Analytic and Algebraic Geometry 3

edited by
Tadeusz Krasiński
Stanisław Spodzieja

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Łódź 2019

Tadeusz Krasiński – University of Łódź
Department of Algebraic Geometry and Theoretical Computer Science
Faculty of Mathematics and Computer Science
90-238 Łódź, 22 Banacha St.
tadeusz.krasinski@wmii.uni.lodz.pl

Stanisław Spodzieja – University of Łódź
Department of Analytic Functions and Differential Equations
Faculty of Mathematics and Computer Science
90-238 Łódź, 22 Banacha St.
stanislaw.spodzieja@wmii.uni.lodz.pl

INITIATING EDITOR
Beata Koźniewska

TYPESETTING
Stanisław Spodzieja

TECHNICAL EDITOR
Leonora Gralka

COVER DESIGN
Michał Jankowski

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Preface

Annual Conferences in Analytic and Algebraic Geometry have been organized by Faculty of Mathematics and Computer Science of the University of Łódź since 1980. Proceedings of these conferences (mainly in Polish) were published in the form of brochures containing educational materials describing current state of branches of mathematics mentioned in the conference title, new approaches to known topics, and new proofs of known results (the all are available on the website: http://konfrogi.math.uni.lodz.pl/). Since 2013 proceedings are published (non-regularly) in the form of monographs. Two volumes have been published so far: Analytic and Algebraic Geometry, 2013 and Analytic and Algebraic Geometry 2, 2017. The content of these volumes consists of new results and survey articles concerning real and complex algebraic geometry, singularities of curves and hypersurfaces, invariants of singularities, algebraic theory of derivations and other topics.

This volume (the third in the series) is dedicated to three mathematicians: Jacek Chądzyński, who died unexpectedly on September 3, 2019 at the age of 80 and two others Tadeusz Krasiński and Andrzej Nowicki who celebrate in 2019 the jubilees of 70th birthdays. These people were closely associated with our conferences in Analytic and Algebraic Geometry. The first one was a founder of these conferences. Thanks to their mathematical vigor and stimulation the conferences become more interesting and fruitful. On next pages we provide short scientific biographies of each of them.

We would like to thank many people for the help in preparing the volume. In particular, Michał Jankowski for designing the cover, referees for preparing reports of all the articles and all participants of the Conferences for their good humor and enthusiasm during the conferences.

Tadeusz Krasiński
Stanisław Spodzieja

November 2019, Łódź

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DEDICATIONS

Professor Jacek Chądzyński (1939 – 2019)

Analytic and Algebraic Geometry 3
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JACEK CHĄDZYŃSKI
SCIENTIFIC BIOGRAPHY

Professor Jacek Chądzyński was born on November 20, 1939 and died on September 3, 2019. He has been associated with the University of Lodz since 1958. Here in the years 1958–1963 he studied mathematics and obtained a master's degree. In the year 1968 he received a PhD degree in mathematics at the Faculty of Mathematics, Physics and Chemistry of the University of Lodz for his thesis "On a certain condition equivalent to the Riemann hypothesis". His supervisor was prof. Zygmunt Charzyński. He obtained the habilitation degree in 1983, also at this Faculty, on the basis of the dissertation "On the stability of holomorphic mappings". He received the title of professor of mathematical sciences in 1990.

From graduation, Jacek Chądzyński worked continuously at the University of Lodz in the following positions: assistant, senior assistant, assistant professor, associate professor and full professor. In the years 1984–2010 he headed the Department of Analytic Functions and Differential Equations. He was a member of the Mathematics Committee of the Polish Academy of Sciences (2003–2007), a real member of the Łódź Scientific Society (1985–2006) and a member of Polish Mathematical Society.

Scientific activity of Jacek Chądzyński was related to holomorphic functions of several variables and complex analytic geometry. He was the author and coauthor of 39 scientific articles. The most important and inspiring results from his achievements include: estimates of the Łojasiewicz exponent (1983), exact formulas for the Łojasiewicz exponent (1988 and 1992), theorems about sets on which the local and global Łojasiewicz exponent are achieved (1997) and theorems on the gradient of a polynomial at infinity (2003) and some results concerning the jacobian conjecture.

Jacek Chądzyński was on a six-month internship at the Moscow Łomonosow University (1969/1970), where he participated in seminars lead by B. W. Shabat, E. M. Chirka, and A. G. Vitushkin. He participated in lectures by A. Andreotti on complex analysis at CIME in Bressanone (1973) and twice in the International Banach Center in Warsaw in semesters: Complex Analysis - conducted by prof.

J. Siciak (1979) and Theory of Singularity - conducted by prof. S. Łojasiewicz (1985). These contacts resulted in the initiation of the research in multidimensional complex analysis and algebraic geometry at the University of Lodz.

In February 1980, Jacek Chądzyński set up a new pioneering seminar on complex algebraic geometry - a discipline popular at the time in the world, but little practiced in the country. Parallelly with this seminar, four departments of the Institute of Mathematics began to organize annual domestic workshops (conferences) on the theory of extremal problems. At each conference, independent seminars were held corresponding to the scientific fields of these departments. From 1981, J. Chądzyński began to lead there a seminar on complex analytic and algebraic geometry. When only his seminar remained, the conference changed its name to: Workshop on Complex Analytic and Algebraic Geometry. So far 40 conferences have been held, of which Jacek Chądzyński was the organizer or co-organizer of 28. They quickly gained great popularity in the country. Thanks to the discussions at these conferences, many important joint scientific papers have been published. Here, close cooperation was established between the University of Lodz and the Jagiellonian University, the Kielce University of Technology, the Nicolaus Copernicus University and recently with the Université Savoie Mont Blanc in Chambery. At these conferences, students and PhD students along with the professors present their results in a friendly atmosphere. On the occasion of the XXX conference the Rector of the Jagiellonian University laid on the hands of Jacek Chądzyński Medal of the Jagiellonian University Plus ratio quam vis.

An undoubted merit of Jacek Chądzyński is the creation of a center of algebraic and analytic geometry in Łódź. These research are successfully continued at Faculty of Mathematics and Computer Science by his successors and a large group of their students. Jacek Chądzyński supervised four PhD thesis, was a reviewer in eight PhD dissertations, five habilitation dissertations, five processes for the title of professor and in three processes for the position of full and extraordinary professor at the Jagiellonian University. He was the manager of three highly rated grants: a three-year CPBP grant and two three-year KBN grants, and the main contractor in the continuation of the last of them.

During over fifty years of work at the University of Lodz, Jacek Chądzyński conducted didactic classes in Faculties of Mathematics, Physics and Economics. In mathematics, he lectured in complex analysis of one and several variables, ordinary and partial differential equations, mathematical analysis and algebraic geometry. He promoted over twenty masters in mathematics. He conducted classes with high mathematical precision and diligence. Students highly appreciated his lectures. For many years he was the tutor of the theoretical studies in the field of mathematics.

Jacek Chądzyński was the author of three academic textbooks: Introduction to complex analysis (5 editions), Introduction to complex analysis. Part II. Holomorphic functions in several variables (3 editions) and Introduction to complex analysis in problems (2 editions). All textbooks have received very positive reviews. The first is used by students in most Polish universities. The other two are also very

popular at Polish universities. In addition to the above textbooks, Jacek Chądzyński was the author of two scripts from complex analysis (7 editions) and ordinary differential equations. He received the minister's award for the first of them.

Organizational achievements of Jacek Chądzyński for the University is also significant. He was, among others, a member of the Senate of the University of Lodz (1990–1993 and 1996–2005), a member of the Statute Committee (1993–2005), a member of the Appeals Committee (1993–2010), a member of the Dean's College at the Faculty of Mathematics, Physics and Chemistry (1990–1993). In 1996 he actively participated in separating the Faculty of Mathematics from the Faculty of Mathematics, Physics and Chemistry.

The professor also conducted social activities in the University of Lodz. He was, among others, a member of the Presidium of the Works Council of the ZNP (Polish Teacher's Union) at the University of Lodz (1976–1980) and a delegate of the Institute of Mathematics at WZD NSZZ "Solidarność" (1981).

He has been awarded the Rector's awards for scientific, didactic and organizational activities several times. He was awarded the Gold ZNP Badge, the Bronze Cross of Merit, the University of Lodz Gold Medal, the Gold Cross of Merit, the University of Łódź Medal "in the Service of Society and Science", the Medal of the 50th Anniversary of the University of Lodz, the Medal of the National Education Committee and the Universitas Lodziensis Merentibus Medal.

From October 1, 2014, Jacek Chądzyński has retired, but he still participated in the seminar. He was also a member of the editorial team of proceedings for conferences on Analytic and Algebraic Geometry. He co-authored two publications on the 70th anniversary of the University of Lodz.

Professor Jacek Chądzyński died unexpectedly on September 3, 2019 at the age of 79 years.

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Professor Tadeusz Krasiński

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TADEUSZ KRASIŃSKI SCIENTIFIC BIOGRAPHY

Professor Tadeusz Krasiński was born in Konstantynów on December 23, 1949. He graduated from elementary school in Konstantynów and secondary school in Łódź. In the years 1968-1973 he studied mathematics (theoretical mathematics) at the University of Łódź. His Msc thesis supervisor was professor Zygmunt Charzyński. In 1981 he received PhD in mathematics from the Institute of Mathematics of the Polish Academy of Sciences under supervision of professor Julian Ławrynowicz. He obtained habilitation degree in 1992 at the Faculty of Mthematics, Physics and Chemistry of the University of Łódź based on the dissertation "Level sets of polynomials in two variables and the jacobian conjecture". In 2010 he received the title of professor.

Tadeusz Krasiński has been working at the University of Łódź since 1977. In the years 2004 - 2010 he also taught mathematics in the State University of Applied Sciences in Płock and during the period 2008-2010 in the University of Humanities and Economics in Łódź. The scope of scientific research of Tadeusz Krasiński covers a variety of issues in mathematics in the field of complex analytic and algebraic geometry, in particular the topics:

1. local and global Łojasiewicz exponent in various aspects,
2. complex Jacobian Conjecture,
3. improper intersections in analytical geometry,
4. bifurcation points at infinity of polynomials.

Tadeusz Krasiński conducts also research into computer science in the field of theoretical computer science, and in particular in the topics:

1. bimolecular computers,
2. automata and formal languages.

Tadeusz Krasiński wrote about eighty scientific papers. He was a supervisor in four PhD dissertations (three in mathematics and one in computer science). He was the head of a KBN grant in the years 2000-2003. He did 3 short scientific visits in

Italy, France and Vietnam and has participated in many international and national conferences.

Tadeusz Krasiński gave lectures on mathematical analysis, complex analysis, mathematical logic for economists, introduction to mathematics, functional analysis, topology, automata and formal languages, theoretical foundations of computer science and monographic lectures on: Riemann surfaces, algebraic curves, algebraic geometry, biomolecular computers.

Tadeusz Krasiński is the author of two textbooks: "Mathematical analysis. Functions of one variable." UŁ Publishing House 2001, and "Automata and Formal Languages" UŁ Publishing House, 2007 (both in Polish).

He has been an active participant in the conferences Analytic and Algebraic Geometry organized by the Faculty of Mathematics and Computer Science of the University of Łódź almost from the beginning (since 1981). Recently he is a main organizer (together with prof. Stanisław Spodzieja) of these conferences.

Prepared by editors

Professor Andrzej Nowicki

Analytic and Algebraic Geometry 3
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ANDRZEJ NOWICKI
SCIENTIFIC BIOGRAPHY

Professor Andrzej Nowicki was born in Żnin on July 26, 1949. He graduated from elementary and secondary school there in Żnin. With great respect, he reminds very good teachers from those years: Irena Sroczyńska, Franciszek Szafrański and Andrzej Wybrański.

In the years 1967-1972 he studied mathematics at the Nicolaus Copernicus University in Toruń. In 1978 he received PhD in mathematics on the thesis "Differential ideals and rings". His supervisor was professor Stanisław Balcerzyk. He obtained habilitation degree in 1995 at the Faculty of Mathematics and Computer Science of the Nicolaus Copernicus University in Toruń, based on the dissertation "Polynomial derivations and their rings of constants". In 2005 he received the title of professor.

From 1972 to the present he lectures on mathematics at the Nicolaus Copernicus University in Toruń. In the years 2002 - 2015 he also taught mathematics in Olsztyn at the University of Information Technology and Management.

His mathematical interests focus on algebraic geometry, theory of commutative rings, differential algebra and differential Galois theory. He is the author or co-author of over 100 scientific papers.

He stayed in Japan three times (for about 2 years), and worked there with japanese mathematicians: Kazuo Kishimoto, Hideyuki Matsumura, Kaoru Motose, Masayoshi Nagata, Yoshihazu Nakai and others. He was also a visiting professor in countries such as Brazil, Spain, Netherlands and China. From 1991 he was frequent guest in Ecole Polytechnique in Paris. From these visits there has been published many scientific articles with Jean Moulin Ollagnier and Jean-Marie Strelcyn.

Andrzej Nowicki is also interested (and even passionate) in elementary number theory. In this topic he published several scientific papers and wrote the series "Podróże po Imperium Liczb" (Journeys Through the Empire of Numbers), consisting of 15 popular science books. He participated many times (as an organizer) in mathematical competitions for schools: Kangur (Kangaroo) and Mathematical Olympiad.

Since 1995 he has been an active participant in the conferences Analytic and Algebraic Geometry organized by the Faculty of Mathematics and Computer Science of the University of Łódź.

He is extremely modest, warm and peaceful person. At the same time he is witty and what many people are saying it is nice to spend time in his company.

Prepared by editors

SCIENTIFIC ARTICLES

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A NOTE ON THE ŁOJASIEWICZ EXPONENT OF NON-DEGENERATE ISOLATED HYPERSURFACE SINGULARITIES

SZYMON BRZOSTOWSKI

Abstract. We prove that in order to find the value of the Łojasiewicz exponent $\l(f)$ of a Kouchnirenko non-degenerate holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singular point at the origin, it is enough to find this value for any other (possibly simpler) function $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, provided this function is also Kouchnirenko non-degenerate and has the same Newton diagram as f does. We also state a more general problem, and then reduce it to a Teissier-like result on (c)-cosecant deformations, for formal power series with coefficients in an algebraically closed field $\mathbb{K}$.

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Key words and phrases. Łojasiewicz exponent, isolated hypersurface singularities, deformations, Kouchnirenko non-degeneracy, integral closure of ideals.

1. Introduction and statement of the result

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function, defined in a neighborhood of 0, with power series expansion $f = \Sigma_{i\in\mathbb{N}_0^n} a_i z^i$, where $a_0 = 0$ and $z^i := (z_1^{i_1} \cdot \ldots \cdot z_n^{i_n})$. The *support of* f is defined as $\operatorname{Supp} f := \{k \in \mathbb{N}_0^n : a_k \neq 0\}$ and its *Newton polyhedron* is $\Gamma_+(f) := \operatorname{conv}(\operatorname{Supp} f + \mathbb{N}_0^n) \subset \mathbb{R}^n_{\geqslant 0}$. The union of the compact faces of $\Gamma_+(f)$ is called the *Newton diagram of* f and denoted by $\Gamma(f)$. If $\Gamma(f)$ touches all the coordinate axes, we say that f is *convenient*. For a face Δ of $\Gamma(f)$, we put $f_\Delta := \Sigma_{i\in\mathbb{N}_0^n\cap\Delta} a_i z^i$. We say that f is *(Kouchnirenko) non-degenerate on* Δ if the system $\{\nabla f_\Delta = 0\}$ has got no solutions in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and ∇ denotes the gradient vector. If f is non-degenerate on all the faces Δ of $\Gamma(f)$, then we simply say that f is *(Kouchnirenko) non-degenerate*.

A basic result of A. G. Kouchnirenko ([8]) says: *if* $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *are two non-degenerate functions with isolated singularities at* 0 *and such that* $\Gamma(f) = \Gamma(g)$*, then their Milnor numbers are equal:* $\mu(f) = \mu(g)$. Moreover, there exists a combinatorial formula (expressed only in terms of the diagram $\Gamma(f)$) for $\mu(f)$.

A fast definition of the *Łojasiewicz exponent* $\l(f)$ of a function f as above is

$$\l(f) = \sup_{\varphi} \frac{\operatorname{ord}(\nabla f \circ \varphi)}{\operatorname{ord} \varphi}, \tag{1}$$

where $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$, $\varphi \neq 0$, are holomorphic paths through the origin (see [9] or [15]) and, as usually, ord of a mapping is the minimum of ord's of its coordinates.

The main observation of this note is the following Kouchnirenko-like result for the Łojasiewicz exponent:

Theorem 1. *For any two Kouchnirenko non-degenerate functions* $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *having isolated singularities at* 0 *and the same Newton diagrams their Łojasiewicz exponents are equal:* $\l(f) = \l(g)$.

In principle, the proof of Theorem 1 is straightforward and consists of four steps (for f and g as in the above statement):

(1) Reduction to the case of convenient singularities.
(2) Application, to a class of simple linear deformations f_s of the given germ f, of topological triviality theorems proved in [4] (or [18]) which imply also so-called Teissier condition (c) for f_s.
(3) Application of the results of [17], which can be restated as follows: if a deformation (f_s) of f satisfies condition (c), then $\l(f_s) = \l(f)$.
(4) Using the Zariski-openness of the set of Kouchnirenko non-degenerate singularities with a given Newton diagram to join f and g by a family of "linear deformations" of the types considered above.

Although we will mostly stick to the above plan, we want to give a somewhat more direct (and, partly, more general) proof of the main result. Namely, we will avoid falling back on the results of J. Damon and T. Gaffney [4] or E.

Yoshinaga [18] and we will directly prove that the condition (c) holds for the aforementioned class of deformations of a Kouchnirenko non-degenerate function f (even if it has a non-isolated singularity). Moreover, this verification will be valid over any algebraically closed field $\mathbb{K}$ (in the formal category).

2. A wider perspective

Although in the previous section we only considered the complex analytic setting, we want to stress that all the definitions and most of the statements given there can be transferred almost verbatim, after obvious changes, to the context of formal power series living in the ring $\mathbb{K}[[z]]$ over an algebraically closed field $\mathbb{K}$ (here $z = (z_1, \ldots, z_n)$ are variables). In particular, an improved version of the Kouchnirenko theorem valid in this context was recently given by P. Mondal (see [11]). Let us remark that for a power series $h \in \mathbb{K}[[z]]$, where $z = (z_1, \ldots, z_n)$ are variables over $\mathbb{K}$, it may sometimes be necessary to introduce $\mathbb{K}$ into the notation, writing e.g. $\Gamma_{\mathbb{K}}(h)$, for otherwise the notation could be misleading if $\mathbb{K}$ happen to contain some symbols that could be treated as variables. Below, we provide the non-so-obvious information in this wider context.

2.1. **The definition of the Łojasiewicz exponent.** The main object of our study requires a modified approach in order to make it more productive. First, let us recall

Definition 2. *Let R be a ring (commutative with unity) and $\mathcal{I}$ be an ideal in R. We say that $r \in R$ is* integral over $\mathcal{I}$ *if r satisfies an equation of the form*

$$r^n + a_1 r^{n-1} + \ldots + a_n = 0,$$

where $a_j \in \mathcal{I}^j$ $(j = 1, \ldots, n)$ and $n \in \mathbb{N}$. The set of all the elements of R that are integral over $\mathcal{I}$ is called the integral closure of $\mathcal{I}$ *and is denoted by $\overline{\mathcal{I}}$.*

We choose [7] as our main source for references for topics concerning integral closure. For now, recall that $\overline{\mathcal{I}}$ is also an ideal in R and that $\overline{\overline{\mathcal{I}}} = \overline{\mathcal{I}}$. Let us state

Definition 3. *Let $\mathbb{K}$ be an algebraically closed field and let $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ be the ring of formal power series with coefficients in $\mathbb{K}$.*

(1) Let $\mathcal{I}$, $\mathcal{J}$ be two ideals in $\mathbb{K}[[z]]$. We define the Łojasiewicz exponent $\mathrm{Ł}_{\mathbb{K}[[z]],\mathcal{J}}(\mathcal{I})$ of $\mathcal{I}$ relative to $\mathcal{J}$ *as*

$$\mathrm{Ł}_{\mathbb{K}[[z]],\mathcal{J}}(\mathcal{I}) := \inf\left\{\tfrac{\alpha}{\beta} : \alpha, \beta \in \mathbb{N} \wedge \mathcal{J}^\alpha \subset \overline{\mathcal{I}^\beta}\right\}.$$

Usually, we will write just $\mathrm{Ł}_{\mathcal{J}}(\mathcal{I})$ in place of $\mathrm{Ł}_{\mathbb{K}[[z]],\mathcal{J}}(\mathcal{I})$.

(2) Let $h \in \mathbb{K}[[z]]$ be a formal power series. We define the Łojasiewicz exponent $\mathfrak{l}(h)$ of h *to be*

$$\mathfrak{l}(h) = \mathfrak{l}_{\mathbb{K}[[z]]}(h) := \mathrm{Ł}_{\mathbb{K}[[z]],\mathfrak{m}}(\nabla h) = \inf\left\{\tfrac{\alpha}{\beta} : \alpha, \beta \in \mathbb{N} \wedge \mathfrak{m}^\alpha \subset \overline{(\nabla h)^\beta \mathbb{K}[[z]]}\right\},$$

where $\mathfrak{m} = (z)\mathbb{K}[[z]]$ is the maximal ideal.

Clearly, in a similar fashion, we may define: $Ł_{\mathbb{C}\{z\},\mathcal{J}}(\mathcal{I})$ for ideals in the convergent power series ring and $ł_{\mathbb{C}\{z\}}(g)$ for holomorphic functions. It was proved in [9] that such definition is in agreement with the one given in the Introduction (cf. formula (1)). This is also an easy consequence of Corollary 7 below, so we prove it in Corollary 5. Still more generally, using Theorem 6 we can show (see [3] for a proof in dimension 2):

Proposition 4. *Let $\mathbb{K}$ be an algebraically closed field and $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \dots, z_n]]$ be the ring of formal power series with coefficients in $\mathbb{K}$. Let $\mathcal{I}$, $\mathcal{J}$ be two ideals in $\mathbb{K}[[z]]$ with $\mathcal{J}$ being proper. Then*

$$Ł_{\mathcal{J}}(\mathcal{I}) = \sup_{\substack{\varphi \in \mathbb{K}[[t]]^n \\ \varphi(0)=0}} \frac{\operatorname{ord}(\varphi^*\mathcal{I})}{\operatorname{ord}(\varphi^*\mathcal{J})}.$$

In the above statement, as is customary, $\varphi^*\mathcal{K} := (h \circ \varphi : h \in \mathcal{K})\mathbb{K}[[t]]$ and, obviously, $\operatorname{ord}(\varphi^*\mathcal{K}) = \min\{\operatorname{ord}(h \circ \varphi) : h \in \mathcal{K}\}$.

Proof. "$\geqslant$" If $\mathcal{J}^\alpha \subset \overline{\mathcal{I}^\beta}$ for some $\alpha, \beta \in \mathbb{N}$, then for any $\varphi \in \mathbb{K}[[t]]^n$ with $\varphi(0) = 0$ we get, by Theorem 6 and properties of order, $\alpha \cdot \operatorname{ord}(\varphi^*\mathcal{J}) \geqslant \beta \cdot \operatorname{ord}(\varphi^*\mathcal{I})$. The ideal $\mathcal{J}$ being proper, $\operatorname{ord}(\varphi^*\mathcal{J}) > 0$ so we infer that $\frac{\alpha}{\beta} \geqslant \frac{\operatorname{ord}(\varphi^*\mathcal{I})}{\operatorname{ord}(\varphi^*\mathcal{J})}$. Passing to the limits with both sides of the last relation, we get the required inequality.

