, one of possible approaches is to study properties of the scheme $D_{m;d,g}$ which parametrizes couples of the form (M, C), where M is a finite scheme of length m and C is a curve of degree d and genus g such that $M \subset C$. If we denote by $m(d, g)$ the maximal number of general points in $\mathbb{P}^3$ with the property that there is a curve of degree d and genus g passing through them, and by $H_{d,g}$ the Hilbert scheme of connected curves C of degree d and genus g in $\mathbb{P}^3$, then we have

$$m(d, g) \leqslant \left[\frac{1}{2}\dim H_{d,g}\right].$$

However, this bound is not sharp in general. If we take $g = (d-1)(d-2)/2$, then curves in $H_{d,g}$ are planar, so $m(d, g) = 3 < \frac{1}{2}H_{d,g}$. In order to cope with this problem, Perrin introduced h^0-stability for locally free sheaves of rank 2 – a model example here is the normal sheaf N_C of a curve $C \subset \mathbb{P}^3$. In this case, we say that N_C is h^0-semi-stable if and only if for every invertible subsheaf L of N_C one has $h^0(L) \leqslant \frac{1}{2}h^0(N_C)$. Building upon the work of Perrin [25], Vogt in [28] has shown recently that there exists a Brill-Noether curve C of degree d and genus g in $\mathbb{P}^3$ passing through a maximum of $2d$ general points apart of a short list of exceptions. Let us recall that a Brill-Noether curve is a member of a unique irreducible component of the Kontsevich space $M_g(\mathbb{P}^3, d)$ which dominates M_g and whose general member is a non-degenerate immersion of a smooth curve. This shows that there is still a room for improvements, see [2].

The postulation problem for codimension 2 subvarieties is directly related to the geometry of Veneroni maps. These are certain birational transformations of higher dimensional projective spaces whose base loci consist of unions of general codimension 2 linear subspaces. Thus the whole machinery of Cremona groups becomes relevant. A special feature of codimension 2 flats is that this approach is neatly connected to free resolutions via the Hilbert-Burch Theorem.

Volume of divisors played a crucial role in Witt-Nyström [29] approach to the duality, postulated by Boucksom-Demailly-Păun-Peternell [6], between the cones $\overline{\mathrm{Eff}}(X)$ of pseudo-effective divisors and the cone of movable curves $\mathrm{Mov}(X)$, under the assumption that X is a projective variety. The key idea was to introduce and use the following transcendental Morse inequality.

Theorem 4.3 (Morse inequality). *Let α and β be two nef classes on a projective manifold X of dimension n, then*

$$\mathrm{vol}(\alpha-\beta) \geqslant \alpha^n - n(\alpha^{n-1}\beta).$$

If we focus on big divisors on a projective variety X, then Khovanskii-Teissier inequality asserts that for big and nef divisors A, B and a movable curve class β one has

$$n(A\cdot B^{n-1})(B\cdot\beta) \geqslant B^n(A\cdot\beta).$$

It is expected that inequalities of this kind hold for cycles of higher codimension. It seems that various generalizations of the Morse-type inequalities to codimension 2-cycles are possible. Towards this direction, one can be guided by ideas and methods introduced by Lehmann and Xiao in [21]. They stated some very natural generalization of the Morse inequalities to convex bodies and support functions. This provides another way of thinking about the volume function based on purely combinatorial and convex geometry methods. This example manifests a natural bridging that can be observed very recently in literature of the subject: highly non-trivial results in algebraic geometry are generalized to the framework of the combinatorial world. It is reasonable to expect that such links are lurking behind in the world of cycles of higher codimension and the mobility function.

Finally, we expect that the mobility function flagged in Section 1, should shed new light on relations expected for codimension 2 cycles. A particularly nice case, we have in mind, is that of surfaces in 4-dimensional projective space (or more generally in 4-dimensional varieties). The study of arrangements of planes in $\mathbb{P}^4$ should be viewed as the degenerate case of this situation. An important, additional tool which one has here at the disposal is the self-intersection formula.

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Analytic and Algebraic Geometry 4
Łódź University Press 2022, 175–183
DOI: https://doi.org/10.18778/8331-092-3.14

AN ESTIMATION OF THE JUMP OF THE MILNOR NUMBER OF PLANE CURVE SINGULARITIES

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ABSTRACT. The jump of the Milnor number of an isolated singularity f_0 is the minimal non-zero difference between the Milnor numbers of f_0 and one of its deformations f_s. We estimate the jump using the Enriques diagram of f_0.