"$\leqslant$" If $Ł_{\mathcal{J}}(\mathcal{I}) > \frac{\alpha}{\beta}$ for some $\alpha, \beta \in \mathbb{N}$, so that $\mathcal{J}^\alpha \not\subset \overline{\mathcal{I}^\beta}$, then Theorem 6 asserts that there exists some $\psi \in \mathbb{K}[[t]]^n$ with $\psi(0) = 0$ such that $\alpha \cdot \operatorname{ord}(\psi^*\mathcal{J}) < \beta \cdot \operatorname{ord}(\psi^*\mathcal{I})$. Hence, $\frac{\alpha}{\beta} < \frac{\operatorname{ord}(\psi^*\mathcal{I})}{\operatorname{ord}(\psi^*\mathcal{J})}$. Similarly as above, this implies the required inequality. □

In the situation of primary interest to us, we can state (see [9])

Corollary 5. *Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function and let $\mathbb{C}\{z\}$, with $z = (z_1, \dots, z_n)$, denote the ring of convergent power series. Then*

$$ł_{\mathbb{C}[[z]]}(g) = ł_{\mathbb{C}\{z\}}(g) = \sup_{\varphi:(\mathbb{C},0)\to(\mathbb{C}^n,0)} \frac{\operatorname{ord}(\nabla g \circ \varphi)}{\operatorname{ord}\varphi}.$$

Proof. The first equality is a consequence of [7, Proposition 1.6.2] (see the proof of Corollary 7 for details). To justify the second one it is enough to repeat the reasoning from the proof of Proposition 4 with Corollary 7 applied in place of Theorem 6. □

Remark. Naturally, in the same way one can show that it holds $Ł_{\mathbb{C}[[z]],\mathcal{J}}(\mathcal{I}) = Ł_{\mathbb{C}\{z\},\mathcal{J}}(\mathcal{I}) = \sup_\varphi \frac{\operatorname{ord}(\varphi^*\mathcal{I})}{\operatorname{ord}(\varphi^*\mathcal{J})}$, where $\mathcal{I}$, $\mathcal{J}$ are ideals in $\mathbb{C}\{z\}$ and $\varphi : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$. Let us also note that in Proposition 4 and Corollary 5 we may restrict φ's to have non-zero components, or even to be polynomials (cf. Theorem 6 and Corollary 7).

2.2. **Testing integral dependence.** Of crucial importance is the following parametric version of the well-known Valuative Criterion of Integral Dependence (see [9] or Corollary 7 for the complex analytic setting; an alternative proof of the theorem stated below, valid in dimension 2 and based on so-called Hamburger-Noether process, can be found in [3, Theorem 21]):

Theorem 6. *Let $\mathbb{K}$ be a field and $\mathcal{I}$ be an ideal in the ring $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ of formal power series with coefficients in $\mathbb{K}$. The following conditions are equivalent for an element $g \in \mathbb{K}[[z]]$:*

(1) $g \in \overline{\mathcal{I}}$,
(2) for any formal parametrization $\varphi \in \overline{\mathbb{K}}[[t]]^n$ with $\varphi(0) = 0$, where $\overline{\mathbb{K}}$ denotes the algebraic closure of the field $\mathbb{K}$, it holds

$$\operatorname{ord}(g \circ \varphi) \geqslant \operatorname{ord}(\varphi^* \mathcal{I}).$$

Moreover, in item 2 we may restrict ourselves to $\varphi \in \overline{\mathbb{K}}[t]^n$ with non-zero components φ_i.

Proof. First note that we may assume that $\mathbb{K} = \overline{\mathbb{K}}$, because by [7, Proposition 1.6.1] we have $\mathcal{I}\overline{\mathbb{K}[[z]]} \cap \mathbb{K}[[z]] = \overline{\mathcal{I}}$. Then, according to [*ibid.*, Proposition 6.8.4], $g \in \overline{\mathcal{I}}$ if, and only if, $g \in \mathcal{I}\mathcal{V}$ for all rank one discrete valuation domains $(\mathcal{V}, \mathfrak{m}_\mathcal{V})$ between $\mathbb{K}[[z]]$ and its field of fractions $\mathbb{K}[[z]]_0$ such that $\mathfrak{m}_\mathcal{V} \cap \mathbb{K}[[z]] = \mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal of $\mathbb{K}[[z]]$. This is the same as saying that $\mathcal{V}$ are regular local rings of Krull dimension 1 (see [*ibid.*, Proposition 6.3.4]). Let $\widehat{\mathcal{V}}$ denote the formal completion of $(\mathcal{V}, \mathfrak{m}_\mathcal{V})$ with respect to the $\mathfrak{m}_\mathcal{V}$-adic topology. Then $\hat{\mathcal{V}}$ is also regular local of the same dimension, hence a valuation domain. Since [*ibid.*, Proposition 6.8.1] asserts that every ideal in a valuation domain is integrally closed, using [*ibid.*, Proposition 1.6.2] we get

$$\mathcal{I}\mathcal{V} = \overline{\mathcal{I}\mathcal{V}} = \overline{\mathcal{I}\hat{\mathcal{V}}} \cap \mathcal{V} = \mathcal{I}\hat{\mathcal{V}} \cap \mathcal{V}.$$

From this it follows that checking whether $g \in \overline{\mathcal{I}}$ is equivalent to testing if $g \in \mathcal{I}\hat{\mathcal{V}}$ for all rank one complete and discrete valuation domains $\hat{\mathcal{V}}$ over $\mathbb{K}[[z]]$ such that $\mathfrak{m}_{\hat{\mathcal{V}}} \cap \mathbb{K}[[z]] = \mathfrak{m}$. Since $\mathbb{K} \subset \hat{\mathcal{V}}$, by the equicharacteristic case of Cohen Structure Theorem (see e.g. [19, Corollary in Chapter VIII, § 12], we get $\hat{\mathcal{V}} \cong \mathbb{L}[[t]]$ for some field $\mathbb{L} \subset \hat{\mathcal{V}}$. We have $\mathfrak{m}_\mathcal{V} \cap \mathbb{K}[[z]] = \mathfrak{m}$, so $\mathbb{L} = \hat{\mathcal{V}}/\mathfrak{m}_{\hat{\mathcal{V}}} \cong \mathcal{V}/\mathfrak{m}_\mathcal{V} \supset \mathbb{K}[[z]]/\mathfrak{m} = \mathbb{K}$ (see e.g. [10, page 63] for the isomorphism). Thus, we may consider $\mathbb{K}$ as a subfield of $\mathbb{L}$. Let $\psi = (\psi_1, \ldots, \psi_n) \in \mathbb{L}[[t]]^n$ be defined by $\psi_i := \iota(z_i)$ $(i = 1, \ldots, n)$, where $\iota : \mathbb{K}[[z]] \to \mathbb{L}[[t]]$ is the inclusion. Using $\iota(\mathfrak{m}) \subset (t)\mathbb{L}[[t]]$ we get $\operatorname{ord}\psi > 0$ and then the condition $g \in \mathcal{I}\hat{\mathcal{V}}$ may be rewritten as $\iota(g) = g \circ \psi \in \iota(\mathcal{I})\mathbb{L}[[t]] = (\psi^*\mathcal{I})\mathbb{L}[[t]]$. Equivalently, $\operatorname{ord} g \circ \psi \geqslant \operatorname{ord}(\psi^*\mathcal{I})$.

"1⇒2" Since g satisfies an equation of integral dependence, it is easy to directly check that $\operatorname{ord}(g \circ \varphi) \geqslant \operatorname{ord}(\varphi^*\mathcal{I})$, for any $\varphi \in \mathbb{K}[[t]]^n$ with $\varphi(0) = 0$.

"∼1⇒∼2" If $g \notin \overline{\mathcal{I}}$ then, by the above characterization, there exists some $\zeta \in \mathbb{L}[[t]]^n$, for some field $\mathbb{L} \supset \mathbb{K}$, such that $\operatorname{ord} g \circ \zeta < \operatorname{ord}(\zeta^*\mathcal{I})$. Moreover, we may

assume e.g. that $g \circ \zeta = t^a + \text{h.o.t.}$. Interpreting the last inequality as a system of algebraic equations over $\mathbb{K}$ with the coefficients of ζ viewed as unknowns, we infer, by Hilbert's Nullstellensatz, that we can find its solution also over the field $\mathbb{K}$. This delivers $\varphi \in \mathbb{K}[[t]]^n$ (even $\varphi \in \mathbb{K}[t]^n$) such that $\operatorname{ord} g \circ \varphi < \operatorname{ord}(\varphi^*\mathcal{I})$. Changing the components of φ by adding to them high enough powers of t we may arrange things so that these components are all non-zero. □

Corollary 7 ([9]). *Let $\mathcal{I}$ be an ideal in the convergent power series ring $\mathbb{C}\{z\}$ with $z = (z_1, \ldots, z_n)$. The following conditions are equivalent for an element $g \in \mathbb{C}\{z\}$:*

(1) $g \in \overline{\mathcal{I}}$,
(2) for any holomorphic curve $\varphi \in \mathbb{C}\{t\}^n$ with $\varphi(0) = 0$ it holds

$$\operatorname{ord}(g \circ \varphi) \geqslant \operatorname{ord}(\varphi^*\mathcal{I}).$$

Moreover, in item 2 we may restrict ourselves to $\varphi \in \mathbb{C}[t]^n$ with non-zero components φ_i.

Proof. We have $g \in \overline{\mathcal{I}} \Leftrightarrow g \in \overline{\mathcal{I}\,\mathbb{C}[[z]]}$, because [7, Proposition 1.6.2] asserts that for any local noetherian ring $(R, \mathfrak{m})$ and an ideal $\mathcal{J} \lhd R$ it holds $\overline{\mathcal{J}\hat{R}} \cap R = \overline{\mathcal{J}}$, where "$\widehat{\ }$" denotes the completion of R with respect to the $\mathfrak{m}$-adic topology. The test in item 2 of Theorem 6 can be performed, in particular, for convergent series φ and then $\operatorname{ord}(\varphi^*\mathcal{I}) = \operatorname{ord}(\varphi^*(\mathcal{I}\,\mathbb{C}[[z]]))$; on the other hand, as the theorem asserts, it is enough to use polynomials from $\mathbb{C}[t]$ for the test (with non-zero components). □

3. The integral closure of the toric gradient ideal of non-degenerate singularities

Here, we prove one of the results given in [18] (see also [16] or [14]) in the more general, formal setting. First, we introduce

Notation. *Let $\mathbb{K}$ be a field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a formal power series. Let $l \in \mathbb{N}^n$ be a vector with positive coordinates.*

(1) If $\varphi \in \mathbb{K}[[t]]^n$, with $\varphi(0) = 0$, is a formal parametrization such that $\operatorname{ord} \varphi_i = l_i$ $(i = 1, \ldots, n)$, then we will say that l is the initial vector of φ *and we will write* $\boxed{\operatorname{vord} \varphi = l}$.
(2) The symbol $\boxed{\operatorname{ord}_l h}$ *will denote $\operatorname{ord}(h \circ \psi)$, where $\operatorname{vord} \psi = l$ and ψ is a parametrization with generic (initial) coefficients. In other words, $\operatorname{ord}_l h$ is the minimum value of all scalar products of l and the vectors from $\Gamma(h) = \Gamma_{\mathbb{K}}(h)$. The face* $\boxed{\Delta = \Delta(l)}$ *of $\Gamma(h)$ for which this minimum is attained is said to be* supported by l, *and l itself is called a* supporting vector of Δ.
(3) $\boxed{\nabla_{\text{tor}} h(z) := \left(z_1 \cdot \frac{\partial h(z)}{\partial z_1}, \ldots, z_n \cdot \frac{\partial h(z)}{\partial z_n}\right)}$ *is the* toric gradient of h. *More generally, if $w = (w_1, \ldots, w_k)$, where $1 \leqslant k \leqslant n$, is a subsequence of the sequence of variables $z = (z_1, \ldots, z_n)$, then we put*

$$\boxed{\nabla^w_{\text{tor}} h(z) := \left(w_1 \cdot \frac{\partial h(z)}{\partial w_1}, \ldots, w_k \cdot \frac{\partial h(z)}{\partial w_k}\right).}$$

We note the following straightforward

Lemma 8. *If $h \in \mathbb{K}[[z]]$ is a non-invertible Kouchnirenko non-degenerate formal power series, then for any formal parametrization $\varphi \in \mathbb{K}[[t]]^n$, with $\varphi(0) = 0$, and such that $l := \operatorname{vord}\varphi$ satisfies $l \in \mathbb{N}^n$, we have*

$$\operatorname{ord}((\nabla_{\operatorname{tor}} h) \circ \varphi) = \operatorname{ord}((\nabla_{\operatorname{tor}} h_{\Delta(l)}) \circ \varphi) = \operatorname{ord}_l(h). \quad \square$$

Clearly, the above-introduced notations, as well as the lemma, are valid in the complex analytic setting.

We have

Proposition 9. *Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a non-invertible Kouchnirenko non-degenerate formal power series. Then*

$$\overline{\nabla_{\operatorname{tor}}(h)\mathbb{K}[[z]]} = \overline{\{z^\alpha : \alpha \in \operatorname{vert}(\Gamma(h))\}\mathbb{K}[[z]]} = \{z^\alpha : \alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n\}\mathbb{K}[[z]].$$

Here and below, "vert" denotes the set of all vertices of a given polyhedron.

Proof. The second equality follows from standard properties of integral closure of monomial ideals and does not require Kouchnirenko non-degeneracy. Namely, by [7, Proposition 1.4.6] we get $\overline{\{z^\alpha : \alpha \in \operatorname{vert}(\Gamma(h))\}\mathbb{K}[z]} = \{z^\alpha : \alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n\}\mathbb{K}[z]$ in the polynomial ring $\mathbb{K}[z]$. But it is immediate to see that both these ideals are unchanged under passage to the ring of formal power series $\mathbb{K}[[z]]$.

We will prove the first equality. Since, obviously, we have $\nabla_{\operatorname{tor}}(h)\mathbb{K}[[z]] \subset \{z^\alpha : \alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n\}\mathbb{K}[[z]]$, we only need to check that any given z^α, where $\alpha \in \operatorname{vert}(\Gamma(h))$, is integral over $\nabla_{\operatorname{tor}}(h)\mathbb{K}[[z]]$. Take $\varphi \in \mathbb{K}[[t]]^n$ such that $\varphi(0) = 0$ and φ has non-zero components, so that $l := \operatorname{vord}\varphi$ satisfies $l \in \mathbb{N}^n$. From Lemma 8 we infer that

$$\operatorname{ord}((\nabla_{\operatorname{tor}} h) \circ \varphi) = \operatorname{ord}_l(h) \leqslant \operatorname{ord}_l(z^\alpha) = \operatorname{ord}(\varphi^*(z^\alpha)).$$

Exploiting the parametric valuative criterion (Theorem 6), we conclude that $z^\alpha \in \overline{\nabla_{\operatorname{tor}}(h)\mathbb{K}[[z]]}$. $\square$

Corollary 10. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic, Kouchnirenko non-degenerate function. Then*

$$\overline{\nabla_{\operatorname{tor}}(f)\mathbb{C}\{z\}} = \overline{\{z^\alpha : \alpha \in \operatorname{vert}(\Gamma(f))\}\mathbb{C}\{z\}} = \{z^\alpha : \alpha \in \Gamma_+(f) \cap \mathbb{N}_0^n\}\mathbb{C}\{z\}.$$

Proof. The first equality follows from Proposition 9 and [7, Proposition 1.6.2] (cf. the proof of Corollary 7). The second equality holds because both involved sets are unchanged when $\mathbb{C}\{z\}$ gets replaced by $\mathbb{C}[[z]]$. $\square$

Comment. Both Proposition 9 and Corollary 10 can be improved by stating that Kouchnirenko non-degeneracy is actually equivalent to the equalities of the various integral closures (for a proof of this result in the analytic case see [18, Theorem 1.7] or [16, Theorem 3.4]; a generalization to mappings can be found in

[14, Corollary 4.5]). The proof is easy: if $(\varrho_1,\dots,\varrho_n)\in(\mathbb{K}^*)^n$ is a solution to some system $\{\nabla h_\Delta=0\}$, then choosing: a vector $l\in\mathbb{N}^n$ such that $\Delta=\Delta(l)$, $\alpha\in\operatorname{vert}(\Gamma(h))\cap\Delta$ and $\varphi(t):=((\varrho_1+\pi\cdot t)\cdot t^{N\cdot l_1},\dots,(\varrho_n+\pi\cdot t)\cdot t^{N\cdot l_n})$ with generic $\pi\in\mathbb{K}$ and $N\gg 1$, we get

$$\operatorname{ord}((\nabla_{\mathrm{tor}}h)\circ\varphi)=\operatorname{ord}((\nabla_{\mathrm{tor}}h_\Delta)\circ\varphi)>\operatorname{ord}_{N\cdot l}h_\Delta=\operatorname{ord}_{N\cdot l}(z^\alpha)=\operatorname{ord}(\varphi^*(z^\alpha)),$$

so that, according to Theorem 6, the monomial z^α is not integral over the ideal generated by $\nabla_{\mathrm{tor}}h$.

As a simple application of the information delivered above, let us note:

Corollary 11. *Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{K}[[z]]=\mathbb{K}[[z_1,\dots,z_n]]$ and let $h\in\mathbb{K}[[z]]$ be a non-invertible Kouchnirenko non-degenerate formal power series. Assume that h is convenient. Set $m_i:=\min\{p:\Gamma_+(z_i^p)\subset\Gamma_+(h)\}$ $(i=1,\dots,n)$ and $m:=\max_{1\leqslant i\leqslant n}\{m_i\}$. Then*

$$\text{ł}(h)\leqslant m-1.$$

The same estimation holds if $h:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ is a holomorphic function.

Proof. Let $\mathfrak{n}$ denote the maximal ideal in $\mathbb{K}[[z]]$. Since we have the containment of ideals $\mathfrak{n}(\nabla h)\mathbb{K}[[z]]\supset(\nabla_{\mathrm{tor}}(h))\mathbb{K}[[z]]$, we infer that

$$\text{Ł}_{\mathfrak{n}}(\mathfrak{n}(\nabla h)\mathbb{K}[[z]])\leqslant\text{Ł}_{\mathfrak{n}}(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]).$$

From Proposition 9 we know that $\overline{\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]}=\{z^\alpha:\alpha\in\Gamma_+(h)\cap\mathbb{N}_0^n\}\mathbb{K}[[z]]$. By assumption, this last set contains some powers of all the variables so that m is indeed well-defined. From Proposition 4 and Theorem 6 it easily follows that $\text{Ł}_{\mathcal{J}}(\mathcal{I})=\text{Ł}_{\overline{\mathcal{J}}}(\overline{\mathcal{I}})$ for any ideals $\mathcal{I}$, $\mathcal{J}$. Using this, we get $\text{Ł}_{\mathfrak{n}}(\nabla_{\mathrm{tor}}(h)\,\mathbb{K}[[z]])\leqslant\text{Ł}_{\mathfrak{n}}(\{z_i^{m_i}\}_{1\leqslant i\leqslant n}\mathbb{K}[[z]])$. But this last number is immediately seen to be equal to m. Thus,

$$\text{Ł}_{\mathfrak{n}}(\mathfrak{n}(\nabla h)\mathbb{K}[[z]])\leqslant m.$$

Now, exploiting Proposition 4, we see that $\text{Ł}_{\mathfrak{n}}(\mathfrak{n}(\nabla h)\mathbb{K}[[z]])=1+\text{Ł}_{\mathfrak{n}}((\nabla h)\mathbb{K}[[z]])=1+\text{ł}(h)$. Combining this with the relation above, we finish the proof in the formal setting. The holomorphic case is treated the same way. □

Comment. Choosing an appropriate parametrization of the form $\varphi(t)=(0,\dots,0,t,0,\dots,0)$ we immediately see that under the above assumptions it actually holds $\text{Ł}_{\mathfrak{n}}(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]])=m$. A proof of this fact for complex analytic mappings can be found in [1, Corollary 3.6] and [14, Theorem 2.7].

4. Constant-Newton-diagram deformations of non-degenerate singularities

In order to prove the main result, we need to have information about special deformations of Kouchnirenko non-degenerate holomorphic functions. Similarly as above, this result turns out to hold more generally – for formal power series with coefficients in an algebraically closed field.

Definition 12. *Let $\mathbb{K}$ be a field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a non-invertible formal power series. Let s be a new variable over $\mathbb{K}[[z]]$. We say that $h_\times \in \mathbb{K}[[s, z]]$ is a* deformation of h *if $h_\times(s, 0) = 0$ and $h_\times(0, z) = h$.*

Definition 13. *We say that a deformation $h_\times \in \mathbb{K}[[s, z]]$ of $h \in \mathbb{K}[[z]]$, where $z = (z_1, \ldots, z_n)$, satisfies* condition (c) *if*

$$\frac{\partial h_\times}{\partial s} \in \overline{(z_1, \ldots, z_n) \cdot \nabla_z(h_\times)\mathbb{K}[[s, z]]}.$$

Comments.

I. Naturally, above it is meant that $\nabla_z(h_\times) := \left(\frac{\partial h_\times}{\partial z_1}, \ldots, \frac{\partial h_\times}{\partial z_n}\right)$. Note that condition (c), as stated, is weaker than the condition

$$\frac{\partial h_\times}{\partial s} \in \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s, z]]}, \tag{2}$$

which we shall actually work with below (cf. Example 16).

II. Teissier in [17] requires that the deformation is not smooth i.e. it must hold $\nabla_z(h_\times)(s, 0) = 0$. We decided to remove this restriction here.

III. If $h_\times \in \mathbb{C}\{s, z\}$ then relation (2) is equivalent to

$$\frac{\partial h_\times}{\partial s} \in \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{C}[[s, z]]} \cap \mathbb{C}\{s, z\} \underset{\text{[7, Prop. 1.6.2]}}{=} \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{C}\{s, z\}} \tag{3}$$

and, similarly, the relation from Definition 13 is equivalent to

$$\frac{\partial h_\times}{\partial s} \in \overline{(z_1, \ldots, z_n) \cdot \nabla_z(h_\times)\mathbb{C}\{s, z\}}.$$

This is the original Teissier condition (c) given in [17, § 2] in the complex analytic setting. We also remark that, according to Teissier, elements of the family $h_\times(\sigma, z)$, for $\sigma \ll 1$, are called *(c)-cosecant*.

The result below shows that simple enough deformations of Kouchnirenko non-degenerate formal power series do satisfy condition (c).

Proposition 14. *Let $\mathbb{K}$ be an algebraically closed field, $\mathbb{K}[[z]] = \mathbb{K}[[z_1, \ldots, z_n]]$ and let $h \in \mathbb{K}[[z]]$ be a non-invertible Kouchnirenko non-degenerate formal power series. Define a deformation $h_\times \in \mathbb{K}[s][[z]]$ of h by the formula $h_\times := h + s \cdot z^\alpha$, where $\alpha \in \Gamma_+(h) \cap \mathbb{N}_0^n$. Then $h_\times$ satisfies Teissier condition (c), and even condition* (2).