1. INTRODUCTION

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an **isolated singularity**, i.e. a function germ for which there exists a representative $\widehat{f_0} : U \to \mathbb{C}$ of f_0, holomorphic in an open neighbourhood U of the point $0 \in \mathbb{C}^n$ such that $\widehat{f_0}(0) = 0$, $\nabla \widehat{f_0}(0) = 0$, $\nabla \widehat{f_0}(z) \neq 0$ for $z \in U \setminus \{0\}$. We put $\nabla f := \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}\right)$. In the sequel a singularity means an isolated singularity.

A **deformation of a singularity** f_0 is the germ of a holomorphic function $f = f(s, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that

(1) $f(0, z) = f_0(z)$,
(2) $f(s, 0) = 0$.

The deformation $f(s, z)$ of the singularity f_0 will also be treated as a family (f_s) of function germs, taking $f_s(z) := f(s, z)$. Since f_0 is an isolated singularity, f_s for sufficiently small s also has isolated singularities near 0 ([GLS06] Theorem 2.6 I). Hence, for sufficiently small s one can define the **Milnor number of** f_s

$$\mu_s := \mu(f_s) = \dim_{\mathbb{C}} \mathcal{O}_n / (\nabla f_s),$$

2010 *Mathematics Subject Classification.* 32S05.
Key words and phrases. singularity, Milnor number, jump of the Milnor number, deformation.

where $\mathcal{O}_n$ is the ring of holomorphic function germs at 0, and (∇f_s) is the ideal in $\mathcal{O}_n$ generated by $\frac{\partial f_s}{\partial z_1}, \dots, \frac{\partial f_s}{\partial z_n}$.

Since the Milnor number is upper semi-continuous in the Zariski topology in families of singularities ([GLS06], Theorem 2.6 I and Proposition 2.57 II), there exists an open neighbourhood S of the point 0 such that

(1) $\mu_s = \text{const.}$ for $s \in S \setminus \{0\}$,
(2) $\mu_0 \geq \mu_s$ for $s \in S$.

The constant difference $\mu_0 - \mu_s$ (for $s \neq 0$) will be called **the jump of the deformation** (f_s) and denoted by $\lambda((f_s))$. The smallest non-zero value among all the jumps of deformations of the singularity f_0 will be called **the jump of the Milnor number of the singularity** f_0 and denoted by $\lambda(f_0)$.

From now on, we will consider only plane curve singularities $f_0 : \left(\mathbb{C}^2, 0\right) \to (\mathbb{C}, 0)$.

The first general result concerning the jump of the Milnor number was obtained by Sabir Gusein-Zade([GZ93]), who proved that there exist singularities f_0 for which $\lambda(f_0) > 1$ and gave some sufficient conditions for which $\lambda(f_0) = 1$. These conditions are in terms of branches and the resolution process of plane surve singularities. In particular from his result follows $\lambda(f_0) = 1$ for irreducible plane singularities.

S. Brzostowski, T. Krasiński and J. Walewska in [BKW21] proved that for the special reducible singularities $f_0^n(x, y) = x^n + y^n$, $n \geq 2$, we have $\lambda(f_0) = \left[\frac{n}{2}\right]$. Determining the jump of a singularity is a difficult task because it is not a topological invariant ([BK14], [dPW95] Section 7.3). For specific classes of deformations i.e. for non-degenerated deformations (it means each element of the family f_s is a non-degenerated singularity in the Kouchnirenko sense [Kou76]) the jump problem was considered in [Bod07], [Wal13], [BKW21], [KW19].

One of the results of this article is an extension of the Gusein-Zade result ([GZ93]) by giving a next sufficient condition for plane curve singularities f_0 under which $\lambda(f_0) = 1$ (Theorem 4.1). Our methods give also the Gusein-Zade conditions.

The second result of the article (Theorem 3.1) is an estimation (from above) of $\lambda(f_0)$ in terms of branches and the resolution process of plane curve singularities using previous result concerning the jump in the case f_0 is a homogenous (quasi-homogenous) singularities ([Zak17],[Zak]).

We obtain both above results in the framework of narrower class of deformations - linear deformations of the form $f_0 + sg$, where g is a holomorphic function in the neighbourhood of 0 such that $g(0) = 0$. We will denote the jump of f_0 for this class of deformations by $\lambda^{lin}(f_0)$. Of course $\lambda(f_0) \leq \lambda^{lin}(f_0)$ and so any estimation of $\lambda^{lin}(f_0)$ from above is automatically an estimation of $\lambda(f_0)$,

To get this formula the **Enriques diagrams** will be used. To any singularity we can assign a weighted Enriques diagram (D, ν) which represents the whole resolution process of this singularity ([CA00] Chapter 3.9). It is a tree with two types of edges and a weight function $\nu : D \to \mathbb{Z}$ on vertices of the diagram. M. Alberich-Carraminñana and J. Roé ([ACR05] Theorem 1.3, Remark 1.4) gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. It means that one singularity is a linear deformation of another. They used a wider class of Enriques diagrams, so-called abstract Enriques diagrams, which are described in Section 2.

In Section 3 we estimate the jump $\lambda(f_0)$ in terms of its Enriques diagram and in Section 4 we give sufficient conditions under which $\lambda^{lin}(f_0) = 1$ and consequently $\lambda(f_0) = 1$.

2. Enriques diagrams

Information about abstract Enriques diagrams can be found in [ACR05] and [KP99]. Moreover in my previous paper [Zak17], in which I gave the estimation of $\lambda^{lin}(f_0)$ for homogeneous singularities, abstract Enriques diagrams are described in more details with examples.

Definition 2.1. *An* ***abstract Enriques diagram*** *(in short an Enriques diagram) is a rooted tree D with a binary relation between vertices, called proximity, which satisfies:*

(1) *The root is proximate to no vertex.*
(2) *Every vertex that is not the root is proximate to its immediate predecessor.*
(3) *No vertex is proximate to more than two vertices.*
(4) *If a vertex Q is proximate to two vertices, then one of them is the immediate predecessor of Q and this is proximate to the other.*
(5) *Given two vertices P, Q with Q proximate to P, there is at most one vertex proximate to both of them.*

The fact that Q is proximate to P we will denote by $Q \to P$. The vertices which are proximate to two points are called **satellite**, the other vertices (except the root) are called **free**. The vertex is **final** if has no successor. To show graphically the proximity relation, Enriques diagrams are drawn according to the following rules:

(1) If Q is a free successor of P, then the edge going from P to Q is smooth and curved and, if P is not the root, it has at P the same tangent as the edge joining P to its predecessor.
(2) The sequence of edges connecting a maximal succession of vertices proximate to the same vertex P are shaped into a line segment, orthogonal to the edge joining P to the first vertex of the sequence.

The example of an abstract Enriques diagram is shown in Figure 1.

We will now introduce few basic notations that are needed in the sequel.

A **weight function** of an abstract Enriques diagram D is any function $\nu : D \to \mathbb{Z}$ defined on vertices of D. A pair (D, ν), where D is an abstract Enriques diagram

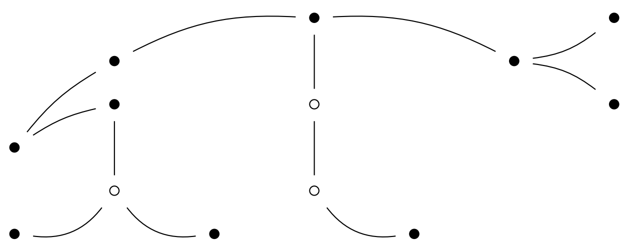

FIGURE 1. The abstract Enriques diagram. Satellite vertices are marked in white

and ν its weight function, is called a **weighted Enriques diagram**. A **consistent Enriques diagram** is a weighted Enriques diagram such that for all $P \in D$

$$\nu(P) \geq \sum_{Q \to P} \nu(Q). \tag{1}$$

A **complete Enriques diagram** is a weighted Enriques diagram such that for all non-final $P \in D$ the equality in (1) holds and for all final $P \in D$ it is a free vertex with weight 1 not proximate to another free vertex with weight 1. To the weight function ν of a weighted diagram D we associate a **system of values**, which is another map $\mathrm{ord}_\nu : D \to \mathbb{Z}$, defined recursively as

$$\mathrm{ord}_\nu(P) := \begin{cases} \nu(P), & \text{if } P \text{ is the root,} \\ \nu(P) + \sum\limits_{P \to Q} \mathrm{ord}_\nu(Q), & \text{otherwise.} \end{cases}$$

For any consistent (D, ν) we define **the Milnor number of** (D, ν) by

$$\mu((D, \nu)) := \sum_{P \in D} \nu(P)(\nu(P) - 1) + 1 - r_D,$$

where $r_D := \sum_{P \in D} r_D(P)$, $r_D(P) := \Big(\nu(P) - \sum_{Q \to P} \nu(Q)\Big)$ for every $P \in D$.