Proof. We must check that $z^\alpha = \frac{\partial h_\times}{\partial s} \in \overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s, z]]}$. Take an arbitrary $\varphi = (\varphi_0, \hat{\varphi}) \in \mathbb{K}[[t]]^{n+1}$ such that $\varphi(0) = 0$. By Proposition 9 we have $z^\alpha \in \overline{\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]]}$, hence using Theorem 6 we get

$$\mathrm{ord}(\hat{\varphi}^* z^\alpha) \geqslant \mathrm{ord}(\hat{\varphi}^*(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[z]])), \tag{4}$$

where we substitute $z = \hat{\varphi}$. This implies

$$\mathrm{ord}(\varphi^*(s \cdot z^\alpha)) > \mathrm{ord}(\varphi^*(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[s, z]])),$$

where we substitute $(s,z)=(\varphi_0,\hat{\varphi})$. Consequently, upon noticing that $\nabla^z_{\mathrm{tor}}(h_\times)=\nabla_{\mathrm{tor}}(h)+sz^\alpha\alpha$,

$$\operatorname{ord}(\varphi^*(\nabla_{\mathrm{tor}}(h)\mathbb{K}[[s,z]]))=\operatorname{ord}(\varphi^*(\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]])). \tag{5}$$

Now, (4) and (5) give

$$\operatorname{ord}(\varphi^*(z^\alpha))\geqslant\operatorname{ord}(\varphi^*(\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]])), \tag{6}$$

where we substitute $(s,z)=(\varphi_0,\hat{\varphi})$. As φ was chosen arbitrarily, Theorem 6 ensures that $z^\alpha\in\overline{\nabla^z_{\mathrm{tor}}(h_\times)\mathbb{K}[[s,z]]}$. This proves the result. □

Remark. Essentially the same proof as the one given above shows that every deformation $h_\times$ of h whose Newton diagram built over the field $\mathbb{K}((s))$ of Laurent series is equal to that of h, that is $\Gamma_+^{\mathbb{K}((s))}(h_\times)=\Gamma_+^{\mathbb{K}}(h)$, also satisfies condition (c).

Corollary 15. *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a holomorphic, Kouchnirenko non-degenerate function. Define a deformation $f_s:(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ of f by the formula $f_s:=f+s\cdot z^\alpha$, where $\alpha\in\Gamma_+(f)\cap\mathbb{N}_0^n$. Then f_s satisfies Teissier condition (c), and even condition* (3).

Proof. Follows from the above proposition and Comment III. on page 35. □

Let us consider the following

Example 16. It is not enough to assume the (weaker) non-degeneracy considered by Mondal (see [11]) in order for Proposition 14 (or Corollary 15) to hold with condition (2) in their assertions. Take the Kouchnirenko's example [8, Remarque 1.21], where $f:=(x+y)^2+xz+z^2\in\mathbb{C}\{x,y,z\}$. Then f is Kouchnirenko degenerate with respect to the vector $l:=(1,1,2)$ supporting the segment $\Delta=\Delta(l)=\operatorname{conv}(\{(1,0,0),(0,1,0)\})\subset\mathbb{R}^3$. Indeed, the system $\{\nabla\operatorname{in}_\Delta f=0\}=\{\nabla(x+y)^2=0\}$ possesses solutions in $(\mathbb{C}^*)^3$. At the same time, f is *Milnor non-degenerate* (see [11, Definition 5.1]) and, consequently, its Milnor number can be read off the Newton diagram of f by the usual Kouchnirenko formula: $\mu(f)=1$.

Consider the deformation $f_s:=f+s\cdot xy\in\mathbb{C}\{s,x,y,z\}$ and let $\varphi(t):=(0,t,-t,0)\in\mathbb{C}\{t\}^4$. Since f_σ, for $0\neq\sigma\ll1$, are Kouchnirenko non-degenerate, we get $\mu(f_\sigma)=\mu(f)$, which by Lê-Saito-Teissier criterion of μ-constancy gives that $\frac{\partial f_s(z)}{\partial s}\in\overline{\nabla_{(x,y,z)}f_s(z)\mathbb{C}\{s,x,y,z\}}$ (see [6]). Nevertheless, this last relation cannot be improved to get condition (2) (or (3)), as the following calculation reveals:

$$\nabla^{(x,y,z)}_{\mathrm{tor}}(f_s)=(2x\cdot(x+y)+x\cdot z+s\cdot x\cdot y,2y\cdot(x+y)+s\cdot x\cdot y,z\cdot x+2z^2),$$

so, substituting $(s,x,y,z)=\varphi(t)$, we get

$$\operatorname{ord}\varphi^*\left(\frac{\partial f_s}{\partial s}\right)=\operatorname{ord}\varphi^*(xy)=2<\infty=\operatorname{ord}\varphi^*(\nabla^{(x,y,z)}_{\mathrm{tor}}(f_s)\mathbb{C}\{s,x,y,z\}).$$

By Corollary 7, $\frac{\partial f_s}{\partial s}$ is not integral over the ideal $\nabla^{(x,y,z)}_{\mathrm{tor}}(f_s)\mathbb{C}\{s,x,y,z\}$, hence condition (3) does not hold for f_s.

Still, $\operatorname{ord}\varphi^*((x,y,z)\cdot\nabla_{(x,y,z)}(f_s)\mathbb{C}\{s,x,y,z\}) = 2$ and one can check that actually f_s does satisfy condition (c), because, as ideals, $\nabla_{(x,y,z)}(f_s)\mathbb{C}\{s,x,y,z\} = (x,y,z)\mathbb{C}\{s,x,y,z\}$ and $\frac{\partial f_s}{\partial s} = xy \in (x,y,z)^2\mathbb{C}\{s,x,y,z\}$.

Finally, let us note that $łf_\sigma) = łf) = 1$ for small σ.

These observations inspire several problems (see Section 6).

5. The proof of the main result

In this section, we go back to the complex analytic world. We need the following result:

Theorem 17 (Teissier)**.** *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a holomorphic function with isolated singular point at 0 and $f_s : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ – its deformation such that $\nabla_z f_s(0,s) = 0$. Assume that f_s satisfies condition (c). Then*

$$łf_\sigma) = łf),$$

for $0 \neq \sigma \ll 1$.

Proof. This is a consequence of two results of B. Teissier. The first one, [17, Théorème 6], asserts that, under the above assumptions, the set of so-called *polar quotients* (see [17]) attached to an isolated singularity is invariant in deformations satisfying condition (c). The second one, [17, §1.7., Corollaire 2], explains that the biggest polar quotient is exactly the Łojasiewicz exponent ł. Hence, the assertion of the theorem follows. □

For convenience, let us state Theorem 1 once again.

Main Theorem. *For any two Kouchnirenko non-degenerate functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ having isolated singularities at 0 and the same Newton diagrams their Łojasiewicz exponents are equal: $łf) = łg)$.*

Proof. Set $f = \Sigma_{i\in\mathbb{N}_0^n} a_i z^i$ and $g = \Sigma_{i\in\mathbb{N}_0^n} b_i z^i$ in a neighborhood of 0.

Firstly, note that we may assume that both f and g are polynomials (of degrees $\leqslant \mu(f)+1 = \mu(g)+1$). This is a consequence of their finite determinacy for the right (biholomorphic) equivalence (see [5, Theorem 9.1.4]).

Secondly, we may make f and g be supported only on $\Gamma = \Gamma(f) = \Gamma(g)$. Indeed, choose e.g. $j \in (\operatorname{Supp} f)\backslash\Gamma$ and consider the deformation $f_s(z) := f(z) - s\cdot a_j z^j$ of f. Then f_σ ($\sigma \in \mathbb{C}$) are all Kouchnirenko non-degenerate functions with the same Newton diagrams as f has, and, actually, all of them differ by only one term, above Γ. In particular, $\mu(f_\sigma) = \mu(f) < \infty$, so all f_σ have isolated singularities at 0. From Corollary 15 we know that locally, at each $\sigma \in [0,1]$, the deformation $f_{s+\sigma} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of f_σ satisfies condition (c) and hence, by Theorem 17, has constant Łojasiewicz exponent. Consequently, $łf_1) = łf_0) = łf)$ and the function f_1 has got one monomial less above Γ than f has. Continuing in this way,

after finitely many steps we will change f into a function without terms above Γ. Similar procedure does the same to g.

Lastly, we essentially repeat the above argument for a monomial z^j with $j \in \Gamma \cap \mathbb{N}_0^n$ and $a_j \neq b_j$ but this time this requires some care. Namely, since the set

$$H := \Big\{(\xi_\alpha)_{\alpha\in\Gamma\cap\mathbb{N}_0^n} : \text{ the function } \sum \xi_\alpha z^\alpha \text{ is Kouchnirenko non-degenerate}\Big\}$$

is Zariski open in $\mathbb{C}^{\#(\Gamma\cap\mathbb{N}_0^n)}$ (see e.g. [8, Théorème 6.1] or [13, Appendix]), we may choose a real (piecewise linear) simple curve δ lying entirely in H and joining the coefficients of f with these of g. Next, covering the curve by a finite number of small enough closed cubes contained in H, we may sequentially modify the curve inside each of these cubes to make it only be built of segments parallel to the coordinate axes in $\mathbb{R}^{2\cdot\#(\Gamma\cap\mathbb{N}_0^n)}$ (see Figure 1). Then, locally, along each of these segments, one can apply the above reasoning to find that the Łojasiewicz exponent is constant there. Consequently, $ł(f) = ł(g)$. □

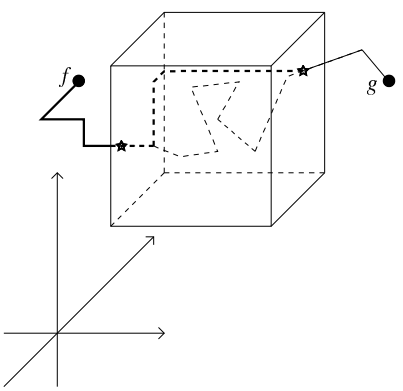

FIGURE 1. Modification of the curve δ (i-th step).

Comment. In the statement of Theorem 1 we can simply assume that the functions $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are Kouchnirenko non-degenerate and have the same Newton diagrams. Indeed, if $\operatorname{ord} f = \operatorname{ord} g = 1$, that is both f and g are smooth at 0, we clearly have $ł(f) = ł(g) = 0$. On the other hand, if e.g. f possesses a non-isolated singularity at 0, then $\Gamma(f) = \Gamma(g)$ cannot be the Newton diagram of any isolated singularity (see [2]), hence g is also non-isolated and $ł(f) = ł(g) = \infty$.

6. Problems

Here we ask several questions that may be worthwhile addressing.

I. *Does Theorem 17 hold for formal power series over an algebraically closed field?*

II. *And what about Theorem 17 with condition (c) replaced by condition* (2)*?*
III. *And what about the Main Theorem?*
IV. *And what about the Łojasiewicz exponent for Milnor non-degenerate singularities?* (consult [11]).
V. For a function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, one can consider the Łojasiewicz exponent $Ł_f(\nabla f)$ of f relative to its gradient. This number is always finite, even in the case of non-isolated singularities. For isolated singularities, we have the identity $Ł_f(\nabla f) = \frac{ł(f)}{1+ł(f)}$ ([17, § 1.7, Corollaire 2]) so this number, too, depends only on the Newton diagram for Kouchnirenko non-degenerate isolated singularities. *Does the same hold in the non-isolated case? And over algebraically closed fields?* Note that, by [12], this is indeed the case in the holomorphic setting in dimension 2.

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Faculty of Mathematics and Computer Science, University of Łódź, ul. Banacha 22, 90-238 Łódź, Poland

E-mail address: szymon.brzostowski@wmii.uni.lodz.pl

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WHEN THE MEDIAL AXIS MEETS THE SINGULARITIES

MACIEJ PIOTR DENKOWSKI

Abstract. In this survey we present recent results in the study of the *medial axes* of sets definable in polynomially bounded o-minimal structures. We take the novel point of view of singularity theory. Indeed, it has been observed only recently that the medial axis — i.e. the set of points with more than one closest point to a given closed set $X \subset \mathbb{R}^n$ (with respect to the Euclidean distance) — reaches some singular points of X bringing along some metric information about them.

1. Introduction

The notion of the *medial axis* or *skeleton* of a domain in the Euclidean space appeared presumably for the first time in the sixties in Blum's article [7] as a central concept for pattern recognition. The main idea was that given a plane bounded domain $D \subset \mathbb{R}^2$, the set of those points $x \in D$ for which the Euclidean distance $d(x, \partial D)$ is realized in more than one point of the boundary ∂D — and this set is often called the skeleton of D for quite obvious reasons — suffices to reconstruct the shape of D, provided we know the distance function along the skeleton. A most common illustration of the skeleton is the propagation of grassfire. If we ignite a fire on the border of a field, then, assuming the fire propagates inwards with uniform speed, at some point the different firefronts will meet and quench to form the skeleton of the field. If the boundary is smooth, then this propagation can be described by a PDE in the type of the eikonal equation.

The medial axis could be also interpreted as the projection of the 'ridge' that forms on the graph of the distance function. And indeed, already from Clarke's paper [10] we may infer that the medial axis coincides with the non-differentiability points of the distance function (see [28], [5]).

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The study of the medial axis (or its variants like the central set of a domain, or conflict sets) on the grounds of singularity theory is motivated not only by the applications in pattern recognition mentioned above, but also by its importance in tomography and robotics (cf. e.g. [12]). Although since the sixties a huge amount of results concerning the medial axis had been amassed thanks to the work of many outstanding mathematicians (cf. e.g. [24], [25], [28], [21], [4]), it is only recently that the special relation existing between the medial axis and the singularities of the set for which it is computed has been observed in [16]. Before that, people concentrated mostly on applications. Also, the results obtained up to now have always been requiring some strong smoothness assumptions (cf. [9], [25]), or, on the contrary, have been rather too general (see also the expository paper [1]). Our setting is that of subanalytic geometry and the theory of o-minimal structures that exclude any topological pathology.

In the present survey we will concentrate on the newly introduced singularity theory approach to the medial axis. Therefore, from the extremely large bibliography on the medial axis we shall extract only those few papers that concern this point of view.

Throughout this paper *definable* means *definable in some polynomially bounded o-minimal structure expanding the field of reals* $\mathbb{R}$ (for a concise presentation of tame geometry see e.g. [11]; for simplicity one can always think about semi-algebraic sets, see also [14]). It is also important to keep in mind that subanalytic sets (see [14] or [15]) do not form an o-minimal structure unless we control them at infinity (or near the boundary). When some local property is studied, this does not play any role and the results obtained for definable sets hold for subanalytic ones. However, from the global point of view the difference is significant (see [14] and [16]).

Some additional references for general results about medial axes can be found in the papers [1], [9] and [21].

2. Basic notions and preliminary results

Consider a closed set $\varnothing \neq X \subsetneq \mathbb{R}^n$ and let $d(x, X)$ denote the Euclidean distance from $x \in \mathbb{R}^n$ to X. Put

$$m(x) = m_X(x) := \{y \in X \mid ||y - x|| = d(x, X)\}.$$

Definition 2.1. We call $m\colon \mathbb{R}^n \to \mathscr{P}(X)$ the *multifunction of closest points*. Its multivaluedness set

$$M_X := \{x \in \mathbb{R}^n \mid \#m(x) > 1\} \subset \mathbb{R}^n \setminus X$$

is called the *medial axis* or *skeleton* (formally we should be adding: 'of $\mathbb{R}^n \setminus X$').

Remark 2.2. By the strict convexity of the norm, it has empty interior: $\operatorname{int} M_X = \varnothing$. If X is definable or subanalytic, then so is M_X (see Theorem 3.3) in which case we also have $\operatorname{int}\overline{M_X} \neq \varnothing$. Outside tame geometry this may not be true as is shown in [21] Example 4A.

Already when $\mathbb{R}^n \setminus X$ does not contain a half-space (i.e. a set described by $\langle x - v, v\rangle \leq 0$, or the reverse, for some $v \neq 0$), the medial axis is nonempty (cf. [5] Theorem 2.27).

We write $\mathbb{B}(x,r) = \mathbb{B}_n(x,r) \subset \mathbb{R}^n$ for the open Euclidean ball centred at x and with radius $r > 0$ and $\mathbb{S}(x,r) := \partial\mathbb{B}(x,r)$ for the sphere. When $r = d(x,X)$, we call the sphere or ball *supporting*. Note that X cannot enter a supporting ball.

Remark 2.3. Any point $x \in \mathbb{B}(a,r)$ where $a \in X$ has its distance $d(x,X)$ realized in $\mathbb{B}(a,2r)$. This is a mere student's exercise, but it plays an important role in many proofs.

There are two other notions closely related to that of the medial axis. The first is the concept of the central set.

Definition 2.4. We call $\mathbb{B}(x,r) \subset \mathbb{R}^n \setminus X$ a *maximal ball* for X, if

$$\mathbb{B}(x,r) \subset \mathbb{B}(x',r') \subset \mathbb{R}^n \setminus X \;\Rightarrow\; x = x', r = r'.$$

The set C_X consisting of of the centres of maximal balls for X is called the *central set* (formally: 'of $\mathbb{R}^n \setminus X$').

Remark 2.5. If $\mathbb{B}(x,r)$ is a maximal ball, then $r = d(x,X)$.

The relation between M_X and C_X is considered folklore ([1]). This result has a practical consequence in that we often work with C_X instead of M_X (see e.g. the proof of Proposition 3.20). The closure $\overline{M_X}$ is sometimes called *cut locus*.

Theorem 2.6. *There is always* $M_X \subset C_X \subset \overline{M_X}$.

Proof. [5] Theorem 2.25; see also [21] for a different proof. □

Both inclusions may be strict:

Example 2.7. If X is the parabola $y = x^2$, then $M_X = \{0\} \times (1/2, +\infty)$ whereas the focal point $(0, 1/2)$ belongs to C_X.

For the second inclusion an example is given in [9]: X is the boundary of the union of $\mathbb{B}_3((0,0,1);1)$ in $\mathbb{R}^3$ with $\mathbb{B}_3((1,0,1/2);1/2)$ and the cylinder $(0,1) \times \mathbb{B}_2((0,1/2);1/2)$ joining the two balls. Then $(0,0,1/2)$ lies in $\overline{M_X}$, but not in C_X.

A third notion closely related to the previous ones is that of *conflict set*. Given two nonempty, closed, disjoint sets $X_1, X_2 \subset \mathbb{R}^n$, their conflict set consists of all the points that are equidistant to X_1 and X_2. This can be extended to more than two sets:

[1] In [9] it is given without any references, though there is no straightforward proof. It is one of many examples of a property that is intuitively clear, but whose proof is quite far from being meretricious.

Definition 2.8. If $X_1, \ldots, X_k \subset \mathbb{R}^n$ are closed, pairwise disjoint, nonempty sets, where $k \geq 2$, and $\varrho(x) := \min_{i=1}^k d(x, X_i)$, then their *conflict set* is defined as

$$\mathrm{Conf}(X_1, \ldots, X_k) = \{x \in \mathbb{R}^n \mid \exists i \neq j \colon d(x, X_i) = d(x, X_j) \leq \varrho(x)\}.$$

Remark 2.9. We assumed here that the sets X_i are pairwise disjoint. This ensures, at least in the definable case, that the dimension of their conflict set does not exceed $n-1$ (cf. [4]). Otherwise, if the sets were assumed too be only pairwise distinct, we would lose some control. For instance, the conflict set of the two intersecting half-lines $\{y = x, x \geq 0\}$ and $\{y = -x, x \leq 0\}$ is the union of the half-line $\{x = 0, y \geq 0\}$ together with the oblique quadrant $\{y \leq -|x|\}$.

The definition,just as the two previous ones, makes sense also in any metric space. In particular, if all the sets X_i are contained in $E \subset \mathbb{R}^n$, we can compute the relative conflict set $\mathrm{Conf}_E(X_1, \ldots, X_k)$ with respect to a given metric in E.

Remark 2.10. For two distinct closed sets X, Y with a unique common point $X \cap Y = \{a\}$, there is

$$M_{X \cup Y} \setminus (\mathrm{Terr}^o(X) \cup \mathrm{Terr}^o(Y)) = \mathrm{Conf}(X, Y) \setminus C(X, Y),$$

where $\mathrm{Terr}^o(X) = \{x \in \mathbb{R}^n \mid d(x, X) < d(x, Y)\}$ is the *open territory* of X and $C(X, Y) = \{x \in \mathrm{Conf}(X, Y) \mid m_X(x) = m_Y(x)\}$. To see '$\supset$' take a point x equidistant to X and Y (so that x does not belong to any of the open territories) but with $m_X(x) \neq m_Y(x)$; then $\# m_{X \cup Y}(x) > 1$. To see '$\subset$' pick a point x from the set on the left-hand side. Then it is equidistant to X and Y and thus it belongs to the conflict set. But $m_X(x) = m_Y(x)$ implies that this set is contained in $X \cap Y = \{a\}$ and so $m_{X \cup Y}(x) = \{a\}$, contrary to the assumptions.

If the intersection $X \cap Y$ has more than one point, there is no such a simple relation between the medial axis and the conflict set as we can see for instance from the example of X being the unit circle in $\mathbb{R}^2$ together with the point $(2, 0)$ and Y just the unit circle.

Finally, let us recall two classical cones we will be using. The *Peano tangent cone* of X at $a \in X$, i.e.

$$C_a(X) = \{v \in \mathbb{R}^n \mid \exists X \ni x_\nu \to a, t_\nu > 0 \colon t_\nu(x_\nu - a) \to v\},$$

and the *Clarke normal cone* of X at a:

$$N_a(X) = \{w \in \mathbb{R}^n \mid \forall v \in C_a(X), \langle v, w \rangle \leq 0\}.$$

Both sets are definable (respectively, subanalytic) in the definable (respectively, subanalytic) case and we have the inequalities $\dim C_a(X) \leq \dim_a X$ and $\dim N_a(X) \geq n - \dim_a X$.

When studying the multifunction $m(x)$ we shall need some notions of continuity. As $m(x)$ is compact-valued,it is natural to make use of the Hausdorff, or more generally Kuratowski limits ([2]). To be more precise, let us recall the Kuratowski

[2]Or Painlevé-Kuratowski limits. As a matter of fact it was P. Painlevé who first introduced this convergence generalizing some previous work of Hausdorff. A similar concept was later considered

upper and lower limit of a multifunction $F\colon \mathbb{R}^n \to \mathscr{P}(\mathbb{R}^m)$; for a detailed study of these limits in o-minimal geometry see [13].

Definition 2.11. If x_0 is an accumulation point of the *domain* $\mathrm{dom}F$ i.e. of the set of points x for which $F(x) \neq \varnothing$, then we define the *Kuratowski lower and upper limit* at x_0 as follows:

- $y \in \liminf_{x\to x_0} F(x)$ iff for any sequence $x_\nu \to x_0$, $x_\nu \neq x_0$, one can find a sequence $F(x_\nu) \ni y_\nu \to y$;
- $y \in \limsup_{x\to x_0} F(x)$ iff there are sequences $x_\nu \to x_0$, $x_\nu \neq x_0$, and $F(x_\nu) \ni y_\nu \to y$.

Clearly, the upper limit contains the lower one and both are closed sets that do not alter if we replace the values of F by their closures. We write $E = \lim_{x\to x_0} F(x)$ or $F(x) \xrightarrow{K} E$ $(x \to x_0)$, if both limits coincide with the set E (that could be empty).

Remark 2.12. In particular,

$$C_a(X) = \limsup_{\varepsilon\to 0+} (1/\varepsilon)(X - a).$$

By the Curve Selection Lemma, in the definable case we can replace the upper limit by the limit itself (see e.g. [20]).

A simple computation shows that

$$(\dagger) \qquad a \in m(x) \;\Rightarrow\; x - a \in N_a(X).$$

This light observation has some heavy consequences, the first one being Nash's Lemma 3.1.

Before discussing what kind of relation there is between the medial axis M_X and the singularities of X we should recall the different classes of regular and singular points:

$$\mathrm{Reg}_k X := \{x \in X \mid X \text{ is a } \mathscr{C}^k - \text{submanifold in a neighbourhood of } x\},$$

for $k \in \mathbb{N}\cup\{\infty, \omega\}$ where $\mathscr{C}^\omega$ denotes analycity (in the latter case we write $\mathrm{Reg}X := \mathrm{Reg}_\omega X$ and $\mathrm{Sng}X := X \setminus \mathrm{Reg}X$ for the singularities.) and we put $\mathrm{Sng}_k X := X \setminus \mathrm{Reg}_k X$.