A **subdiagram** of an abstract Enriques diagram D is a subtree $D_0 \subset D$ with the same proximity relation such that if $Q \in D_0$ then its predecessor belongs to D_0.

In the class of weighted Enriques diagrams, we introduce equivalence relation. We say that weighted diagrams (D, ν) and (D', ν') are **equivalent** if they differ at most in free vertices of weight 1. The equivalence class of (D, ν) is denoted by $[(D, \nu)]$ and called the **type** of (D, ν). Of course, the Milnor number is constant in the class $[(D, \nu)]$.

A **minimal Enriques diagram** is a consistent Enriques diagram (D, ν) with:

(1) no free vertices of weight 0,
(2) no free vertices of weight 1 except for these such $P \in D$ for which there exists a satellite vertex $Q \in D$ satisfying $Q \to P$.

It is easy to see ([Zak17], Theorem 2.12) that

Theorem 2.2. *Let (D, ν) be a consistent weighted diagram. There exists exactly one minimal diagram which belongs to $[(D, \nu)]$.*

The theory of Enriques diagrams has its roots in the theory of plane curve singularities. The embedded resolution of a plane curve singularity using blow-ups can be explicitly presented as a complete Enriques diagram. A precise description can be found in [CA00] Chapter 3.8 and Chapter 3.9. Two plane curve singularities are topologically equivalent if and only if their Enriques diagrams are isomorphic (as graphs). For the Enriques diagram (D, ν) of a plane curve singularity f_0, the weight function represents the orders of the consecutive proper transforms of f_0 while the system of values – the orders of the total transforms. The number $r_D(P)$ equals to the number of branches at P of a proper transform of f_0 for which next blow-up at P "resolve" these branches. Hence, r_D is the number of branches of f_0. Moreover (D, ν) is complete. We need only the next fact which easily follows from these results.

Theorem 2.3 ([CA00] Theorem 3.8.6)**.** *There exists a bijection between minimal Enriques diagrams and topological types of singularities.*

In the paper [ACR05], M. Alberich-Carramiñana and J. Roé gave a necessary and sufficient condition for two Enriques diagrams of singularities to be linear adjacent. This is the key result we will use in the sequel. First we give definitions.

Definition 2.4. *Let (D, ν) and (D', ν') be weighted Enriques diagrams, with (D', ν') consistent. We will write $(D', \nu') \geq (D, \nu)$ when there exist isomorphic subdiagrams $D_0 \subset D$, $D'_0 \subset D'$ with an isomorphism (that preserves proximity relations)*

$$i : D_0 \to D'_0$$

such that the new weight function $\kappa : D \to \mathbb{Z}$ for D, defined by

$$\kappa(P) := \begin{cases} \nu'(i(P)), & P \in D_0 \\ 0, & P \notin D_0 \end{cases}$$

satisfies

$$\mathrm{ord}_\nu(P) \leq \mathrm{ord}_\kappa(P)$$

for any $P \in D$.

Definition 2.5. *Let $[(D, \nu)]$ and $[(\widetilde{D}, \widetilde{\nu})]$ be types of Enriques diagrams. $[(\widetilde{D}, \widetilde{\nu})]$ is **linear adjacent** to $[(D, \nu)]$ if there exists a consistent Enriques diagram $(D', \nu') \in [(\widetilde{D}, \widetilde{\nu})]$ such that $(D', \nu') \geq (D_{min}, \nu_{min})$, where (D_{min}, ν_{min}) is the minimal diagram of type $[(D, \nu)]$.*

Theorem 2.6 ([ACR05] Theorem 1.3 and Remark 1.4)**.** *Let $[(D, \nu)]$ and $[(\widetilde{D}, \widetilde{\nu})]$ be types of consistent Enriques diagrams. The following conditions are equivalent:*

(1) $[(\widetilde{D}, \widetilde{\nu})]$ *is linear adjacent to* $[(D, \nu)]$.

(2) *For every singularity* f_0 *whose Enriques diagram belongs to* $[(\widetilde{D}, \widetilde{\nu})]$*, there exists a linear deformation* (f_s) *of* f_0 *such that the Enriques diagram of a generic element* f_s *belongs to* $[(D, \nu)]$.
(3) *There exists a singularity* f_0 *whose Enriques diagram belongs to* $[(\widetilde{D}, \widetilde{\nu})]$ *and a linear deformation* (f_s) *of* f_0 *such that the Enriques diagram of a generic element* f_s *belongs to* $[(D, \nu)]$.