Example 2.13. For a plane analytic curve $\Gamma \subset \mathbb{R}^2$ through the origin we have $0 \in \mathrm{Sng}\Gamma$ if and only if either Γ has a cusp at zero, or there is an integer $k \geq 1$ such that $0 \in \mathrm{Reg}_k\Gamma \cap \mathrm{Sng}_{k+1}\Gamma$ and all the possibilities can occur (cf. [5] Example 3.1):
Take two relatively prime integers $p > q$ with q odd and such that for a given k, we have $k < p/q < k + 1$ and consider the curve Γ defined by $y^q = x^p$. Then $0 \in \mathrm{Reg}_k\Gamma \cap \mathrm{Sng}_{k+1}\Gamma$. For instance the function $y = x^{5/3}$ has analytic graph and is $\mathcal{C}^1$ but not $\mathcal{C}^2$ smooth at the origin.

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by Vietoris. On the other hand, Kuratowski was the first to present a thorough exposition of the theory in metric spaces in his memorable book on topology.

A major role in the theory is played by the important but apparently somewhat forgotten Poly-Raby Theorem:

Theorem 2.14 ([27])**.** *Let $X \subset \mathbb{R}^n$ be a closed, nonempty set and $\delta(x) := \operatorname{dist}(x, X)^2$. Then for any $k \geq 2$ or $k \in \{\omega, \infty\}$,*

$$\operatorname{Reg}_k X = \{x \in \mathbb{R}^n \mid \delta \text{ is of class } \mathscr{C}^k \text{ in a neighbourhood of } x\} \cap X.$$

Remark 2.15. We have to assume here $k \geq 2$ as is easily seen from the example of $X = (-\infty, 0]$ in $\mathbb{R}$.

As it happens, M_X coincides with the set of non-differentiability points of $\delta(x)$ (see [5]). This can be derived from some results concerning the Clarke subdifferential from [10]. Let us recall briefly Clarke's subdifferential of a locally Lipschitz function $f\colon U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ open. By the Rademacher Theorem, the set D_f of differentiability points of f is dense in U. Hence, we can define the *Clarke subdifferential* $\partial f(x)$ at any point $x \in U$ as the convex hull $\operatorname{cvx}\nabla_f(x)$ of the set $\nabla_f(x)$ of all the possible limits of the gradients $\nabla f(x_\nu)$ for sequences $D_f \ni x_\nu \to x$. It is easy to see that $\partial f(x)$ is a compact set and by [10] it reduces to a singleton $\{y\}$ iff $x \in D_f$ and $\nabla f|_{D_f}$ is continuous at y (and then $\partial f(x) = \{y\}$). In order to compute $\partial f(x)$ we may restrict ourselves to any dense subset of D_f (see [10]). A more detailed study of the multifunction $x \mapsto \partial f(x)$ in the definable setting is presented in [20].

Theorem 2.16. *We have for any point $x \in \mathbb{R}^n$,*

(1) $\partial\delta(x) = \{2(x - y) \mid y \in \operatorname{cvx} m(x)\}$;
(2) *The following conditions are equivalent:*
   (a) $x \in M_X$;
   (b) $\#\partial\delta(x) > 1$;
   (c) $x \notin D_\delta$;
   (d) $x \notin D_d \cup X$.
(3) $\nabla\delta(x) = 0 \Leftrightarrow x \in X$;
(4) $\nabla\delta$ *is continuous in* $D_\delta = \mathbb{R}^n \setminus M_X$;
(5) *If* $x \notin M_X \cup X$, *then* $x \in D_d$ *and* $\nabla d(x) = \frac{x - m(x)}{d(x)}$.

Proof. (5) is a refinement of a result shown already in [10]. According to [28], (1) can be deduced from [10] Theorem 2.1, but a self-contained proof of all the point can be found in [5] Theorem 2.23 and Lemma 2.21. □

3. Medial axis and singularities

3.1. **The medial axis of singular sets.** The starting point of the new theory is an old result of J. Nash from his famous work [26].

Lemma 3.1 ([26])**.** *Let X be a $\mathscr{C}^k$-submanifold of an open set $\Omega \subset \mathbb{R}^n$ where $k \geq 2$, or $k \in \{\infty, \omega\}$. Then there exists an arbitrarily small neighbourhood $U \subset \Omega$ of X such that*

(i) $m|_U$ *is univalued i.e. each point* $x \in U$ *has a unique closest point* $m(x) \in X$*;*
(ii) *the function* $m\colon U \ni x \mapsto m(x) \in X$ *is of class* $\mathscr{C}^{k-1}$*, or, respectively,* $\mathscr{C}^k$ *with* $k \in \{\infty, \omega\}$*.*

Proof. An elementary proof can be found in [16]. It is based on the fact that in the case considered here, by (†) we have

$$a \in m(x) \;\Rightarrow\; x - a \in (T_a X)^{\perp}$$

where $T_a X$ denotes the tangent space of X at a. Given a local parametrization $\varphi(t)$ of X at a, its partial derivatives span the tangent space and thus the proof reduces to applying the Implicit Function Theorem to the function $(t, x) \mapsto \left(\langle x - \varphi(t), \frac{\partial \varphi}{\partial t_i}(t)\rangle\right)_{i=1}^{d}$ (where $d = \dim X$) at the point $(\varphi^{-1}(a), a)$ and then using the function $t(x)$ found to get $m(x) = \varphi(t(x))$. □

Remark 3.2. As observed by S. G. Krantz and H. R. Parker, for a finite k, we cannot expect a better class than $\mathscr{C}^{k-1}$ and we have to start from $k \geq 2$. It is easy to check this using the example of $y = |x|^{3/2}$ which will also prove useful later on.

The Nash Lemma already on its own raises two natural questions:

Problem 1.

(1) *What happens when we let* X *have singularities?*
(2) *What is the structure of the exceptional set of points for which there is more than one closest point?*

The first question leads us naturally towards the setting of subanalytic geometry or o-minimal structures, while the second one is a natural way of introducing the medial axis M_X into the picture.

The general singular counterpart of the Nash Lemma solving Problem 1 is the following theorem with parameter for a set definable in some o-minimal structure. Given $X \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ we denote by X_t its section at the point t i.e. the set $\{x \in \mathbb{R}^n \mid (t, x) \in X\}$. Let $\pi_k(t, x) = t$.

Theorem 3.3 ([16] Theorem 2.1)**.** *Let* $X \subset \mathbb{R}^k_t \times \mathbb{R}^n_x$ *be a nonempty set with locally closed* t*-sections and* $Y := \pi_k(X)$*. Assume that the set* X *is definable (in a not necessarily polynomially bounded o-minimal structure). Then there exists a definable set* $W \subset \mathbb{R}^k \times \mathbb{R}^n$ *with open* t*-sections and such that* $X_t \subset W_t$ *is closed in* W_t *and* $m_t(x) \neq \varnothing$ *for* $x \in W_t$*, where*

$$m_t(x) := \{y \in X_t\colon ||x - y|| = \operatorname{dist}(x, X_t)\}, \quad (t, x) \in W,$$

and moreoever

(1) *the multifunction* $m(t, x) := m_t(x)$ *is definable* ([3])*;*

[3] i.e. its *graph* $\{(t, x, y) \in W \times X \mid y \in m(t, x)\}$ is definable.

(2) *If* $M_t = \{x \in W_t \mid \#m(t,x) > 1\}$, *then the set*

$$M := \bigcup_{t\in Y} \{t\} \times M_t \subset W$$

is definable with nowheredense sections M_t *and in particular* $m\colon W \setminus M \to \mathbb{R}^n$ *is a definable function;*

(3) *for any integer* $p \geq 2$, *there is a definable set* $F^p \subset W$ *containing* M *and such that each* F^p_t *is closed and nowheredense; moreover,* $X_t \setminus F^p_t = \mathrm{Reg}_p X_t$ *and*

$$m(t,\cdot) \text{ is } \mathscr{C}^{p-1} \text{ in a neighbourhood of } x \in W_t \setminus \overline{M_t} \Leftrightarrow\ x \notin F^p_t.$$

Remark 3.4. The Poly-Raby Theorem 2.14 is most useful for the proof. On the other hand, since the Rolin-Le Gal result on the existence of o-minimal structures that do not admit $\mathscr{C}^\infty$ cellular cell decompositions we know that we cannot expect to take $p = \infty$ in the theorem above.

When the parameter t is fixed we recognize here the multifunction $x \mapsto m(t,x)$ of the closest points to the set X_t. The section M_t of M is the set of non-unicity (multivaluedness) of this multifunction — the medial axis in the open set W_t. In (3) this set is extended to a set 'eating out' the singularities of class $\mathscr{C}^p$ of the set X_t; this extension is defined by the class $\mathscr{C}^{p-1}$ of the function $m(t,\cdot)$ (univalued in the open set being the complement of $\overline{M_t}$). What is more, everything here depends in a definable way on the multidimensional parameter t.

This good dependence on the parameter is all the more an important feature of the result as it is no longer true when we turn to the subanalytic case. The reason for this is the fact that the function $(t,x) \mapsto d(x,X_t)$ is not in general subanalytic when X is such, although the distance itself $x \mapsto d(x,X)$, as is known, is subanalytic in $\mathbb{R}^n$ (see Raby's Theorem 4.3 and Example 4.4 in [14]); it is one of the most important results in subanalytic geometry. We will illustrate this using an example from the survey [14] (this is a modified version of the example from [16] Remark 3.3).

Example 3.5. Consider

$$X = \{(x,1/x) \mid x > 0\} \cup \bigcup_{n=1}^{+\infty} \{(1/n,-n)\} \subset \mathbb{R}\times\mathbb{R}.$$

Although this set is subanalytic, the set $M = \bigcup\{(1/n,0)\}$ is not.

Nevertheless, there is a subanalytic analogue of the last theorem once we get rid of the parameters.

Theorem 3.6 ([16] Theorem 3.2)**.** *Let* $X \subset \mathbb{R}^n$ *be subanalytic, nonempty and locally closed. Then there exists a subanalytic neighbourhood* $W \supset X$ *in which* X *is closed and*

(1) *the multifunction* $m(x) = \{y \in X\colon ||x-y|| = \mathrm{dist}(x,X)\} \neq \varnothing$, *for* $x \in W$, *is subanalytic;*

(2) *the set* $M_X = \{x \in W \colon \#m(x) > 1\}$ *is subanalytic and nowheredense (in particular* $m \colon W \setminus M_X \to \mathbb{R}^n$ *is a globally subanalytic function);*
(3) *there is a nowheredense, subanalytic set* $F \subset W$ *closed in* W *and such that* $M_X \subset F$, $F \cap X = \mathrm{Sng}X$ *and* $x \in W \setminus \overline{M_X}$ *is a point of analycity of* m *if and only if* $x \in W \setminus F$.

Let us note that in (3) we obtain the analycity of m, which is a consequence of the well-known Tamm Lemma (its geometric proof not requiring the use of Hironaka's desingularization was given by K. Kurdyka in the eighties).

We should stress the fact that the proof of the theorem above cannot be obtained by a simple cutting off of X using an increasing sequence of cubes in order to apply the preceding result to the globally definable sets $X_\nu = X \cap [-\nu, \nu]^n \subset \mathbb{R}^n$ obtained in this way. Indeed, in general there is no equality $M_X = \bigcup M_\nu$, where M_ν is the medial axis defined for X_ν.

Example 3.7. Take X to be the union of half-circles $\{x^2 + (y-\nu)^2 = (3/4)^2, y \leq \nu\}$; then $(0, \nu) \in M_\nu \setminus M_{\nu+1}$ and in particular $(0, \nu) \notin M_X$.

3.2. **Reaching of singularities.** The Nash Lemma 3.1 implies that

$$(\ddagger) \qquad \overline{M_X} \cap X \subset \mathrm{Sng}_2 X.$$

Already in [16] we observed that some singular points are reached by the medial axis.

Example 3.8. ([5] Example 3.1). Consider in $\mathbb{R}^2$ the sets

$$X_1 := \{y = x^2\},\ X_2 := \{y = |x|^{3/2}\} \text{ and } X_3 := \{y = (1 + \mathrm{sgn}x)x^2\}.$$

Then $0 \in \mathrm{Reg}_1 X_i \cap \mathrm{Sng}_2 X_i$ for $i = 2, 3$, whereas $X_1 = \mathrm{Reg}_2 X_1$.

It is easy to see that M_{X_1} is the half-line $\{0\} \times (1/2, +\infty)$ and so it does not meet X_1. On the other hand, $M_{X_2} = \{0\} \times (0, +\infty)$ reaches the $\mathscr{C}^1$-singularity of X_2. But again M_{X_3} stays away from it. This is due to the fact that although both X_2 and X_3 have the same kind of singularity, their geometric 'radii of curvature' (see below — the reaching radius) are different.

We are led to the following natural question:

Problem 2. *Characterise the points of*

$$\overline{M_X} \cap \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X \quad \textit{and} \quad \overline{M_X} \cap \mathrm{Sng}_1 X.$$

Remark 3.9. If we think of the example of a quadrant $X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}$, we see that even for $\mathscr{C}^1$-singular points the question whether the medial axis reaches them or not is not obvious at all.

Partial answers to the Problem were given in [5] where several techniques were developed that should allow to thoroughly solve the question. In particular, an important tool is the newly introduced *reaching radius* ([5] Definition 4.24):

Let $V_a = N_a(X) \cap \mathbb{S}(0,1)$ denote the intersection of the normal cone to X at a with the unit sphere ([4]).

Definition 3.10. We define the *weak reaching radius*

$$r'(a) = \inf_{v \in V_a} r_v(a)$$

where

$$r_v(a) = \sup\{t \geq 0 \mid a \in m(a + tv)\}$$

is the *directional reaching radius* (or *v-reaching radius*). Next we put

$$\tilde{r}(a) = \liminf_{X \setminus \{a\} \ni x \to a} r'(x)$$

for the *limiting reaching radius*. Finally, we define the *reaching radius* as

$$r(a) = \begin{cases} r'(a), & a \in \mathrm{Reg}_2 X, \\ \min\{r'(a), \tilde{r}(a)\}, & a \in \mathrm{Sng}_2 X. \end{cases}$$

Remark 3.11. If $a \in \mathrm{int} X$, then $N_a(X) = \{0\}$ and so $V_a = \varnothing$ which gives $r'(a) = +\infty$ (as the infimum over the empty set).

Of course, if X is a hypersurface, then at $a \in \mathrm{Reg}_1 X$, we have $V_a = \{\nu(a), -\nu(a)\}$ where ν is a local unit normal vector field.

Example 3.12. The idea is that the reaching radius should vanish only at points attained by the medial axis ([5]). The reason why we consider the biggest lower bound of the radii in all possible normal directions at a is that we have to take into account the curvature and obtain a possibly finite number, e.g. for $X = \{y = x^2\} \subset \mathbb{R}^2$ we have $r'(0) = r_{(0,1)}(0) = 1/2 < r_{(-1,0)}(0) = +\infty$.

The need for considering also the limiting radius comes from the fact that for $X = \{y = |x|\}$ we have $r'(0) = +\infty$, while $\liminf_{X \setminus \{0\} \ni x \to 0} r'(x) = 0$.

On the other hand, if $X = ((-\infty, -1] \cup [1, +\infty)) \times \{0\}$, then $\tilde{r}(-1, 0) = +\infty$, while using the directions from the normal cone we see that $\inf_{v \in V_0} r_v(-1, 0) = 1$. This explains the final minimum in the definition.

By the Nash Lemma, $r(a) > 0$ at points $a \in \mathrm{Reg}_2 X$. On the other hand, by [5] Theorem 4.28 and Lemma 4.27, $X_+ := r^{-1}(+\infty) \cap \mathrm{Reg}_1 X$ is either void, or a connected component of $\mathrm{Reg}_1 X$ and $X \subset T_a X + a$ for any $a \in X_+$. More importantly, by [5] Theorem 4.33, $M_X \neq \varnothing$ implies $X \setminus r^{-1}(+\infty) \neq \varnothing$.

Theorem 3.13. *For a definable X, the function $r \colon X \to [0, +\infty]$ is definable ([6]) and $a \in \overline{M_X} \cap X$ iff $r(a) = 0$.*

---

[4] If $a \in m(x)$, then $x - a$ belongs to the normal cone to X at a, cf. (†).

[5] The notion is thus different from what is known as Federer's *reach* $\rho(X) := \inf\{d(x, M_X) \mid x \in X\}$, see [22] and [5] Subsection 4.3. It is rather awkward to use the distance $d(x, M_X)$ itself as it does not bring along enough geometric information, even though it has some interesting properties too, see [5] Corollaries 4.18 and 4.19.

[6] I.e. $r^{-1}(+\infty)$ is definable and the restriction $r|_{X \setminus r^{-1}(+\infty)}$ is a definable function.

Remark 3.14. A major role in the proof is played by the so called *proximal inequality*. We say that $v \in V_a$ is proximal for X, if for some $r > 0$, $m(a + rv) = \{a\}$. This is equivalent to the following inequality:

$$(\#) \qquad \exists r > 0 \colon \forall x \in X,\ \langle x - a, v\rangle \leq \frac{1}{2r}||x - a||^2.$$

Example 3.15. It would be helpful, if $r'(a) = 0$ implied $\tilde{r}(a) = 0$. Unfortunately, the example of $X = \{(x, y, z) \mid z = 0, y \leq |x|^{3/2}\}$ shows that there may be $r'(a) = 0$ and $\tilde{r}(a) > 0$. Here $\mathrm{Sng}_1 X = \{(x, |x|^{3/2}, 0) \mid x \in \mathbb{R}\}$, so that $r' \equiv +\infty$ along $\mathrm{Reg}_1 X$.

On the other hand, if X is the graph of $f(x, y) = y|x|^{3/2}$, then $r'(0) > 0$, while $\tilde{r}(0) = 0$, in particular, $r'(0, y) = 0$ for $y \neq 0$.

Although the definition of the reaching radius seems rather technical, the Theorem above is often quite easy to apply.

Example 3.16. Consider the surface $X = \{z^3 = xy(x^4 + y^4)\}$ in $\mathbb{R}^3$ from [22]. The interesting thing here is that at each point $a \in X$ the tangent cone is flat, i.e. a plane. However, there is a discontinuity of the tangents at the origin (where the tangent is the (x, y)-plane) if we move along the x- or y-axis where the tangent planes are vertical (they contain the z-axis). Thus the origin lies in $\mathrm{Sng}_1 X$. It is easy to see that $r'(0) > 0$, but $\tilde{r}(0) = 0$ so that $0 \in \overline{M_X}$, by the last Theorem.

3.3. **Stability of the medial axis with application to the reaching of singularities.** Almost since its introduction the medial axis M_X has been known as being highly unstable under small deformations of X. F. Chazal and R. Soufflet illustrated this in [9] with a most simple example: the medial axis of a circle is its centre, but even the smallest 'protuberance' on the circle leads to the medial axis becoming a whole segment. The paper [9] is entirely devoted to showing that under some hypotheses on X there is a kind of stability of the medial axis for $\mathcal{C}^2$ deformations expressed by means of map images. However, that kind of approach consists actually in looking at the initial and the final states only — with a black box in between, where the actual deformation takes place. Even from the point of view of applications it seems more natural to see the deformation as a continuous process. Which is more, there is no need for it to be smooth. This is best expressed using the Kuratowski convergence of sets and indeed lets us have some insight of what is happening to the medial axis.

Let $\pi(t, x) = t$ for $(t, x) \in \mathbb{R}^k \times \mathbb{R}^n$. We have the following type of semicontinuity of the medial axis ([7]):

Theorem 3.17 ([18] Theorem 4.1)**.** *Assume that $X \subset \mathbb{R}^k \times \mathbb{R}^n$ is definable with closed t-sections, 0 is an accumulation point of $\pi(X)$ and $X_t \xrightarrow{K} X_0$. Then for*

[7] Recently, we have obtained with A. Denkowska a similar but more detailed result based partly on [13] for conflict sets and also for *Voronoi diagrams* which are medial axes of finite sets. The result is as yet unpublished.

$M = \{(t,x) \mid \#m(t,x) > 1\}$, where $m(t,x) = \{a \in \mathbb{R}^n \mid a \in X_t \colon ||x-a|| = d(x,X_t)\}$, we have

$$\liminf_{\pi(M)\ni t\to 0} M_t \supset M_0$$

where we posit $\liminf_{\pi(M)\ni t\to 0} M_t = \varnothing$ when $0 \notin \overline{\pi(M)\setminus\{0\}}$.

Remark 3.18. The Theorem implies that 0 cannot be an isolated point of $\pi(M) = \{t \mid M_t \neq \varnothing\}$, i.e. $M_0 = \varnothing$, if $0 \notin \overline{\pi(M)\setminus\{0\}}$.

The proof depends heavily on the Curve Selection Lemma for which the definability assumption is unavoidable. Whether there is a general counterpart of this result remains an open question.

Example 3.11 from [18] shows that we can hardly expect a better result even in the quite regular situation when we are dealing with a convergent definable one-parameter family of graphs:

Example 3.19. Consider the set $X = \{(t,x,y) \in \mathbb{R}\times\mathbb{R}^2 \mid y = t|x|\}$. It is definable, we have $X_t \xrightarrow{K} X_0$, but

$$M_t = \begin{cases} \{(x,y) \mid x = 0, y > 0\}, & t > 0, \\ \varnothing, & t = 0, \\ \{(x,y) \mid x = 0, y < 0\}, & t < 0, \end{cases}$$

so that there is no convergence.

The Theorem above combined with the following recent observation of A. Białożyt [3] enables us to prove a refined version of Theorem 4.6 from [5] on reaching a certain type of singular points.

Proposition 3.20 (A. Białożyt)**.** *Let $V \subset \mathbb{R}^n$ be a real cone ([8]). Then $M_V \neq \varnothing$ if and only if V is not a convex set.*

Proof. See [3]. If V is convex, then $m_V(x)$ is obviously univalued and $M_V = \varnothing$. On the other hand, it is easy to see that by homothety $M_V \cup \{0\}$ is a real cone, too. Assume that V is non-convex and take $x \neq y$ in V such that $[x,y] \cap V = \{x,y\}$. If the midpoint z of the segment $[x,y]$ is not in M_V take $a = m_V(z)$ and write $B_t := \mathbb{B}(a + t(z-a), td(z,V))$. Then by the choice of z, we conclude that there must be

$$\sup\{t \geq 1 \mid B_t \subset \mathbb{R}^n \setminus V\} < +\infty,$$

which implies that the central set $C_V \neq \varnothing$ and we are done due to Theorem 2.6. □

Theorem 3.21. *Assume that $X \subset \mathbb{R}^n$ is a definable with a non-convex tangent cone $V := C_0(X)$. Then $0 \in \overline{M_X}$ and $C_0(M_X) \supset \overline{M_V}$.*

[8] A real cone $V \subset \mathbb{R}^n$ is a union of half-lines starting from the origin, i.e. for any $t \geq 0$, $tV \subset V$ and we assume that $V \neq \varnothing$ by definition.

Proof. As noted in Remark 2.12, in the definable case we know that V is the Kuratowski limit when $t \to 0^+$ of the dilatations $(1/t)X$, $t > 0$. Hence by Theorem 3.17,

$$M_V \subset \liminf_{t\to 0^+} M_{(1/t)X}.$$

By homothety, we have $M_{(1/t)X} = (1/t)M_X$ and so the limit inferior is actually a limit and coincides with $C_0(M_X)$. Finally, as observed [20], for a definable set we have $C_0(E) = \lim_{t\to 0^+}(1/t)E$ also in the case when $0 \notin \overline{E}$ in which case the limit is empty. Therefore, since we know by Proposition 3.20 that $0 \in \overline{M_V}$ (as $M_V \cup \{0\}$ is a cone,by homothety), we obtain the result sought for. □

In general we can hardly expect equality between $C_0(M_X)$ and $\overline{M_V}$ in the last theorem:

Example 3.22. Consider a calyx-shaped X, i.e. the union of the horn $x^2 + y^2 + z^3 = 0$ together with $z = ||(x, y)||$ (Euclidean norm). Then $V = C_0(X)$ consists of $\{x = y = 0, z \leq 0\}$ together with $z = ||(x, y)||$ and so it is non-convex and the last Theorem applies. However, M_X contains the z-axis without the origin, so that $C_0(M_X)$ contains the whole z-axis, whereas the cone $M_V \cup \{0\}$ intersected with the z-axis is just the half-line $\{x = y = 0, z \geq 0\}$ which means that $C_0(M_V) = \overline{M_V} = M_V \cup \{0\}$ does not contain the half-line $\{x = y = 0, z < 0\}$.

3.4. **The plane case.** The plane case is far from being plain, if we may indulge in a little pun. Let us recall the following classical fact.

Lemma 3.23. *If $X \subset \mathbb{R}^2$ is a definable curve such that $0 \in X$ and the germ $(X \setminus \{0\}, 0)$ is connected, i.e. X has a single branch ending at the origin, then the tangent cone $C_0(X)$ is a half-line that we can identify with $\mathbb{R}_+ \times \{0\}^{n-1} \subset \mathbb{R}^n$ in properly chosen coordinates and X is near zero the graph of a definable $\mathscr{C}^1$ function $f\colon [0, \varepsilon) \to \mathbb{R}$ with $f(0) = 0$ and $f'(0) = 0$.*

In the situation from this Lemma, for $0 < t \ll 1$, we can write $f(t) = at^\alpha + o(t^\alpha)$ with $a \neq 0, \alpha \geq 1$, provided $f \not\equiv 0$.

Definition 3.24. We say that X as in the Lemma above is *superquadratic* at zero iff $f \not\equiv 0$ and $\alpha < 2$ (cf. [5, Section 3.3]).

Remark 3.25. The definability of f allows us also to assume that f has constant convexity on $[0, \varepsilon)$ and is $\mathscr{C}^2$ on $(0, \varepsilon)$.

The choice of the adjective *superquadratic* in view of the fact that we require $\alpha < 2$ may seem a little strange. Its geometrical origin is shown in the following easy lemma.

Lemma 3.26 ([5] Lemma 3.17)**.** *If $\gamma\colon [0, \varepsilon) \to [0, +\infty)$ is superquadratic with $\gamma(0) = \gamma'(0) = 0$, then for any $r > 0$ the disc $D_r := \mathbb{B}((0, r), r) \subset \{y > 0\}$ tangent to the x-axis at zero contains points of γ inside.*

Proof. It follows from the obvious observation that if $g\colon [0,r) \to \mathbb{R}_+$ denotes the usual parametrization of the lower part of the circle ∂D_r through zero, then $g(x) = \frac{1}{2r}x^2 + o(x^2)$ near zero. At the same time $\gamma(x) = ax^\alpha + o(x^\alpha)$ with $a > 0$ and $\alpha \in (0,2)$ and so there must be $g(x) < \gamma(x)$ for small x. □

Following [6] we will give here the correct version of [5] Proposition 3.24. It reads:

Proposition 3.27 ([5] Proposition 3.24 – correct version)**.** *Assume that $X \subset \mathbb{R}^2$ is a definable curve such that $0 \in X$ and the germ $(X \setminus \{0\}, 0)$ is connected. Then $0 \in \overline{M_X}$ if and only if X is superquadratic at zero.*

Proof. If X is superquadratic at zero, then by Lemma 3.26, the weak reaching radius $r'(0,0)$ is zero and so the reaching radius $r(0,0)$ is zero, too. By Theorem 3.13, it means that $0 \in \overline{M_X}$.

If X is not superquadratic at zero, then either $f \equiv 0$, or $\alpha \geq 2$, where f is the function from Lemma 3.23. In both cases f has a $\mathscr{C}^2$ extension by 0 through zero and the Nash Lemma leads to the conclusion that $0 \notin \overline{M_X}$. □

Lemma 3.28. *If $X \subset \mathbb{R}^2$ is definable with $\dim_0 X = 1$ and $0 \in \overline{M_X} \cap X$, then $\dim_0 M_X = 1$.*

Proof. Since the assumptions imply that $0 \in \overline{M_X} \setminus M_X$, then by the Curve Selection Lemma, $\dim_0 M_X \geq 1$. On the other hand, M_X has empty interior, whence $\dim_0 M_X < 2$. □

We may complete now the previous Proposition with a metric statement.

Proposition 3.29 ([6] Proposition 2.3)**.** *Assume that X is as in the previous Proposition and $0 \in \overline{M_X} \cap X$. Then the tangent cone $C_0(M_X)$ is the half-line perpendicular to $C_0(X)$ lying on the same side of $C_0(X)$ as X near zero. To be more precise, if X near zero is the graph of $f\colon [0,\varepsilon) \to \mathbb{R}$ and f is, say, convex, then $C_0(M_X) = \{0\} \times [0,+\infty)$.*

Proof. From the previous Lemma we know that $\dim_0 M_X = 1$. By Lemma 3.23, we assume that X is the graph of a convex definable function $f\colon [0,+\infty) \to \mathbb{R}$ of class $\mathscr{C}^1$ that is $\mathscr{C}^2$ on $(0,\varepsilon)$, $f(0) = f'(0) = 0$ and f is superquadratic at the origin (by Proposition 3.27). Thanks to the convexity, for some neighbourhood U of the origin, we have $M_X \cap U \subset \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\}$.

Take any sequence $M_X \ni a_\nu \to 0$ such that $a_\nu/||a_\nu|| \to v$. For each index we pick a point $b_\nu \in m(a_\nu) \setminus \{0\}$. Then $b_\nu \to 0$ (cf. [20] Lemma 8.5). Moreover, $v_\nu := (a_\nu - b_\nu)/d(a_\nu, X)$ is a unit normal vector to X at b_ν and for each $\theta \in [0, d(a_\nu, X))$, b_ν is the unique closest point in X to $b_\nu + \theta v_\nu$ and so the unit vector v_ν is proximal, which implies, as in the proof of Theorem 4.35 in [5], the proximal inequality (#):

$$\forall c \in X,\ \langle c - b_\nu, v_\nu \rangle \leq \frac{1}{2d(a_\nu, X)} ||c - b_\nu||^2.$$

From this, after multiplying both sides by $d(a_\nu, X)$ and taking $c = 0$, we obtain

$$\frac{1}{2}||b_\nu||^2 \leq \langle a_\nu, b_\nu \rangle,$$

whence $||b_\nu||/||a_\nu|| \leq 2\cos\alpha_\nu$, where $\alpha_\nu = \angle(b_\nu, a_\nu)$. In particular all the angles α_ν are acute.

Since $||b_\nu|| \to 0$, we obtain $b_\nu/||b_\nu|| \to (1,0)$, for $C_0(X) = [0,+\infty) \times \{0\}$. Our proof will be accomplished, if we show that $\alpha_\nu \to \pi/2$, since $\alpha_\nu = \angle(b_\nu/||b_\nu||, a_\nu/||a_\nu||) \to \angle((1,0), v)$. As the angles are acute, we immediately get $\angle((1,0), v) \in [0, \pi/2]$.

We know that X is superquadratic at zero, which implies that for any $y > 0$, the origin does not belong to $m((0,y))$, by Lemma 3.26. If $b \in m((0,y))$, then b is the unique closest point for any point from the segment $[(0,y), b] \setminus \{(0,y)\}$. As earlier, by [20] Lemma 8.5, $b \to 0$ when $y \to 0^+$. Then the set $Y := \{b \in X \mid \exists y > 0\colon\ b \in m((0,y))\}$ is definable and $0 \in \overline{Y} \setminus Y$. Therefore, by the Curve Selection Lemma, Y coincides with X in a neighbourhood of zero that we may take to be a ball $\mathbb{B}(0,R)$.

In particular, we can find $r, \rho > 0$ such that there is a continuous definable surjection $[0,r) \ni y \mapsto F(y) \in X \cap \mathbb{B}(0,\rho)$ satisfying $F(y) \in m((0,y))$. Then, for any $(x,y) \in \mathbb{B}(0,\rho/2)$ such that $x > 0, y > f(x)$, the distance $d((x,y), X)$ is realized in $\mathbb{B}(0,\rho) \cap X$. If b is a closest point to (x,y), then the vector $(x,y) - b$ is normal to X at b, but as $b = F(y')$ for some $y' \in [0,r)$, we conclude that $(x,y) \in [(0,y'), b]$ and so $m((x,y)) = \{b\}$. Therefore,

$$M_X \cap \{(x,y) \in \mathbb{B}(0,\rho/2) \mid x > 0, y > 0\} = \varnothing.$$

This means that $M_X \cap \mathbb{B}(0,\rho/2) \subset \{(x,y) \in \mathbb{R}^2 \mid y \geq 0, x \leq 0\}$, whence $\angle((1,0), v) \in [\pi/2, \pi]$. Summing up, we obtain $\angle((1,0), v) = \pi/2$ as required. □

If we are dealing with a $\mathscr{C}^1$-smooth curve, a so called 'rolling disc' argument yields:

Theorem 3.30. *Assume that* $0 \in \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X$. *Then* $0 \in \overline{M_X}$ *iff* X *is superquadratic at the origin.*

Proof. [5] Theorem 3.19. □

In the presence of at least two branches, we have the following result for a $\mathscr{C}^1$-singularity:

Theorem 3.31. *Let* $X \subset \mathbb{R}^2$ *be a definable curve with* $0 \in \mathrm{Sng}_1 X$ *and assume that the germ* $(X \setminus \{0\}, 0)$ *has at least two connected components. Then* $0 \in \overline{M_X}$.

Proof. [5] Theorem 3.21. □

Remark 3.32. Due to the Nash Lemma, Proposition 3.27 together with Theorems 3.30 and 3.31 completely solve Problem 2 in the plane.

Using the main result of Birbrair and Siersma from [4], which is the following Theorem, we are able to compute in the case of plane curves the tangent cone to M_X at a point $a \in X$ reached by the medial axis.

Theorem 3.33 (Birbrair-Siersma [4])**.** *Let $X_1, \ldots, X_k \subset \mathbb{R}^n$ be closed, definable, pairwise disjoint, nonempty sets such that 0 belongs to $K := \mathrm{Conf}(X_1, \ldots, X_k)$ and let $S := \mathbb{S}(0, d(0, X_1))$ be the supporting sphere at 0. Then $C_0(K)$ is the cone spanned over the conflict set $\mathrm{Conf}_S(\tilde{X}_1, \ldots, \tilde{X}_k)$ in the sphere, where $\tilde{X}_i := X_i \cap S$.*

Here, what we mean by a *cone spanned over* a subset E of the sphere S centred at zero is the set $\bigcup\{\mathbb{R}_+ v \mid v \in E\}$. The conflict set *in the sphere* is computed with respect to the geodesic metric in S (cf. Remark 2.9).

If $(X, 0) \subset \mathbb{R}^2$ is a definable pure one-dimensional closed germ, then $X \setminus \{0\}$ consist of finitely many branches $\Gamma_0, \ldots, \Gamma_{k-1}$ ending at zero and dividing a small ball $\mathbb{B}(0, r)$ into k regions. For $k > 1$, if we enumerate the branches in a consecutive way, we can call these open regions $D(\Gamma_i, \Gamma_{i+1})$, $i \in \mathbb{Z}_k$. Assuming that $0 \in \overline{M_X}$, we say that a pair of *consecutive* branches Γ_i, Γ_{i+1} *contributes* to M_X at zero, if $0 \in \overline{M_X \cap D(\Gamma_i, \Gamma_{i+1})}$.

Let $1 \leq c \leq k$ be the number of contributing regions. For each such region $D(\Gamma_i, \Gamma_{i+1})$ we have two half-lines ℓ_i, ℓ_{i+1} tangent to Γ_i, Γ_{i+1} at zero, respectively. These half-lines define an oriented angle $\alpha(i, i+1) \in [0, 2\pi]$, consistent with the region ([9]).

As we know that M_X is one-dimensional, the germ $(\overline{M_X}, 0)$ consists of finitely many branches ending at zero. For a definable curve germ $(E, 0)$, we will denote by $b_0(E)$ the number of its branches at the origin.

Combining [5] Theorem 3.27 with Propositions 3.27 and 3.29, we obtain the following Tangent Cone Theorem.

Theorem 3.34 ([5] Theorem 3.27 – extended version, [6] Theorem 2.4)**.** *Assume that $0 \in \overline{M_X} \cap X$ where X is a pure one-dimensional closed definable set in the plane. Then,*

1. *either $b_0(X) = 1$, in which case $b_0(M_X) = 1$ and $C_0(M_X)$ is the half-line perpendicular to $C_0(X)$ lying on the same side of $C_0(X)$ as X near zero,*
2. *or $b_0(X) = k > 1$, in which case $b_0(M_X) \leq c + 1$ where c is the number of contributing regions, and $C_0(M_X)$ is the union of the bisectors of all the pairs of half-lines forming up $C_0(X)$ given by pairs of consecutive branches delimiting regions that contribute to M_X at zero with possibly one exception: there is at most one contributing region $D(\Gamma_i, \Gamma_{i+1})$ with angle $\alpha(i, i+1) > \pi$ in which case at least one of the curves Γ_i, Γ_{i+1} is superquadratic at zero and $M_{i,i+1} = M_X \cap D(\Gamma_i, \Gamma_{i+1})$ has at most two branches at zero and*

[9]Note that it may happen that $\alpha(i, i+1) = 2\pi$; indeed, if X consists of the two branches $\Gamma_0 = [0, +\infty) \times \{0\}$ and the superquadratic $\Gamma_1 = \{y = x^{3/2}, x \geq 0\}$, then both regions $D(\Gamma_0, \Gamma_1)$ and $D(\Gamma_1, \Gamma_0)$ are contributing. The angles are 0 and 2π, respectively.

$C_0(M_{i,i+1})$ consists of one or two half-lines orthogonal to the corresponding tangent ℓ_i or ℓ_{i+1}.

Proof. (1) is the statement of Proposition 3.29. To see that M_X near zero consists of one branch we consider the situation from the proof of Proposition 3.29. In particular $M_X \cap \mathbb{B}(0,r) \subset \{(x,y) \in \mathbb{R}^2 \mid y \geq 0, x \leq 0\}$. Suppose that there are (at least) two different branches M_1, M_2 ending at zero. Then one of them, say M_1, lies in the region delimited by the other one, i.e. M_2, and $\{0\} \times [0, +\infty)$. Take a point $a \in M_2$. Then $m(a)$ contains a non-zero point b. Then, if a is sufficiently near zero, the segment $[a,b]$ intersects M_1. If c belongs to the intersection, then $m(c) = \{b\}$, contrary to $c \in M_X$.

As for (2), we can repeat the argument from the proof in [5] Theorem 3.27 with only one additional case to consider. Let $D(\Gamma_0, \Gamma_1)$ be a contributing region. The same type of argument as above shows that M_X has only one branch in $D(\Gamma_0, \Gamma_1)$ ending at zero ([10]). Let $\alpha = \alpha(0,1) \in [0, 2\pi]$ be the oriented angle consistent with $D(\Gamma_0, \Gamma_1)$.

If $\alpha \in [0, \pi)$, we proceed as in [5] Theorem 3.27: for $a \in M_X$ near zero, $m(a)$ cannot contain zero and has points both from Γ_0 and Γ_1 — these tend to zero when $a \to 0$. The set $M_X \cap D(\Gamma_0, \Gamma_1)$ coincides with the conflict set of Γ_0, Γ_1 (compare the proof of Theorem 3.21 in [5]) and the Birbrair-Siersma Theorem quoted above gives the result as in the original proof in [5].

If $\alpha = \pi$, then $\Gamma = \Gamma_0 \cup \Gamma_1$ is a $\mathscr{C}^1$ curve and $M_X \cap D(\Gamma_0, \Gamma_1)$ reaches the origin iff Γ is superquadratic at zero ([11]). But then no point from the normal cone at zero can have its distance realized at the origin (cf. Lemma 3.26) and so we are in a position that allows us to repeat the argument based on the Birbrair-Siersma Theorem just as in [5].

If $\alpha > \pi$ (clearly, there can be only one such contributing region), then the only possibility that the region $D(\Gamma_0, \Gamma_1)$ be contributing is that at least one of the two delimiting curves be superquadratic at zero and $\mathbb{B}(0,r) \setminus D(\Gamma_0, \Gamma_1)$ be non-convex. In this case we are exactly in the situation from Proposition 3.29 and the result follows. Of course, $M_X \cap D(\Gamma_0, \Gamma_1)$ may have two branches at zero which explains why we have $b_0(M_X) \leq c + 1$. □

The need for taking $c+1$ in (2) is illustrated by the following example from [6].

Example 3.35. Rotate the superquadratic curve $y = x^{3/2}$, $x \geq 0$ by $\pi/6$ anticlockwise and the curve $y = -x^{3/2}$, $x \geq 0$ by the same angle clockwise, obtaining two curves Γ_0, Γ_1 with tangent half-lines at zero $y = (1/\sqrt{3})x, x \geq 0$ and

[10] If there were only two branches of M_X in $D(\Gamma_0, \Gamma_1)$ ending at zero, it could happen that along each of them the segments joining the points to the points realizing their distance would not intersect the other branch. In that case we pick a point a in between the two branches of M_X and the segment $[a, m(a)]$ must intersect one of the branches in a point c. Then $m(a) \in m(c)$ but there is a point $b \in m(c) \setminus m(a)$ and the triangle inequality shows that $||a - b|| < ||a - m(a)||$, which is a contradiction.

[11] I.e. $D(\Gamma_0, \Gamma_1)$ is near zero the epigraph of a superquadratic function.

$y = -(1/\sqrt{3})x, x \geq 0$, respectively. Let $X = \Gamma_0 \cup \Gamma_1$. Then we have two contributing regions: $D(\Gamma_1, \Gamma_0)$ with $\alpha(1,0) = \pi/3$ and $D(\Gamma_0, \Gamma_1)$ with $\alpha(1,2) = 5\pi/3$. The medial axes has three branches at zero: the half-line $[0, +\infty) \times \{0\}$ and two curves symmetric with respect to $(-\infty, 0] \times \{0\}$, living in the quadrants $\{x \leq 0, y \geq 0\}$ and $\{x \leq 0, y \geq 0\}$, respectively. Then

$$C_0(M_X) = ([0, +\infty) \times \{0\}) \cup \{y = -\sqrt{3}x, x \leq 0\} \cup \{y = \sqrt{3}x, x \leq 0\}.$$

In the non-definable setting the tangent cone $C_0(M_X)$ when M_X reaches the set X at zero may be quite big.

Example 3.36. ([5] Example 3.28). Consider $X = \{0\} \cup \bigcup_{\nu=1}^{+\infty} \{(x_\nu, 0)\} \subset \mathbb{R}^2$ where $x_\nu = 1/\nu$. Then

$$M_X = \bigcup_{\nu=1}^{+\infty} \left\{ \frac{x_\nu + x_{\nu+1}}{2} \right\} \times \mathbb{R}$$

and so $0 \in \overline{M_X}$, but $C_0(M_X) = \{(x, y) \mid x \geq 0\}$, while $C_0(X) = [0, +\infty) \times \{0\}$.

Example 3.37. Let $X = \{(x, x^2) \mid x \in [0, 1)\} \cup \{(x, x^3) \mid x \in [0, 1)\}$. Then M_X near zero is clearly a curve lying between the two branches of $X \setminus \{0\}$. Since these branches have a common tangent $[0, +\infty) \times \{0\}$ at zero, this line is also the tangent cone of M_X at the origin.

From these results, we obtain a symmetry property of plane analytic curves.

Proposition 3.38 ([5] Corollary 3.30)**.** *Let $X \subset \mathbb{R}^2$ be a real-analytic curve germ at zero and such that $X \setminus \{0\}$ consists of only two branches and $0 \in \overline{M_X}$. Then in a neighbourhood U of zero, the medial axis M_X is a half-line that is a symmetry axis of $X \cap U$.*

Proof. In view of the preceding results, there are two possibilities ([12]): either $0 \in \mathrm{Sng}_1 X$, or $0 \in \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X$ with X superquadratic at the origin. In the first case, by [22] Corollary 5.6 we know that $C_0(X)$ is a half-line ℓ, that we may assume to be $\{0\} \times [0, +\infty)$, whereas in the second one it is a line L that we assume to be the x-axis. Using the definition of the tangent cone, we may assume in both cases that in a neighbourhood of the origin X is a graph over an interval $(-\varepsilon, \varepsilon)$ in the x-axis. Consider $F = 0$ to be an analytic equation of X in the same neighbourhood.

Let h be the branch over $(-\varepsilon, 0]$ and g the branch over $[0, \varepsilon)$. They both are $\mathcal{C}^1$ at zero and due to the Puiseux Theorem, for some integer $p > 0$, $g(t^p)$ has an analytic extension through zero onto $(-\delta, \delta)$ for some $\delta \in (0, \varepsilon)$. Then, we obtain

$$F(s, g(s)) \equiv 0, \quad s \in [0, \sqrt[p]{\delta}).$$

Therefore, by the identity principle, this holds true for $|s| < \sqrt[p]{\delta}$. But we may repeat the same argument with h and so we conclude that $g(-s) = h(s)$ for $s \in (0, \delta)$ (if

[12] Note that both may occur: $y^3 = x^4$ is $\mathscr{C}^1$ regular at zero but superquadratic at this point, cf. Example 2.13.

δ was chosen < 1) which gives the symmetry sought after (since the germ of M_X at zero depends only on the germ of X at this point) and proves that M_X is a half-line near zero, as well. □

Remark 3.39. This result implies that for instance the superquadratic curve $y = \mathrm{sgn}(x)|x|^{3/2}$ is not analytic at the origin.

4. Superquadratic points

Motivated by the situation in the plane, we may introduce a notion of superquadraticity in higher dimensions. The first natural step would be the following.

Definition 4.1 ([5] Definition 3.9)**.** If X is the graph of a non-negative continuous function f at $x_0 \in X$, then we call X *superquadratic* at this point, if the function $g_f(r) := \max_{x\in\mathbb{S}(x_0,r)} f(x)$ is superquadratic, i.e. it can be written near zero as $g(r) = ar^\alpha + o(r^\alpha)$ with $\alpha < 2$.

On the other hand, a geometric interpretation as in e Lemma 3.26 suggests that it might be a good idea to consider a notion of order of vanishing.

Definition 4.2 ([5] Definition 3.10)**.** We define the *order at zero* of a continuous definable function germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ as

$$\mathrm{ord}_0 f = \sup\{\eta > 0 \mid |f(x)| \leq \text{const.}||x||^\eta, ||x|| \ll 1\},$$

if $f \not\equiv 0$, and $\mathrm{ord}_0 f := +\infty$ otherwise.

Remark 4.3. Since we are in a polynomially bounded o-minimal structure, the Łojasiewicz inequality ensures the well-posedness of the definition. It is a mere exercise to prove that in one variable $g(t) = at^\alpha + o(t^\alpha)$ is written precisely with $\alpha = \mathrm{ord}_0 g$ and $|g(t)| \leq \text{const.}|t|^\alpha$.

By the methods used by Bochnak and Risler in [8] Theorem 1, it is easy to show that the least upper bound in the definition is in fact attained.

The inequality defining the order is satisfied with any exponent $\alpha \leq \mathrm{ord}_0 f$ and it makes sense also for a vector-valued f; then it is written as $||f(x)|| \leq \text{const.}||x||^\eta$. In the latter case, $\mathrm{ord}_0 f$ coincides with the minimal order of the components f_i of $f = (f_1, \ldots, f_k)$ ([13]).