This theorem was also formulated using prime divisors by J. Fernández de Bobadilla, M. Pe Pereira and P. Popescu-Pampu in Theorem 3.25 ([dBPPP17]).

Theorems 2.3 and 2.6 imply the following corollary:

Corollary 2.7. $\lambda^{lin}(f_0)$ *is a topological invariant.*

3. Estimation of the jump of the Milnor number for linear deformation

Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be a singularity and (D, ν) its minimal Enriques diagram. The jump of the Milnor number for linear deformation can be estimated as follows.

Theorem 3.1.

$$\lambda^{lin}(f_0) \leq \min\{l(P) : P \text{ - } \textit{a leaf in } D\},$$

where l(P) can be read from the table

$\nu(P)$ \ *vertex P*	*root*	*free*	*satellite*
1	-	-	*1*
2	*1*	*1*	*2*
≥ 3	$\nu(P) - 2$	$\nu(P) - 1$	$\nu(P)$

Proof. Let $D_L = \{L_1, \dots, L_m\}$ be a set of leaves of (D, ν). For each $i = 1, \dots, m$ we will define the diagram (E_i, λ_i) by a modification of (D, ν), for which the difference of the Milnor number of (E_i, λ_i) and (D, ν) is equal to $l(L_i)$. If $\nu(L_i) = 1$ we remove only the L_i from (D, ν) and this will be (E_i, λ_i). If $\nu(L_i) = 2$ and L_i is the root, then E_i will have only one vertex with weight 1. If $\nu(L_i) = 2$ and L_i is not a root we change the weight of L_i to 1 and add one additional satellite vertex W with weight 1, so that $W \to L_i$ (Figure 2(a)) and this will be (E_i, λ_i).

If $\nu(L_i) \geq 3$ we change the weight of L_i to $\nu(L_i) - 1$ and add new vertices free U and satellite $W_1, \dots, W_{\nu(L_i)-3}$ (if $\nu(L_i) = 3$ there is no W_j vertices), all proximate to L_i. The weight of new vertices are: $\lambda_i(U) = 2$, $\lambda_i(W_j) = 1$ (for

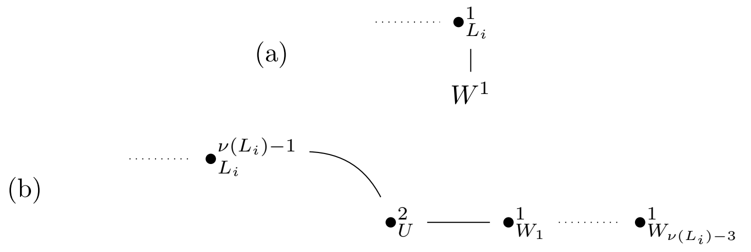

FIGURE 2. The Enriques diagram (E, λ)

$j = 1, \ldots, \nu(L_i) - 3)$. The proximity relation between new vertices is

$$\begin{aligned}
&W_{\nu(L_i)-3} \to W_{\nu(L_i)-4}, L_i \\
&\ldots \\
&W_2 \to W_1, L_i \\
&W_1 \to U, L_i \\
&U \to L_i,
\end{aligned}$$

see Figure 2(b).

It is easy to check that (E_i, λ_i) is a minimal (and hence consistent) diagram and that $(E_i, \lambda_i) \notin [(D, \nu)]$. From the above detailed description of (E_i, λ_i) we easily show that $[(D, \nu)]$ is linear adjacent to $[(E_i, \lambda_i)]$.

Now we may compute the Milnor number of (E_i, λ_i). It is easy to notice that

$$r_{E_i} = \begin{cases} r_D + 1, & \text{if } \nu(L_i) = 1 \\ r_D - 1, & \text{if } \nu(L_i) = 2, L_i \text{ is a root} \\ r_D - 2 + w_{L_i}, & \text{if } \nu(L_i) = 2, L_i \text{ is not a root} \\ r_D - d + 2 + w_{L_i}, & \text{if } \nu(L_i) \geq 3 \end{cases},$$

where w_{L_i} is a number of vertices to which L_i is proximate to. Then we get $\mu\left((E_i, \lambda_i)\right) = \mu\left((D, \nu)\right) - l(L_i)$. Since this formula is true for every $i = 1, \ldots, m$ and from Theorem 2.6 we get $\lambda^{lin}(f_0) \leq \min\limits_{i=1,\ldots,m} l(L_i)$. □

Hence, we get a corollary for the general jump $\lambda(f_0)$.

Corollary 3.2.

$$\lambda(f_0) \leq \min\{l(P) : P \text{ - } \textit{a leaf in } D\}.$$

Remark 3.3. *In Theorem 3.1 the estimation cannot be replace by an equality. Let consider the singularity $f_0(x, y) = x^8 + y^5$, its minimal Enriques diagram (D, ν) is shown in Figure 3. It is easy to check that $[(D, \nu)]$ is linear adjacent to $[(E, \lambda)]$ shown in Figure 4. Since $\mu((D, \nu)) - \mu((E, \lambda)) = 22 - 21 = 1$, we have $\lambda^{lin}(f_0) = 1$. On the other hand from Theorem 3.1 we get only such an estimation $\lambda^{lin}(f_0) \leq 3 - 1 = 2$.*

$$\bullet^{5} \quad \frown \quad \bullet^{3}$$

Figure 3. Minimal Enriques diagram of $f_0(x, y) = x^8 + y^5$

$$\bullet^{4} \quad \frown \quad \bullet^{4}$$

Figure 4. Enrqiues diagram (E, λ)

4. Singularities with the Milnor number 1

In this Section we gave next sufficient conditions for plane curve singularities f_0 under which $\lambda(f_0) = 1$. In [GZ93] Gusain-Zade proved that if for a singularity f_0 there exists a maximal exceptional divisor which intersects no more than three other components of the total preimage of the curve $f_0 = 0$, then $\lambda(f_0) = 1$. In terms of Enriques diagram this condition is equivalent to the first three conditions of the next theorem. We add the next one condition.

Theorem 4.1. *Let $f_0 : \mathbb{C}^2 \to \mathbb{C}$ be a singularity and (D, ν) its minimal diagram. If one of below conditions is true:*

1. *there exists a leaf $P \in D$ such that P is satellite with weight 1,*
2. *the diagram D contains only root with weight 2,*
3. *there exists a leaf $P \in D$ such that P is free with weight 2,*
4. *$\nu(R_D) \geq 2 + \sum_{P \to R_D} \nu(P)$ and there exists $P \in D$ such that $\nu(P) = \nu(R_D) - 2$,*

then $\lambda(f_0) = \lambda^{lin}(f_0) = 1$.

Proof. If (D, ν) satisfies one of first three conditions then from Theorem 3.1 we get immediately that $\lambda(f_0) = \lambda^{lin}(f_0) = 1$.

If (D, ν) satisfies the fourth condition we will construct (E, λ) such that, $[(D, \nu)]$ is linear adjacent to $[(E, \lambda)]$ and $\mu((E, \lambda)) = \mu((D, \nu)) - 1$. Let $\{P_1, \ldots, P_m\}$ will be the set of vertices of the diagram D, where P_1 is a root, and $\nu(P_2) = \nu(P_1) - 2$. We can assume that $\nu(P_2) < \nu(\tilde{P}_2)$ where $\tilde{P}_2$ is a predecessor of P_2. Indeed, otherwise (if their weights are the same) we take $\tilde{P}_2$ instead of P_2. We put $E = D$ with changed weights λ,

$$\lambda(P_i) = \begin{cases} \nu(P_1) - 1, & \text{if } i = 1 \\ \nu(P_2) + 1, & \text{if } i = 2 \\ \nu(P_i), & \text{if } i \geq 3 \end{cases} .$$

The diagram E is consistent and it is easy to check that $[(D, \nu)]$ is linear adjacent to $[(E, \lambda)]$. Since $r_E = r_D - 1$, $\mu((E, \lambda)) = \mu((D, \nu)) - 1$. □

Remark 4.2. *The singularity f_0 from Remark 3.3 is an example of a singularity that does not meet the first three conditions and meets the last one.*

Remark 4.3. *The above theorem seems to describe all Enriques diagrams of singularity such that $\lambda^{lin}(f_0) = 1$.*

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The volume is dedicated to two mathematicians: Wojciech Kucharz, who celebrates 70th anniversary in 2022 and Tadeusz Winiarski, who celebrated the 80th anniversary in 2020.

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ISBN 978-83-8331-092-3
9 788383 310923