Remark 4.4. For a given function germ $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ the definition of differentiability at zero gives readily the following two implications:

$$f \text{ is differentiable at } 0 \text{ and } \nabla f(0) = 0 \Rightarrow \mathrm{ord}_0 f \geq 1,$$

and

$$\mathrm{ord}_0 f \geq 2 \Rightarrow \ f \text{ is differentiable at } 0 \text{ and } \nabla f(0) = 0.$$

The example of $f(x) = |x|^{3/2}$ shows that there may be $f'(0) = 0$ and $\mathrm{ord}_0 f \in (1, 2)$.

[13] Also in this case the upper bound is attained. If $|f_i(x)| \leq c_i||x||_i^\theta$ for $||x|| \ll 1$ where $c_i > 0$ and $\theta_i = \mathrm{ord}_0 f_i$, then $\max|f_i(x)| \leq (\max c_i)||x||^{\min\theta_i}$ whence $\mathrm{ord}_0 f \geq \min\theta_i$. On the other hand, for the Euclidean norm we have $||f(x)|| \leq |f_i(x)|$ for any i, whence $\mathrm{ord}_0 f \leq \theta_i$.

From a practical point of view it is natural to consider also the following notion.

Definition 4.5 ([5] Definition 3.12)**.** We call *sectional order at zero* for a definable function $f\colon (\mathbb{R}^n,0)\to(\mathbb{R},0)$, $f\not\equiv 0$, the number

$$s_0(f)=\inf\{\alpha>0 \mid f(tv)=at^\alpha+o(t^\alpha), 0\le t\ll 1, v\in\mathbb{S}^{n-1}\colon f|_{\mathbb{R}_+v}\not\equiv 0\}.$$

The relations between these three notions are given in Proposition 3.13 from [5]:

Proposition 4.6. *Consider a non-constant, continuous, definable germ* $f\colon(\mathbb{R}^n,0)\to(\mathbb{R},0)$*. Then for the following three conditions:*

(1) $s_0(f)<2$*;*
(2) $\operatorname{ord}_0 f<2$*;*
(3) $|f|$ *is superquadratic at* 0*;*

we have (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3)*.*

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate. Indeed, if (2) does not hold, then in a neighbourhood of zero, $|f(x)|\le C||x||^2$ for some $C>0$. Thus for $f(tv)$ we have for all $t\ge 0$ small enough, $|at^\alpha+o(t^\alpha)|\le Ct^2$ which implies $\alpha\ge 2$ (divide both sides by t^α and take $t\to 0^+$) and so $s_0(f)\ge 2$. If (3) does not hold, then $|f(x)|\le g_{|f|}(||x||)=a||x||^\alpha+o(||x||^\alpha)$ for some $\alpha\ge 2$. But as $\operatorname{ord}_0 g_{|f|}=\alpha$, we obtain $|f(x)|\le \text{const.}||x||^\alpha$ and so $\operatorname{ord}_0 f\ge 2$.

Furthermore, to see that (3) $\Rightarrow$ (2) suppose that $\operatorname{ord}_0 f\ge 2$ and consider the definable set $A=\{(r,x)\in[0,\varepsilon]\times\mathbb{R}^n \mid ||x||=r, g_{|f|}(||x||)=|f(x)|\}$. Then 0 is an accumulation point of A and so there is a continuous definable selection $r\mapsto(r,\gamma(r))\in A$.

Then $g_{|f|}(r)=|f(\gamma(r))|$ and it follows from the definition of the order of vanishing (note that for small r, the values $\gamma(r)$ are near zero) that $\operatorname{ord}_0 g_{|f|}\ge\operatorname{ord}_0 f$ and so $\operatorname{ord}_0 g_{|f|}\ge 2$ as required. □

Example 4.7 ([5] Example 3.15)**.** The implication (2) $\Rightarrow$ (1) does not hold in general. To see this consider the semi-algebraic function

$$f(x,y)=\begin{cases}0, & x\le 0 \text{ or } y\le 0,\\ \frac{y}{x}, & 0<y\le x \text{ and } x^2+y^2>\frac{y^2}{x^2},\\ (x^2+y^2)\frac{x}{y}, & 0<y\le x \text{ and } x^2+y^2\le\frac{y^2}{x^2},\\ f(y,x), & 0<x<y.\end{cases}$$

It is easy to check that f is continuous. Clearly, $f|_{\mathbb{R}_+v}\not\equiv 0$ iff $v\in\mathbb{S}^2\cap\{x,y>0\}=:S$ in which case $f(tv)=t^2(v_1/v_2)$ for $0\le t\le v_2/v_1$ (for greater t's we get $f(tv)=v_2/v_1$), where $v=(v_1,v_2)$. Hence $s_0(f)=2$.

But if there were $\operatorname{ord}_0 f\ge 2$, then we would have in a neighbourhood of zero, $f(x,y)\le C||(x,y)||^2$ for some constant $C>0$. In particular, this would hold for $(x,y)=tv$ for any $v\in S$ and all $t\in(0,\varepsilon)$ with an appropriate $\varepsilon>0$. However, this would lead to $v_1/v_2\le C$ which yields a contradiction when we make $(v_1,v_2)\in S$ tend to $(1,0)$.

Remark 4.8. The equivalence (2) $\Leftrightarrow$ (3) in the last Proposition allows us to extend the Definition 4.1 to any hypersurface being the graph of a definable function.

Definition 4.9. A set $X \subset \mathbb{R}^{n+1}$ is said to be *superquadratic at* a point $a \in X$, if in some coordinates it can be written in a neighbourhood of a as the graph of a superquadratic function of n variables.

The reason why we confine ourselves — at least for the moment — to hypersurfaces is that the superquadraticity introduced above has a further geometric characterisation similar to Lemma 3.26. First, let us introduce the (open) *bi-ball* (or *bidisc* when we are in the plane) *in the direction* $v \in \mathbb{S}^{n-1}$ as the open set

$$b_v(a,r) := \mathbb{B}(a - rv, r) \cup \mathbb{B}(a + rv, r)$$

where $r > 0$.

Proposition 4.10 ([5] Proposition 3.18)**.** *Let $X \subset \mathbb{R}^n$ be a closed definable set such that the tangent cone $C_0(X)$ is a linear hyperplane and $X \cap U$ is a graph over it, for some neighbourhood U of $0 \in X$. Then the following assertions are equivalent:*

1. *X is superquadratic at the origin.*
2. *For any $r > 0$, $b_{\nu(0)}(0,r) \cap X \neq \varnothing$ where $\nu(0)$ is a unit normal to X at 0.*

Proof. Choose coordinates in $\mathbb{R}^n = \mathbb{R}^{n-1}_x \times \mathbb{R}_t$ so that $C_0(X) = \{t = 0\}$ and write $X = \Gamma_f$ in a neighbourhood of zero. Fix $\nu(0) = (0,1)$.

We start with (1) $\Rightarrow$ (2). Suppose that for some $r > 0$, $b_{\nu(0)}(0,r) \cap X = \varnothing$. This implies that for all $x \in \mathbb{R}^{n-1}$ sufficiently close to zero, we have $(x, |f(x)|) \notin \mathbb{B}(r\nu(0), r)$. On the other hand, observe that for $0 < ||x|| < r$ we have $(x, (1/r)||x||^2) \in \mathbb{B}(r\nu(0), r)$. Summing up, in a neighbourhood of zero, $|f(x)| \leq (1/r)||x||^2$ which means by Proposition 4.6 that X is not superquadratic.

In order to prove (2) $\Rightarrow$ (1) assume that X is not superquadratic at zero. Then by Proposition 4.6 we conclude that $\operatorname{ord}_0 f \geq 2$, i.e. $|f(x)| \leq c||x||^2$ for $||x|| < \varepsilon$ where $c, \varepsilon > 0$ are constants. Observe that for any $0 < r < 1/(2c)$, the graph of $t = c||x||^2$ does not enter the ball $\mathbb{B}((0,r), r)$. This readily implies that $X \cap b_{\nu(0)}(0,r) = \varnothing$, provided we have taken $r < \min\{1/(2c), \varepsilon\}$. □

Now, thanks to this result and in view of Theorem 3.13 (we need only to use the directional reaching radius in this case) we easily obtain the following Proposition:

Proposition 4.11. *If X satisifes the assumptions of the previous Proposition, then*

$$X \text{ is superquadratic at the origin } \Rightarrow\ 0 \in \overline{M_X}.$$

The converse to the implication above does not hold.

Example 4.12 ([5] Remark 4.18)**.** Consider $X = \{z = y|x|^{3/2}\}$ which is the graph of a $\mathscr{C}^1$ function $z = f(x,y)$ in $\mathbb{R}^3$. We easily check that $\operatorname{ord}_0 f \geq 2$ so that X is not superquadratic at the origin, but as it is such along all the other points of the y-axis, we have $0 \in \overline{M_X}$ by the preceding Proposition.

Remark 4.13. Clearly, in view of the last Proposition, if a point $a \in X$ belongs to the closure of superquadratic points in X, then it belongs to $\overline{M_X}$.

The converse, unfortunately, is not true and the question of the relation between superquadraticity of $\mathscr{C}^1$-smooth hypersurfaces in at least three dimensions and the reaching of singularities by the medial axis is settled by the following clever Example of A. Białożyt [3]:

Example 4.14 (A. Białożyt)**.** Let X be the graph of the function

$$f(x,y) = \begin{cases} \frac{y^2}{x}, & |y| < x^3, x > 0; \\ 2x^2|y| - x^5, & |y| \geq x^3, x > 0; \\ 0, & x \leq 0. \end{cases}$$

Then we can check that f is of class $\mathscr{C}^1$, it is not superquadratic at any point and yet $0 \in \overline{M_X} \cap X$ as X contains a suitable part of a rotated cone.

More results about superquadraticity, its generalization for sets of codimension greater than 1 and how can that be exploited in the context of Problem 2 will be published in [3] where the following theorem is shown:

Theorem 4.15 (A. Białożyt)**.** *If $X \subset \mathbb{R}^k_x \times \mathbb{R}^n_y$ is definable with $0 \in \mathrm{Reg}_1 X \cap \mathrm{Sng}_2 X$ and $C_0(X) = \mathbb{R}^k \times \{0\}^n$, then $0 \in \overline{M_X}$ provided X is superquadratic at 0 in the sense that $g_X(\varepsilon) = a\varepsilon^\alpha + o(\varepsilon^\alpha)$ with $a \neq 0$ and $\alpha < 2$ where $g_X(\varepsilon) := \max\{||y||\colon (x,y) \in X, ||x|| = \varepsilon\}$. Moreover, if $\dim_0 X = 1$, then the converse holds: $0 \in \overline{M_X}$ implies X is superquadratic at the origin.*

Remark 4.16. Although the result above, Theorem 3.21 and Remark 4.13 give an answer to Problem 2 for a quite large family of singularities, still much work has to be done in order to definitely settle the question. It seems that Theorem 3.13 should lead to some advances.

5. On the multifunction of closest points

Another question related to the medial axis in the setting of singularity theory is what can be said about the the metric properties of the multifunction $m(x)$. It appears that $m(x)$ satisfies some Łojasiewicz-type inequalities ([5] Proposition 2.16 — see below; here our recent results [20] prove useful). Note that the distance of X along M_X encodes some metric information about the singularities. This, together with the study of the link of M_X, $\mathrm{lk}(M_X, a) = M_X \cap \partial\mathbb{B}(a,\varepsilon)$ (by the Local Conical Structure Theorem it does not depend on $\varepsilon > 0$ sufficiently small), provides some information about the tangent cone of M_X at $a \in \overline{M_X} \cap X$.

Let us begin with a general semicontinuity result that holds regardless of the definability of X.

Proposition 5.1 ([5] Proposition 2.17)**.** *The multifunction $m(x)$ is upper semicontinuous: $\limsup_{D \ni x \to x_0} m(x) = m(x_0)$ at any point $x_0 \in \mathbb{R}^n$ and for*

any dense subset D of $\mathbb{R}^n$. Along M_X we usually only have an inclusion: $\limsup_{M_X \ni x \to x_0} m(x) \subset m(x_0)$.

Proof. See [5] and [16] for the second part of the statement. □

Another useful fact is the following observation that also holds in general.

Proposition 5.2. *Let $U \subset \mathbb{R}^n$ be open and nonempty. Assume that there is a continuous selection $\mu\colon U \to X$ for $m(x)$, i.e. for any $x \in U$, $\mu(x) \in m(x)$. Then $\mu = m|_U$, i.e. $m(x)$ is univalent on U.*

Proof. [5] Proposition 2.19. □

Following [20] we will recall the different possible notions of a fibre of a multifunction $F\colon \mathbb{R}^m \to \mathscr{P}(\mathbb{R}^n)$. For $a \in \mathrm{dom}F$, we consider

- $F^{-1}(F(a)) = \{x \in \mathbb{R}^m \mid F(x) = F(a)\}$ *the (strong) pre-image*;
- $F^*(F(a)) = \{x \in \mathrm{dom}F \mid F(x) \subset F(a)\}$ *the lower pre-image*;
- $F_*(F(a)) = \{x \in \mathbb{R}^m \mid F(x) \supset F(a)\}$ *the upper pre-image*;
- $F^\#(F(a)) = \{x \in \mathbb{R}^m \mid F(x) \cap F(a) \neq \varnothing\}$ *the weak pre-image.*

Finally, we may consider a *point pre-image* defined for a point $y \in F(a)$ as the section $(\Gamma_F)_y := \{x \in \mathbb{R}^m \mid y \in F(x)\}$. Obviously,

$$F^\#(F(a)) = \bigcup_{y \in F(a)} (\Gamma_F)_y.$$

Apart from $m(x)$ we introduced in [5] two more multifunctions of interest, namely, *the normal set* multifunction

$$N(a) = \{x \in \mathbb{R}^n \mid a \in m(x)\} = \{x \in \mathbb{R}^n \mid ||x - a|| = d(x, X)\},\ a \in X$$

and the *univalued normal set* multifunction

$$N'(a) = \{x \in \mathbb{R}^n \mid m(x) = \{a\}\}, \quad a \in X.$$

Proposition 5.3 ([5] Propositions 2.2, 2.6)**.** *In the introduced setting*

(1) $a \in N'(a) \subset N(a)$;
(2) *$N(a)$ is closed, convex and definable (respectively, subanalytic), actually $X \ni a \mapsto N(a)$ is a definable (resp. subanalytic) multifunction, when X is definable (resp. subanalytic);*
(3) $N(a) \subset N_a(X) + a$;
(4) $x \in N'(a) \ \Rightarrow\ [a, x] \subset N'(a)$
and $x \in N(a) \setminus \{a\} \ \Rightarrow\ [a, x) \subset N'(a)$;
(5) *For any non-isolated $a \in X$,* $\limsup_{X \ni b \to a} N(b) \subset N(a)$;
(6) *$N'(a)$ is convex and definable/subanalytic (as a set and as a multifunction of $a \in X$) when X is definable/subanalytic;*
(7) $N(a) = \overline{N'(a)}$.

Clearly, we have

$$M_X = \bigcup_{a\in X} N(a) \setminus N'(a) = \bigcup_{a\in X} N(a) \setminus \bigcup_{a\in X} N'(a) = \mathbb{R}^n \setminus \bigcup_{a\in X} N'(a)$$

cf. [5] Theorem 27.

The different types of pre-images of $m(x)$, $N(a)$ or $N'(a)$ can be explicitly computed, see [5] Subsection 2.2 and the Proposition below.

The Kuratowski convergence of closed subsets of $\mathbb{R}^n$ is metrizable, and thus by the results of [20] Section 6 we have the following Łojasiewicz-type inequalities in the definable setting:

Proposition 5.4 ([5] Proposition 2.16)**.** *Let F denote either the closed multifunction $N(x)$, $x \in X$, or the compact one $m(x)$, $x \in \mathbb{R}^n$. Then for any point x_0 in the domain of F, there are constants $C, \ell > 0$ such that in a neighbourhood of x_0,*

$$\operatorname{dist}_{HK}(F(x), F(x_0)) \geq C d(x, F^\bullet(F(x_0)))^\ell$$

where dist_{HK} denotes the Hausdorff-Kuratowski distance ([14]) and $F^\bullet(F(x_0))$ stands for any of the pre-images introduced above. In particular,

1. $\operatorname{dist}_K(N(x), N(x_0)) \geq C||x - x_0||^\ell$ *for all $x \in X$ near $x_0 \in X$;*
2. $\operatorname{dist}_H(m(x), m(x_0)) \geq C d(x, N'(x_0))^\ell$ *for all $x \in \mathbb{R}^n$ near $x_0 \in X$;*
3. $\operatorname{dist}_H(m(x), m(x_0)) \geq C d(x, N(x_0))^\ell$ *for all $x \in \mathbb{R}^n$ near $x_0 \in X$;*
4. $\operatorname{dist}_H(m(x), m(x_0)) \geq C d(x, \bigcup_{y\in m(x_0)} N(y))^\ell$ *for all $x \in \mathbb{R}^n$ near $x_0 \in \mathbb{R}^n$;*
5. $\operatorname{dist}_H(m(x), m(x_0)) \geq C d(x, M_X)^\ell$ *for all $x \in \mathbb{R}^n$ near $x_0 \in M_X$.*

Fix a definable or subanalytic closed, nonempty proper subset X of $\mathbb{R}^n$ and put

$$M_X(k) = \{x \in M_X \mid \dim m(x) = k\}.$$

These sets are obviously definable or subanalytic, respectively.

Theorem 5.5 ([16] Theorem 4.10, Theorem 4.13)**.** *In the setting considered,*

1. *If $k = n - 1$ (which is the maximal dimension possible), then $\dim M_X(n - 1) = 0$, i.e. $M_X(n - 1)$ is isolated;*
2. *In general $k + \dim M_X(k) \leq n - 1$ and the inequality may be strict already for $k = 1$, $n = 3$.*

Remark 5.6. Point (1) in the theorem above was obtained earlier by Albano and Cannarsa in [2] in a slightly different form and for a general closed set X using the Hausdorff dimension (it coincides with the analytic dimension for a subanalytic set). Point (2), on the contrary, proved in [16] using methods typically from tame geometry, is still waiting for a general counterpart.

Example 5.7. ([16] Example 4.15). Consider $X = \{x^2 + y^2 + z^2 = 1, yz = 0\}$ in $\mathbb{R}^3$. Then $m(0) = X$ and so $0 \in M_X(1)$ and clearly it is the only point in this set. Therefore $1 + \dim M_X(1) < 3 - 1 = 2$.

[14] For compact sets it is the usual Hausdorff distance, for closed ones, it is the metric giving the Kuratowski convergence.

Using the last Theorem, A. Białożyt shows in [3]:

Theorem 5.8 (A.Białożyt)**.** *In the definable setting, for any $a \in M_X$, there is a neighbourhood U of a such that*

$$\dim_a M_X = n - 1 - \min\{\dim m(b) \mid b \in M_X \cap U\}.$$

We have to move around a due to the Białożyt wristwatch example:

Example 5.9 (A. Białożyt)**.** Let X be the boundary of $\mathbb{B}_2(0,3) \cup ((-1,1) \times \mathbb{R})$ in $\mathbb{R}^2$. Then for $a = 0$, we have $\dim m(a) = 1$ and $\dim_a M_X = 1$, as $M_X = \{0\} \times \mathbb{R}$. Only taking any other point $b \in M_X \setminus \{a\}$ gives the equality from the Theorem.

Further studies on the subject are presented in [3].

6. Closing remarks

Several results concerning the topological structure of the medial axis are known. For instance in [21] Theorem 1.B it is shown that for a domain $D \subset \mathbb{R}^n$ that does not contain any half-space (cf. Remark 2.2) the set $M_X \cap D$ is connected. This intuitive result has an astonishingly intricate proof (see [17] for a self-contained one; see also the inspiring paper [23]). Moreover, Fremlin proves also an interesting fact [21] Proposition 1.F which hints at the fact that the medial axis should not have 'bad' singularities itself, i.e. no cusps are allowed (this is partly confirmed by the results of [4]). Also Yomdin in his very nice paper [28] presented a general structural result concerning the medial axis, however the proof is fallacious as it is based on a non-existing (and most probably false) version of a Lipschitz Implicit Function Theorem (LIFT). We discuss this problem in [19] obtaining a correct LIFT which, nonetheless, allows us to reprove Yomdin's stability result only in a generic case in $\mathbb{R}^3$.

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Jagiellonian University, Faculty of Mathematics and Computer Science, Institute of Mathematics, Łojasiewicza 6, 30-348 Kraków, Poland

E-mail address: `maciej.denkowski@uj.edu.pl`

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NEGATIVE CURVES ON SPECIAL RATIONAL SURFACES

MARCIN DUMNICKI, ŁUCJA FARNIK, KRISHNA HANUMANTHU,
GRZEGORZ MALARA, TOMASZ SZEMBERG, JUSTYNA SZPOND,
AND HALSZKA TUTAJ-GASIŃSKA

Abstract. We study negative curves on surfaces obtained by blowing up special configurations of points in $\mathbb{P}^2$. Our main results concern the following configurations: very general points on a cubic, 3–torsion points on an elliptic curve and nine Fermat points. As a consequence of our analysis, we also show that the Bounded Negativity Conjecture holds for the surfaces we consider. The note contains also some problems for future attention.

1. Introduction

Negative curves on algebraic surfaces are an object of classical interest. One of the most prominent achievements of the Italian School of algebraic geometry was Castelnuovo's Contractibility Criterion.

Definition 1.1 (Negative curve)**.** *We say that a reduced and irreducible curve C on a smooth projective surface is* negative*, if its self-intersection number C^2 is less than zero.*

Example 1.2 (Exceptional divisor, (-1)-curves)**.** Let X be a smooth projective surface and let $P \in X$ be a closed point. Let $f : \mathrm{Bl}_P X \to X$ be the blow up of X at the point P. Then the exceptional divisor E of f (i.e., the set of points in $\mathrm{Bl}_P X$ mapped by f to P) is a negative curve. More precisely, E is rational and $E^2 = -1$. By a slight abuse of language we will call such curves simply (-1)–curves.

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Castelnuovo's result asserts that the converse is also true; for example, see [10, Theorem V.5.7] or [1, Theorem III.4.1].

Theorem 1.3 (Castelnuovo's Contractibility Criterion)**.** *Let Y be a smooth projective surface defined over an algebraically closed field. If C is a rational curve with $C^2 = -1$, then there exists a smooth projective surface X and a projective morphism $f : Y \to X$ contracting C to a point on X. In other words, Y is isomorphic to $\mathrm{Bl}_P X$ for some point $P \in X$.*

The above result plays a pivotal role in the Enriques-Kodaira classification of surfaces.

Of course, there are other situations in which negative curves on algebraic surfaces appear.

Example 1.4. Let C be a smooth curve of genus $g(C) \geq 2$. Then the diagonal $\Delta \subset C \times C$ is a negative curve as its self-intersection is given by $\Delta^2 = 2 - 2g$.

It is quite curious that it is in general not known if for a general curve C, there are other negative curves on the surface $C \times C$, see [12]. It is in fact even more interesting, that there is a direct relation between this problem and the famous Nagata Conjecture, which was observed by Ciliberto and Kouvidakis [5].

There is also a connection between negative curves and the Nagata Conjecture on general blow ups of $\mathbb{P}^2$. We recall the following conjecture about (-1)-curves which in fact implies the Nagata Conjecture; see [4, Lemma 2.4].

Conjecture 1.5 (Weak SHGH Conjecture)**.** *Let $f : X \to \mathbb{P}^2$ be the blow up of the projective plane $\mathbb{P}^2$ in general points $P_1, \ldots, P_s$. If $s \geq 10$, then the only negative curves on X are the (-1)–curves.*

On the other hand, it is well known that already a blow up of $\mathbb{P}^2$ in 9 general points carries *infinitely* many (-1)–curves.

One of the central and widely open problems concerning negative curves on algebraic surfaces asks whether on a fixed surface negativity is bounded. More precisely, we have the following conjecture (BNC in short). See [2] for an extended introduction to this problem.

Conjecture 1.6 (Bounded Negativity Conjecture)**.** *Let X be a smooth projective surface. Then there exist a number τ such that*

$$C^2 \geq \tau$$

for any reduced and irreducible curve $C \subset X$.

If the Conjecture holds on a surface X, then we denote by $b(X)$ the largest number τ such that the Conjecture holds. It is known (see [2, Proposition 5.1]) that if the negativity of reduced and irreducible curves is bounded below, then the negativity of all reduced curves is also bounded below.

Conjecture 1.6 is known to fail in the positive characteristic; see [8, 2]. In fact Example 1.4 combined with the action of the Frobenius morphism provides a counterexample. In characteristic zero, Conjecture 1.6 is open in general. It is easy to prove BNC in some cases; see Remark 3.7 for an easy argument when the anti-canonical divisor of X is $\mathbb{Q}$-effective. However, in many other cases the conjecture is open. In particular the following question is open and answering it may lead to a better understanding of Conjecture 1.6.

Question 1.7. Let X, Y be smooth projective surfaces and suppose that X and Y are birational and Conjecture 1.6 holds for X. Does then Conjecture 1.6 hold for Y also?

As a special case of this question, one can ask whether Conjecture 1.6 holds for blow ups of $\mathbb{P}^2$. Since the conjecture clearly holds for $\mathbb{P}^2$, it is interesting to consider the blow ups of $\mathbb{P}^2$. If the blown up points are general, then one has Conjecture 1.5 stated above. On the other hand, it is also interesting to study blow ups of $\mathbb{P}^2$ at special points.

In this paper, we consider some examples of such special rational surfaces and completely list all the negative curves on them. In particular, we focus on blow ups of $\mathbb{P}^2$ at certain points which lie on elliptic curves. Our main results classify negative curves on such surfaces; see Theorems 2.4, 3.3 and 3.6. As a consequence, we show that Conjecture 1.6 holds for such surfaces. Additionally we provide effective optimal values of the number $b(X)$.

2. Very general points on a cubic

In this section we study negative curves on blow ups of $\mathbb{P}^2$ at an arbitrary number s of very general points on a plane curve of degree 3. This situation was studied in detail by Harbourne in [9]. Before stating our main result we need to recall some notation. For the first notion, see [6, Definition 5] or [7] where this property is called *adequate* rather than standard.

Definition 2.1 (Standard form)**.** *Let $P_1, \dots, P_s$ be points in $\mathbb{P}^2$. Let Γ be a plane curve of degree d with $m_i := \operatorname{mult}_{P_i} \Gamma$, for $i = 1, \dots, s$. We say that Γ is in the* standard form *if*

- *the multiplicities $m_1, \dots, m_s$ form a weakly decreasing sequence and*
- *$d \geq m_1 + m_2 + m_3$.*

Gimigliano showed in [7, page 25] that if the points $P_1, \dots, P_s$ are general in $\mathbb{P}^2$, then any curve Γ can be brought to the standard form by a finite sequence of standard Cremona transformations.

Theorem 2.2 (Gimigliano)**.** *Let $P_1, \dots, P_s$ be general points in $\mathbb{P}^2$. Let Γ be a curve of degree d passing through points $P_1, \dots, P_s$ with multiplicities $m_1, \dots, m_s$. Then there exists a birational transformation σ of $\mathbb{P}^2$ and general points $P'_1, \dots, P'_s$*

and a curve Γ' *of degree* d' *passing through* $P'_1, \ldots, P'_s$ *with multiplicities* $m'_1, \ldots, m'_s$ *such that*

- Γ' *is in a standard form;*
- $\Gamma' = \sigma(\Gamma)$*;*
- $d^2 - \sum_{i=1}^{s} m_i^2 = (d')^2 - \sum_{i=1}^{s} (m'_i)^2$.

We recall also the following Lemma, which is modeled on [7, Lemma 3.2].

Lemma 2.3. *Let* $d \geq m_1 \geq \ldots \geq m_r \geq 0$ *and* $t \geq n_1 \geq \ldots \geq n_r \geq 0$ *be integers. Further assume that* $d \geq m_1 + m_2$, $3d \geq m_1 + \ldots + m_r$ *and* $t \geq n_1 + n_2 + n_3$. *Then* $dt \geq \sum_i m_i n_i$.

Proof. We first note that if $m_3 = 0$, then the lemma follows easily. Indeed, $d \geq m_1 + m_2$, $t \geq n_1 + n_2 + n_3$ imply $dt \geq m_1 n_1 + m_2 n_2$.

We now induct on d. If at any point we have $m_3 = 0$, we are done by the above argument.

The base case is $d = 0$, which is easy.

Suppose the statement is true for $d - 1$. Given $d, m_1, m_2, \ldots, m_r$ satisfying the hypothesis, consider $d - 1, m_1 - 1, m_2 - 1, m_3 - 1, m_4, \ldots, m_r$. Note that $m_3 > 0$.

Then the tuple $(d-1, m_1-1, m_2-1, m_3-1, m_4, \ldots, m_r)$ satisfies the hypothesis, after permuting the m_i if necessary. If $m_4 = d$, then $m_1 = m_2 = m_3 = m_4 = d$ and this violates $3d \geq m_1 + \ldots + m_r$. So $m_i < d$ for all $i \geq 4$.

By induction hypothesis,

$$(d-1)t \geq (m_1 - 1)n_1 + (m_2 - 1)n_2 + (m_3 - 1)n_3 + m_4 n_4 + \ldots + m_r n_r$$

implies

$$dt - \sum_i m_i n_i \geq t - n_1 - n_2 - n_3 \geq 0.$$

□

Now we are in a position to prove our first result.

Theorem 2.4 (Very general points on a cubic)**.** *Let* D *be an irreducible and reduced plane cubic and let* $P_1, \ldots, P_s$ *be very general points on* D. *Let* $f : X \longrightarrow \mathbb{P}^2$ *be the blow up at* $P_1, \ldots, P_s$. *If* $C \subset X$ *is any reduced and irreducible curve such that* $C^2 < 0$, *then*

a) C *is the proper transform of* D, *or*
b) C *can be brought by a Cremona transformation to the proper transform of a line in* $\mathbb{P}^2$ *through any two of the points* $P_1, \ldots, P_s$, *or*
c) C *is an exceptional divisor of* f.

Proof. Assume that C is a reduced and irreducible curve on X different from the curves mentioned in cases a), b) or c). Then $C = dH - m_1 E_1 - \ldots - m_s E_s$, for

some $d \geq 1$ and $m_1, \ldots, m_s \geq 0$. Here $H = f^\star(\mathcal{O}_{\mathbb{P}^2}(1))$ and $E_i = f^{-1}(P_i)$ are the exceptional divisors of f.

Intersecting C with the proper transform of D we get

$$3d \geq m_1 + \ldots + m_r. \tag{2.1}$$

Let $\Gamma = f(C)$ be the image of C on $\mathbb{P}^2$. Then Γ has a singularity of order at least m_i at p_i for $i = 1, \ldots, s$. By Theorem 2.2, we can assume that Γ is in the standard form, so that

$$d \geq m_1 + m_2 + m_3 \text{ and } m_1 \geq m_2 \geq \ldots \geq m_s. \tag{2.2}$$

Now inequalities (2.1) and (2.2) allow us to use Lemma 2.3 with $t = d$ and $n_i = m_i$ for $i = 1, \ldots, s$. We get

$$d^2 \geq m_1^2 + m_2^2 + \ldots + m_r^2,$$

which is equivalent to $C^2 \geq 0$. This shows that the only negative curves on X are the curves listed in a), b) or c). □

Corollary 2.5. *Let X be a surface as in Theorem 2.4 with $s > 0$. Then Conjecture 1.6 holds for X and we have*

$$b(X) = \min\{-1,\ 9 - s\}.$$

3. Special points on a cubic

In this section, we consider blow ups of $\mathbb{P}^2$ at 3-torsion points of an elliptic curve as well as the points of intersection of the Fermat arrangement. In order to consider these two cases, we deal first with the following numerical lemma which seems quite interesting in its own right.

Lemma 3.1. *Let $m_1, \ldots, m_9$ be nonnegative real numbers satisfying the following 12 inequalities:*

$$m_1 + m_2 + m_3 \leq 1, \tag{3.1}$$
$$m_4 + m_5 + m_6 \leq 1, \tag{3.2}$$
$$m_7 + m_8 + m_9 \leq 1, \tag{3.3}$$
$$m_1 + m_4 + m_7 \leq 1, \tag{3.4}$$
$$m_2 + m_5 + m_8 \leq 1, \tag{3.5}$$
$$m_3 + m_6 + m_9 \leq 1, \tag{3.6}$$
$$m_1 + m_5 + m_9 \leq 1, \tag{3.7}$$

$$m_2 + m_6 + m_7 \leq 1, \tag{3.8}$$

$$m_3 + m_4 + m_8 \leq 1, \tag{3.9}$$

$$m_1 + m_6 + m_8 \leq 1, \tag{3.10}$$

$$m_2 + m_4 + m_9 \leq 1, \tag{3.11}$$

$$m_3 + m_5 + m_7 \leq 1. \tag{3.12}$$

Then $m_1^2 + \cdots + m_9^2 \leq 1$.

Proof. Assume that the biggest number among $m_1, \ldots, m_9$ is $m_1 = 1 - m$ for some $0 \leq m \leq 1$.

Consider the following four pairs of numbers

$$p_1 = (m_2, m_3),\ p_2 = (m_4, m_7),\ p_3 = (m_9, m_5),\ p_4 = (m_6, m_8).$$

These are pairs such that together with m_1 they occur in one of the 12 inequalities. In each pair one of the numbers is greater or equal than the other. Let us call this bigger number a *giant*. A simple check shows that there are always three pairs, such that their giants are subject to one of the 12 inequalities in the Lemma.

Without loss of generality, let p_1, p_2, p_3 be such pairs. Also without loss of generality, let m_2, m_4 and m_9 be the giants. Thus $m_2 + m_4 + m_9 \leq 1$. Assume that also m_6 is a giant.

Inequality $m_2 + m_3 \leq m$ implies that

$$m_2^2 + m_3^2 = (m_2 + m_3)^2 - 2m_2m_3 \leq m(m_2 + m_3) - 2m_2m_3.$$

Observe also that

$$(m_2 + m_3)^2 - 4m_2m_3 \leq m(m_2 - m_3).$$

Analogous inequalities hold for pairs p_2, p_3 and p_4. Therefore

$$\begin{gathered} m_2^2 + m_3^2 + m_4^2 + m_7^2 + m_5^2 + m_9^2 \leq \\ \leq m(m_2 + m_4 + m_9 + m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9 \leq \\ \leq m + \big[m(m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9\big]. \end{gathered}$$

But we have also

$$\begin{gathered} m_2^2 + m_3^2 + m_4^2 + m_7^2 + m_5^2 + m_9^2 = \\ = (m_2 + m_3)^2 + (m_4 + m_7)^2 + (m_5 + m_9)^2 - 2m_2m_3 - 2m_4m_7 - 2m_5m_9 = \\ = (m_2 + m_3)^2 - 4m_2m_3 + (m_4 + m_7)^2 - 4m_4m_7 + \\ + (m_5 + m_9)^2 - 4m_5m_9 + 2m_2m_3 + 2m_4m_7 + 2m_5m_9 \leq \\ \leq m(m_2 - m_3) + m(m_4 - m_7) + m(m_9 - m_5) + 2m_2m_3 + 2m_4m_7 + 2m_5m_9 \leq \\ \leq m - \big[m(m_3 + m_7 + m_5) - 2m_2m_3 - 2m_4m_7 - 2m_5m_9\big], \end{gathered}$$

which obviously gives

$$m_2^2 + m_3^2 + m_4^2 + m_7^2 + m_5^2 + m_9^2 \leq m.$$

Since

$$m_6^2 + m_8^2 \leq m_6^2 + m_6 m_8 \leq m_6(m_6 + m_8) \leq (1-m)m,$$

we get that the sum of all nine squares is bounded by

$$(1-m)^2 + m + (1-m)m = 1. \qquad \square$$

If we think of numbers $m_1, \ldots, m_9$ as arranged in a 3×3 matrix

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix},$$

then the inequalities in the Lemma 3.1 are obtained considering the horizontal, vertical and the triples determined by the condition that there is exactly one element m_i in every column and every row of the matrix (so determined by permutation matrices). Bounding sums of only such triples allows us to bound the sum of squares of all entries in the matrix. It is natural to wonder, if this phenomena extends to higher dimensional matrices. One possible extension is formulated as the next question.

Problem 3.2. Let $M = (m_{ij})_{i,j=1\ldots k}$ be a matrix whose entries are non-negative real numbers. Assume that all the horizontal, vertical and permutational k-tuples of entries in the matrix M are bounded by 1. Is it true then that the sum of squares of all entries of M is also bounded by 1?

3.1. **Torsion points.** We now consider a blow up of $\mathbb{P}^2$ at 9 points which are torsion points of order 3 on an elliptic curve embedded as a smooth cubic.

Theorem 3.3 (3–torsion points on an elliptic curve)**.** *Let D be a smooth plane cubic and let $P_1, \ldots, P_9$ be the flexes of D. Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $P_1, \ldots, P_9$. If C is a negative curve on X, then*

a) *C is the proper transform of a line passing through two (hence three) of the points $P_1, \ldots, P_9$, or*
b) *C is an exceptional divisor of f.*

Proof. It is well known that there is a group law on D such that the flexes are 3–torsion points. Since any line passing through two of the torsion points automatically meets D in a third torsion point, there are altogether 12 such lines. The torsion points form a subgroup of D which is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$. We can pick this isomorphism so that

$$P_1 = (0,0),\ P_2 = (1,0),\ P_3 = (2,0),$$
$$P_4 = (0,1),\ P_5 = (1,1),\ P_6 = (2,1),$$
$$P_7 = (0,2),\ P_8 = (1,2),\ P_9 = (2,2).$$

This implies that the following triples of points are collinear:

$$(P_1, P_2, P_3),\ (P_4, P_5, P_6),\ (P_7, P_8, P_9),\ (P_1, P_4, P_7),$$
$$(P_2, P_5, P_8),\ (P_3, P_6, P_9),\ (P_1, P_5, P_9),\ (P_2, P_6, P_7),$$

$$(P_3, P_4, P_8),\ (P_1, P_6, P_8),\ (P_2, P_4, P_9),\ (P_3, P_5, P_7).$$

Let C be a reduced and irreducible curve on X different from the exceptional divisors of f and the proper transforms of lines through the torsion points. Then C is of the form

$$C = dH - k_1E_1 - \ldots - k_9E_9,$$

where $E_1, \ldots, E_9$ are the exceptional divisors of f and $k_1, \ldots, k_9 \geq 0$ and and $d > 0$ is the degree of the image $f(C)$ in $\mathbb{P}^2$.

For $i = 1, \ldots, 9$, let $m_i = \frac{k_i}{d}$. Since C is different from proper transforms of the 12 lines distinguished above, taking the intersection product of C with the 12 lines, and dividing by d, we obtain exactly the 12 inequalities in Lemma 3.1. The conclusion of Lemma 3.1 implies then that

$$C^2 = d^2 - \sum_{i=1}^{9} m_i^2 \geq 0,$$

which finishes our argument. □

Corollary 3.4. *Let X be a surface as in Theorem 3.3. Then Conjecture 1.6 holds for X and we have*

$$b(X) = -2.$$

Of course, there is no reason to restrict to 3–torsion points. In particular there is the following natural question, which we hope to come back to in the near future.

Problem 3.5. For $m \geq 4$, decide whether the Bounded Negativity Conjecture holds on the blow ups of $\mathbb{P}^2$ at all the m–torsion points of an elliptic curve embedded as a smooth cubic.

3.2. **Fermat configuration of points.** The 9 points and 12 lines considered in the above subsection form the famous Hesse arrangement of lines; see [11]. Any such arrangement is projectively equivalent to that obtained from the flex points of the Fermat cubic $x^3 + y^3 + z^3 = 0$ and the lines determined by their pairs. Explicitly in coordinates we have then

$$P_1 = (1 : \varepsilon : 0),\ P_2 = (1 : \varepsilon^2 : 0),\ P_3 = (1 : 1 : 0),$$
$$P_4 = (1 : 0 : \varepsilon),\ P_5 = (1 : 0 : \varepsilon^2),\ P_6 = (1 : 0 : 1),$$
$$P_7 = (0 : 1 : \varepsilon),\ P_8 = (0 : 1 : \varepsilon^2),\ P_9 = (0 : 1 : 1)$$

for the points and

$$x = 0,\ y = 0,\ z = 0,\ x+y+z = 0, x+y+\varepsilon z = 0,\ x+y+\varepsilon^2 z = 0,\ x+\varepsilon y+z = 0,$$
$$x+\varepsilon^2 y+z = 0,\ x+\varepsilon y+\varepsilon z = 0,\ x+\varepsilon y+\varepsilon^2 z = 0,\ x+\varepsilon^2 y+\varepsilon z = 0, x+\varepsilon^2 y+\varepsilon^2 z = 0$$

for the lines, where ε is a primitive root of unity of order 3.

Passing to the dual plane, we obtain an arrangement of 9 lines defined by the linear factors of the Fermat polynomial

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3) = 0.$$

These lines intersect in triples in 12 points, which are dual to the lines of the Hesse arrangement. The resulting dual Hesse configuration has the type $(9_4, 12_3)$ and it belongs to a much bigger family of Fermat arrangements; see [14]. Figure 1 is an attempt to visualize this arrangement (which cannot be drawn in the real plane due to the famous Sylvester-Gallai Theorem; for instance, see [13]).

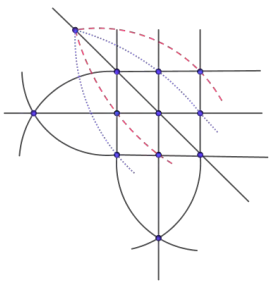

FIGURE 1. Fermat configuration of points

It is convenient to order the 9 intersection points in the affine part in the following way:

$$\begin{array}{lll} Q_1 = (\varepsilon : \varepsilon : 1), & Q_2 = (1 : \varepsilon : 1), & Q_3 = (\varepsilon^2 : \varepsilon : 1), \\ Q_4 = (\varepsilon : 1 : 1), & Q_5 = (1 : 1 : 1), & Q_6 = (\varepsilon^2 : 1 : 1), \\ Q_7 = (\varepsilon : \varepsilon^2 : 1), & Q_8 = (1 : \varepsilon^2 : 1), & Q_9 = (\varepsilon^2 : \varepsilon^2 : 1). \end{array}$$

With this notation established, we have the following result.

Theorem 3.6 (Fermat points)**.** *Let $f : X \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at $Q_1, \ldots, Q_9$. If C is a negative curve on X, then*

a) *C is the proper transform of a line passing through two or three of the points $Q_1, \ldots, Q_9$, or*
b) *C is an exceptional divisor of f.*

Proof. The proof of Theorem 3.3 works with very few adjustments.

Let us assume, to begin with, that C is a negative curve on X, distinct from the curves listed in the theorem. Then

$$C = dH - k_1E_1 - \ldots - k_9E_9,$$

for some $d > 0$ and $k_1, \ldots, k_9 \geq 0$. We can also assume that d is the smallest number for which such a negative curve exists. As before, we set

$$m_i = \frac{k_i}{d} \quad \text{for} \quad i = 1, \ldots, 9.$$

Then the inequalities (3.1) to (3.9) follow from the fact that C intersects the 9 lines in the arrangement non-negatively.

If one of the remaining inequalities (3.10), (3.11) or (3.12) fails, then we perform a standard Cremona transformation based on the points involved in the failing inequality. For example, if (3.10) fails, we make Cremona based on points Q_1, Q_6 and Q_8. Note that these points are *not* collinear in the set-up of our Theorem.

Since C is assumed not to be a line through any two of these points, its image C' under Cremona is a curve of strictly lower degree, negative on the blow up of $\mathbb{P}^2$ at the 9 points. The points $Q_1, \ldots, Q_9$ remain unchanged by the Cremona because, as already remarked, all dual Hesse arrangements are projectively equivalent, see [16]. Then C' is again a negative curve on X of degree strictly lower than d, which contradicts our choice of C such that $C \cdot H$ is minimal.

Hence, we can assume that the inequalities (3.10), (3.11) and (3.12) are also satisfied. Then we conclude exactly as in the proof of Theorem 3.3. □

Remark 3.7. If we are interested only in the bounded negativity property on X, the assertion follows from the fact, that $-K_X$ is $\mathbb{Q}$-effective. Indeed, if $C \subset X$ is a reduced and irreducible curve, from the genus formula we get

$$1 + \frac{C \cdot (C + K_X)}{2} = g(C) \geq 0,$$

so

$$C^2 \geq -2 - CK_X.$$

The bounded negativity follows from the fact that $-CK_X$ may be negative only in finite number of cases.

Having classified all the negative curves on the blow up of $\mathbb{P}^2$ at the 9 Fermat points, it is natural to wonder about the negative curves on blow ups of $\mathbb{P}^2$ arising from the other Fermat configurations. Note that the argument given in Remark 3.7 is no longer valid, since $-K_X$ is not nef nor effective. So it will be interesting to ask whether BNC holds for such surfaces. We pose the following problem.

Problem 3.8. For a positive integer m, let $Z(m)$ be the set of all points of the form

$$(1 : \varepsilon^\alpha : \varepsilon^\beta),$$

where ε is a primitive root of unity of order m and $1 \leq \alpha, \beta \leq m$. Let $f_m : X(m) \to \mathbb{P}^2$ be the blow up of $\mathbb{P}^2$ at all the points of $Z(m)$. Is the negativity bounded on $X(m)$? If so, what is the value of $b(X(m))$?

We end this note by the following remark which discusses bounded negativity for blow ups of $\mathbb{P}^2$ at 10 points.

Remark 3.9. Let X denote a blow up of $\mathbb{P}^2$ at 10 points. As mentioned before, if the blown up points are general, then Conjecture 1.5 predicts that the only negative curves on X are (-1)-curves. This is an open question. On the other hand, let us consider a couple of examples of special points.

Let X be obtained by blowing up the 10 nodes of an irreducible and reduced rational nodal sextic. Such surfaces are called *Coble surfaces* (these are smooth rational surfaces X such that $|-K_X| = \varnothing$, but $|-2K_X| \neq \varnothing$). Then it is known that BNC holds for X. In fact, we have $C^2 \geq -4$ for every irreducible and reduced curve $C \subset X$; see [3, Section 3.2].

Now let X be the blow up of 10 double points of intersection of 5 general lines in $\mathbb{P}^2$. Then $-K_X$ is a big divisor and by [15, Theorem 1], X is a *Mori dream space*. For such surfaces, the submonoid of the Picard group generated by the effective classes is finitely generated. Hence BNC holds for X ([8, Proposition I.2.5]).

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(Marcin Dumnicki) Jagiellonian University, Faculty of Mathematics and Computer Science, Łojasiewicza 6, PL-30-348 Kraków, Poland

E-mail address: Marcin.Dumnicki@uj.edu.pl

(Łucja Farnik) Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: Lucja.Farnik@gmail.com

(Krishna Hanumanthu) Chennai Mathematical Institute, H1 SIPCOT IT Park, Siruseri, Kelambakkam 603103, India

E-mail address: krishna@cmi.ac.in

(Grzegorz Malara) Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: grzegorzmalara@gmail.com

(Tomasz Szemberg) Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: tomasz.szemberg@gmail.com

(Justyna Szpond) Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: szpond@up.krakow.pl

(Halszka Tutaj-Gasińska) Jagiellonian University, Faculty of Mathematics and Computer Science, Łojasiewicza 6, PL-30-348 Kraków, Poland

E-mail address: halszka.tutaj-gasinska@im.uj.edu.pl

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GELFOND-MAHLER INEQUALITY FOR MULTIPOLYNOMIAL RESULTANTS

ALEKSANDRA GALA-JASKÓRZYŃSKA, KRZYSZTOF KURDYKA, KATARZYNA RUDNICKA, AND STANISŁAW SPODZIEJA

Abstract. We give a bound of the height of a multipolynomial resultant in terms of polynomial degrees, the resultant of which applies. Additionally we give a Gelfond-Mahler type bound of the height of homogeneous divisors of a homogeneous polynomial.

1. Introduction

Let $f \in \mathbb{Z}[u]$, where $u = (u_1, \dots, u_N)$ is a system of variables and $\mathbb{Z}$ is the ring of integers, be a nonzero polynomial of the form

$$f(u) = \sum_{|\nu| \leqslant d_f} a_\nu u^\nu, \tag{1}$$

where $a_\nu \in \mathbb{Z}$, $u^\nu = u_1^{\nu_1} \cdots u_N^{\nu_N}$ and $|\nu| = \nu_1 + \cdots + \nu_N$ for $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{N}^N$ and $\mathbb{N}$ denotes the set of nonnegative integers. By the *height* of the polynomial f we mean

$$H(f) := \max\{|a_\nu| : \nu \in \mathbb{N}^N,\ |\nu| \leq d_f\}.$$

Let $f_1, \dots, f_r \in \mathbb{Z}[u]$ be nonzero polynomials, and let d_j be the degree of $f = f_1 \cdots f_r$ with respect to u_j for $j = 1, \dots, N$.

A.P. Gelfond [3] obtained the following bound.

Theorem 1.1 (Gelfond)**.**

$$H(f_1) \cdots H(f_r) \leqslant 2^{d_1 + \cdots + d_N - k} \sqrt{(d_1 + 1) \cdots (d_N + 1)} H(f) \tag{2}$$

where k is the number of variables u_j that genuinely appear in f.

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K. Mahler [6] introduced a measure $M(f)$ of a polynomial $f \in \mathbb{C}[u]$ (currently called *Mahler measure*, see Section 2.1) and in [7] reproved (2) and proved the following

Theorem 1.2 (Mahler)**.** *Under notations of Theorem 1.1,*

$$H(f) \leqslant 2^{d_1+\cdots+d_N-k} M(f). \tag{3}$$

Moreover,

$$L_1(f_1)\cdots L_1(f_r) \leqslant 2^{d_1+\cdots+d_N} M(f) \leqslant 2^{d_1+\cdots+d_N} L_1(f), \tag{4}$$

where $L_1(f) := \sum_{|\nu|\leqslant d_f} |a_\nu|$ *is the* L_1-norm *of a polynomial* f *of the form* (1)*.*

The aim of the article is to obtain a similar to the above-described estimates for the height, L_1-norms and Mahler's measures of a resultant for systems of homogeneous forms. More precisely let $d_0, \ldots, d_n$ be fixed positive integers and let $f_0, \ldots, f_n$ be a system of homogeneous polynomials in $x = (x_0, \ldots, x_n)$ with indeterminate coefficients of degrees $d_0, \ldots, d_n$ in x, respectively. By a resultant $\mathrm{Res}_{d_0,\ldots,d_n}$ we mean the unique irreducible polynomial in the coefficients of $f_0, \ldots, f_n$ with integral coefficients such that for any specializations $f_{0,a_0}, \ldots, f_{n,a_n}$ of $f_0, \ldots, f_n$, the value $\mathrm{Res}_{d_0,\ldots,d_n}(f_{0,a_0}, \ldots, f_{n,a_n})$ is equal to zero if and only if the polynomials $f_{0,a_0}, \ldots, f_{n,a_n}$ have a common nontrivial zero. For basic notations and properties of the resultants, see Section 3.1 and for more detailed description on the resultant see for instance [2]. The main result of this paper is Theorem 3.12 which says that:

$$\begin{aligned} M(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*+1)^{nK_n d_*^n}, \\ H(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*+1)^{n(K_n+n+1)d_*^n - n(n+1)}, \\ L_1(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*+1)^{n(K_n+n+1)d_*^n}, \end{aligned}$$

where $K_n = e^{n+1}/\sqrt{2\pi n}$ and $d_* = \max\{d_0, \ldots, d_n\}$. Moreover if $n \geqslant 2$ and $d_* \geqslant 4$ then we have the following estimates:

$$\begin{aligned} M(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*)^{nK_n d_*^n}, \\ H(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*)^{n(K_n+n+1)d_*^n - n(n+1)}, \\ L_1(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*)^{n(K_n+n+1)d_*^n}. \end{aligned}$$

Note that the above estimates of $L_1(\mathrm{Res}_{d_0,\ldots,d_n})$ are not a direct consequences of the estimates of $H(\mathrm{Res}_{d_0,\ldots,d_n})$ (see Remark 3.13).

M. Sombra in [9], as a corollary from a study of the height of the mixed sparse resultant, gave an estimation of $H(\mathrm{Res}_{d,\ldots,d})$:

$$H(\mathrm{Res}_{d,\ldots,d}) \leqslant (d+1)^{n(n+1)!d^n}.$$

Since $K_n + n + 1 = n + 1 + e^{n+1}/\sqrt{2\pi n} < (n+1)!$ for $n \geqslant 3$, so the estimation (26) is more explicit than the above for $n \geqslant 3$.

The paper is organized as follows. In Section 2 we collect basic notations concering the Mahler measure of a polynomial and we prove a Mahler type bounds for

the height and the L_1-norm of multihomogeneous polynomials (see Lemma 2.2). The proof of Theorem 3.12 we give in Section 3. The crucial role in the proof plays an estimation of the L_1 norm of the Macaulay discriminant of a coefficients matrix for a powers of polynomials $f_0, \dots, f_n$ (see Lemma 3.9).

Additionally, in Section 4 we give Corollaries 4.1 and 4.2 which are versions of Theorems 1.1 and 1.2 for the multihomogeneous and homogeneous polynomials cases.

2. Auxiliary results

2.1. **Notations.** Let $f \in \mathbb{C}[u]$, where $u = (u_1, \dots, u_N)$ is a system of variables, be a nonzero polynomial of the form

$$f(u) = \sum_{|\nu| \leqslant d_f} a_\nu u^\nu, \tag{5}$$

where for $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{N}^N$ the coefficient a_ν is a complex number and we put $|\nu| = \nu_1 + \cdots + \nu_N$ and $u^\nu = u_1^{\nu_1} \cdots u_N^{\nu_N}$.

In this section I denotes the interval $[0, 1]$ and i the imaginary unit (i.e., $i^2 = -1$). Let $\mathbf{e} : I^N \to \mathbb{C}^N$ be a mapping defined by

$$\mathbf{e}(\mathbf{t}) = (\exp(2\pi t_1 i), \dots, \exp(2\pi t_N i)) \quad \text{for } \mathbf{t} = (t_1, \dots, t_N) \in I^N.$$

For a complex polynomial $f \in \mathbb{C}[u]$, the number

$$M(f) = \exp\left(\int_{I^N} \log|f(\mathbf{e}(\mathbf{t}))| d\mathbf{t}\right)$$

is called the *Mahler measure* of f (see [7]). A significient property of the Mahler measure is the following (see [7]): for $f, g \in \mathbb{C}[u]$,

$$M(fg) = M(f)M(g). \tag{6}$$

Moreover, if $f \in \mathbb{Z}[u]$, $f \neq 0$, then (see for instance [8, Corollary 2]),

$$M(f) \geqslant 1. \tag{7}$$

By L_2-*norm* of a polynomial $f \in \mathbb{C}[u]$ we mean

$$L_2(f) = \left(\int_{I^N} |f(\mathbf{e}(\mathbf{t}))|^2 d\mathbf{t}\right)^{1/2}.$$

For a polynomial $f \in \mathbb{C}[u]$ of the form (5) we have

$$L_2(f) = \left(\sum_{|\nu| \leqslant d_f} |a_\nu|^2\right)^{1/2}, \tag{8}$$

By Jensen's inequality we obtain

$$M(f) \leqslant L_2(f). \tag{9}$$

2.2. **Mahler type inequalities for multihomogeneous polynomials.** By analogous argument as in [7] we obtain the following lemma.

Lemma 2.1. *Let $f \in \mathbb{C}[u]$, where $u = (u_1, \dots, u_N)$, be a homogeneous polynomial of degree $d_f > 0$ of the form*

$$f(u) = \sum_{|\nu|=d_f} a_\nu u^\nu.$$

Then there are homogeneous polynomials $f_{k_1,\dots,k_\ell} \in \mathbb{C}[u_{\ell+1}, \dots, u_N]$, with $\deg f_{k_1,\dots,k_\ell} = d_f - k_1 - \dots - k_\ell$ for $k_1 + \dots + k_\ell \leqslant d_f$, $\ell = 1, \dots, N$, such that

$$f(u_1, \dots, u_N) = \sum_{k_1=0}^{d_f} f_{k_1}(u_2, \dots, u_N) u_1^{k_1}$$

$$f_{k_1,\dots,k_{\ell-1}}(u_\ell, \dots, u_N) = \sum_{k_\ell=0}^{d_f-k_1-\dots-k_{\ell-1}} f_{k_1,\dots,k_\ell}(u_{\ell+1}, \dots, u_N) u_\ell^{k_\ell}.$$

Moreover, for any $\nu = (\nu_1, \dots, \nu_N) \in \mathbb{N}^N$, $|\nu| = d_f$, we have

$$|a_\nu| = |f_\nu| \leqslant \binom{d_f - \nu_1 - \dots - \nu_{N-1}}{\nu_N} M(f_{\nu_1,\dots,\nu_{N-1}}),$$

$$M(f_{\nu_1}) \leqslant \binom{d_f}{\nu_1} M(f),$$

$$M(f_{\nu_1,\dots,\nu_\ell}) \leqslant \binom{d_f - \nu_1 - \dots - \nu_{\ell-1}}{\nu_\ell} M(f_{\nu_1,\dots,\nu_{\ell-1}}),\ 2 \leqslant \ell \leqslant N.$$

In particular,

$$|a_\nu| \leqslant \binom{d_f}{\nu_1} \binom{d_f - \nu_1}{\nu_2} \cdots \binom{d_f - \nu_1 - \dots - \nu_{N-1}}{\nu_N} M(f) \leqslant \binom{d_f}{\nu_1, \dots, \nu_N} M(f) \leqslant N^{d_f - 1} M(f)$$

and so,

$$H(f) \leqslant N^{d_f-1} M(f),$$
$$L_1(f) \leqslant N^{d_f} M(f).$$

Let now m, $d_0, \dots, d_n$ be fixed positive integers, $n \in \mathbb{N}$, and let

$$u_{(m,j)} = (u_{m,j,\nu} : \nu \in \mathbb{N}^{m+1},\ |\nu| = d_j), \quad j = 0, \dots, n,$$

be systems of variables. In fact $u_{(m,j)}$ is a system of

$$N_{m,d_j} := \binom{d_j + m}{m}$$

variables.

From Lemma 2.1, by a similar method as in [7], we obtain the following Mahler type inequalities for multihomogeneous polynomials.

Lemma 2.2. *Let $f \in \mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ be a nonzero polynomial such that f is homogeneous as a polynomial in each system of variables $u_{(m,j)}$. Then for any polynomial $g \in \mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ which divides f in $\mathbb{Z}[u_{(m,0)}, \dots, u_{(m,n)}]$ and have degree e_j with respect to system $u_{(m,j)}$ for $j = 0, \dots, n$, we have*

$$H(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j - 1} \right) M(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j - 1} \right) M(f)$$

and

$$L_1(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j} \right) M(g) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{e_j} \right) M(f).$$

Proof. For simplicity $u_{(m,j)}$ we denote by $u_{(j)}$ and N_{m,d_j} – by N_j for $j = 0, \dots, n$. Let $g \in \mathbb{Z}[u_{(0)}, \dots, u_{(n)}]$ be a divisor of f in $\mathbb{Z}[u_{(0)}, \dots, u_{(n)}]$ and let $g_1 = f/g$. By the assumptions, g is a homogeneous polynomial as a polynomial in each $u_{(j)}$ of some degree e_j for $j = 0, \dots, n$. Let

$$\mathscr{I} = \{\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathbb{N}^{N_0} \times \dots \times \mathbb{N}^{N_n} : |\eta^{(j)}| = e_j \text{ for } j = 0, \dots, n\}.$$

The polynomial g is of the form

$$g(u_{(0)}, \dots, u_{(n)}) = \sum_{\eta \in \mathscr{I}} C_\eta J_\eta,$$

where $C_\eta \in \mathbb{Z}$ and $J_\eta = u_{(0)}^{\eta^{(0)}} \cdots u_{(n)}^{\eta^{(n)}}$ for $\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathscr{I}$. So, we may write

$$g(u_{(0)}, \dots, u_{(n)}) = \sum_{|\eta^{(0)}| = e_0} g_{1,\eta^{(0)}}(u_{(1)}, \dots, u_{(n)}) u_{(0)}^{\eta^{(0)}},$$

where $g_{1,\eta^{(0)}} \in \mathbb{Z}[u_{(1)}, \dots, u_{(n)}]$ for $\eta^{(0)} \in \mathbb{N}^{N_0}$, $|\eta^{(0)}| = e_0$. By induction for $j = 1, \dots, n$ we may write

$$g_{j,\eta^{(j-1)}}(u_{(j)}, \dots, u_{(n)}) = \sum_{|\eta^{(j)}| = e_j} g_{j+1,\eta^{(j)}}(u_{(j+1)}, \dots, u_{(n)}) u_{(j)}^{\eta^{(j)}},$$

where $g_{j+1,\eta^{(j)}} \in \mathbb{Z}[u_{(j+1)}, \dots, u_{(n)}]$ for $\eta^{(j)} \in \mathbb{N}^{N_j}$, $|\eta^{(j)}| = e_j$. Then any coefficient C_η, $\eta \in \mathscr{I}$, is a coefficient of some polynomial $g_{n,\eta^{(n-1)}}$. Then applying $n+1$ times Lemma 2.1, we obtain

$$H(g) \leqslant N_0^{e_0 - 1} \cdots N_n^{e_n - 1} M(g)$$

and

$$L_1(g) \leqslant N_0^{e_0} \cdots N_n^{e_n} M(g).$$

Since g_1 have integral coefficients, by (7) we have $M(g_1) \geqslant 1$. Then (6) gives the assertion. □

3. Height of a multipolynomial resultant

3.1. **Basic notations on a multipolynomial resultant.** Recall some notations and facts concerning the resultant for several homogeneous polynomials (see [2], see also [1]).

In this section $x = (x_0, \dots, x_n)$ is a system of $n+1$ variables.

Let $d_0, \dots, d_n$ be fixed positive integers and let $u_{(0)}, \dots, u_{(n)}$ be systems of variables of the form

$$u_{(j)} = (u_{j,\nu} : \nu \in \mathbb{N}^{n+1},\ |\nu| = d_j), \quad j = 0, \dots, n, \tag{10}$$

In fact $u_{(m,j)}$ is a system of

$$N_{d_j} := \binom{d_j + n}{n} \tag{11}$$

variables.

Let $f_0, \dots, f_n \in \mathbb{C}[u_{(0)}, \dots, u_{(n)}, x]$ be homogeneous polynomials in x of degrees $d_0, \dots, d_n$, respectively of the forms

$$f_j(u_{(0)}, \dots, u_{(n)}, x) = \sum_{\substack{\nu \in \mathbb{N}^{n+1} \\ |\nu| = d_j}} u_{j,\nu} x^\nu, \quad j = 0, \dots, n.$$

In fact $f_j \in \mathbb{Z}[u_{(j)}, x]$.

For any $a_j = (a_{j,\nu} : \nu \in \mathbb{N}^{n+1},\ |\nu| = d_j) \in \mathbb{C}^{N_{d_j}}$ by, $f_{j,a_j} \in \mathbb{C}[x]$ we denote the *specialization* of f_j, i.e., the polynomial $f_{j,a_j}(x) = f_j(a_j, x)$.

Fact 3.1 ([2], Chapter 13)**.** *There exists a unique polynomial $P_{d_0,\dots,d_n} \in \mathbb{Z}[u_{(0)}, \dots, u_{(n)}]$ such that:*

(i) *For any $a_0 \in \mathbb{C}^{N_{d_0}}, \dots, a_n \in \mathbb{C}^{N_{d_n}}$*

$$P_{d_0,\dots,d_n}(a_0, \dots, a_n) = 0 \ \Leftrightarrow\ f_{0,a_0}, \dots, f_{n,a_n} \text{ have a common nontrivial zero.}$$

(ii) *For $a_0 \in \mathbb{C}^{N_{d_0}}, \dots, a_n \in \mathbb{C}^{N_{d_n}}$ such that $f_{0,a_0} = x_0^{d_0}, \dots, f_{n,a_n} = x_n^{d_n}$,*

$$P_{d_0,\dots,d_n}(a_0, \dots, a_n) = 1.$$

(iii) *$P_{d_0,\dots,d_n}$ is irreducible in $\mathbb{C}[u_{(0)}, \dots, u_{(n)}]$.*

The polynomial $P_{d_0,\dots,d_n}$ in Fact 3.1 is called *resultant* or *multipolynomial resultant* and denoted by $\mathrm{Res}_{d_0,\dots,d_n}$ or shortly by Res. We will also write $\mathrm{Res}(f_{0,a_0}, \dots, f_{n,a_n})$ instead of $\mathrm{Res}(a_0, \dots, a_n)$.

Fact 3.2 ([2], Proposition 1.1 in Chapter 13)**.** *For any $j = 0, \dots, n$ the resultant $\mathrm{Res}_{d_0,\dots,d_n}$ is a homogeneous polynomial in $u_{(j)}$ of degree $d_0 \cdots d_{j-1} d_{j+1} \cdots d_n$.*

Set

$$\delta = d_0 + \cdots + d_n - n,$$

and let

$$S_j = \{\nu = (\nu_0, \dots, \nu_n) \in \mathbb{N}^{n+1} : |\nu| = \delta,\ \nu_0 < d_0, \dots, \nu_{j-1} < d_{j-1}, \nu_j \geqslant d_j\} \quad \text{for } j = 0, \dots, n.$$

Fact 3.3. *The sets $S_0, \dots, S_n$ are mutually disjoint and*

$$\{\nu \in \mathbb{N}^{n+1} : |\nu| = \delta\} = S_0 \cup \cdots \cup S_n. \tag{12}$$

Consider the following system of equations

$$\begin{cases} \frac{x^\nu}{x_0^{d_0}} f_0(u_{(0)}, x) = 0 \quad \text{for } \nu \in S_0 \\ \quad \vdots \\ \frac{x^\nu}{x_n^{d_n}} f_n(u_{(n)}, x) = 0 \quad \text{for } \nu \in S_n. \end{cases} \tag{13}$$

Any of the above equation is homegenous of degree δ and depends on

$$N_{d_0,\dots,d_n} = \binom{d_0 + \cdots + d_n}{n}$$

monomials of degree δ. Let's arrange these monomials in a sequence $J_1, \dots, J_N$. Then (13) one can consider as a system of N linear equations with N indeterminates $J_1, \dots, J_N$. Denote by $\mathscr{D}_{d_0,\dots,d_n}$ the matrix of this system of equations and by $D_{d_0,\dots,d_n}$ – the determinat of $\mathscr{D}_{d_0,\dots,d_n}$. From Fact 3.3 and the definition of $D_{d_0,\dots,d_n}$ we easily obtain the following fact.

Fact 3.4. *For $a_j \in \mathbb{C}^{N_{d_j}}$ such that $f_{j,a_j}(x) = x_j^{d_j}$, $j = 0, \dots, n$, we have*

$$|D_{d_0,\dots,d_n}(a_0, \dots, a_n)| = 1,$$

In particular, $D_{d_0,\dots,d_n} \neq 0$.

Proof. Indeed, by Fact 3.3, for the assumed specializations f_{j,a_j}, $j = 0, \dots, n$, the matrix $\mathscr{D}_{d_0,\dots,d_n}(f_{0,a_0}, \dots, f_{n,a_n})$ have in any row and any column exactly one nonzero entry equal to 1. □

From the definition of $D_{d_0,\dots,d_n}$ we see that $D_{d_0,\dots,d_n}$ is a homogeneous polynomoal in $u_{(j)}$ of degree equal to the number of elements $\#S_j$ of S_j and the total degree equal to $N_{d_0,\dots,d_n}$. Moreover, we have the following Macaulay result [5, Theorem 6] (see also [4] and [2, Theorem 1.5 in Chapter 13] for Caley determinantal formula).

Fact 3.5. *The polynomial $D_{d_0,\dots,d_n}$ is divisible by* $\mathrm{Res}_{d_0,\dots,d_n}$ *in $\mathbb{Z}[u_{(0)}, \dots, u_{(n)}]$.*

Put

$$d_* = \max\{d_0, \dots, d_n\}.$$

From the definition of the polynomial $D_{d_0,\dots,d_n}$ we obtain

Lemma 3.6. $L_1(D_{d_0,\dots,d_n}) \leqslant N_{d_0}^{\#S_0} \cdots N_{d_n}^{\#S_n} \leqslant \binom{d_*+n}{n}^{\binom{(n+1)d_*}{n}}$.

Proof. Let $D = D_{d_0,\dots,d_n}$ and $N_j = N_{d_j}$. Monomials of D are of the form

$$J_\eta = C_\eta u_{(0)}^{\eta^{(0)}} \cdots u_{(n)}^{\eta^{(n)}},$$

where $C_\eta \in \mathbb{Z}$ for $\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathbb{N}^{N_0} \times \ldots \times \mathbb{N}^{N_n}$ and $|\eta^{(j)}| = \#S_j$ for $j = 0, \dots, n$. Let $\eta^{(j)} = (\eta_{j,1}, \dots, \eta_{j,N_j})$. Then from definition of D,

$$|C_\eta| \leqslant \prod_{j=0}^{n} \binom{\#S_j}{\eta_{j,1}} \binom{\#S_j - \eta_{j,1}}{\eta_{j,2}} \cdots \binom{\#S_j - \eta_{j,1} - \cdots - \eta_{j,N_j-1}}{\eta_{j,N_j}} = \prod_{j=0}^{n} \binom{\#S_j}{\eta_{j,1}, \dots, \eta_{j,N_j}},$$

so

$$L_1(D) \leqslant \sum_{\substack{\eta=(\eta^{(0)},\dots,\eta^{(n)}) \in \mathbb{N}^{N_0+\cdots+N_n} \\ |\eta^{(k)}|=\#S_k \text{ for } k=0,\dots,n}} \prod_{j=0}^{n} \binom{\#S_j}{\eta_{j,1}, \dots, \eta_{j,N_j}} \leqslant N_0^{\#S_0} \cdots N_n^{\#S_n},$$

which gives the first inequality in the assertion. Since $N_j \leqslant \binom{d_*+n}{n}$ and $\#S_0 + \cdots + \#S_n = N_{d_0,\dots,d_n} \leqslant \binom{(n+1)d_*}{n}$, then we obtain the second inequality in the assertion. □

3.2. **Multipolynomial resultant for powers of polynomials.** Take any $k \in \mathbb{Z}$, $k > 0$. The resultant $\mathrm{Res}_{kd_0,\dots,kd_n}$ and the discriminant $D_{kd_0,\dots,kd_n}$ are polynomials with integer coefficients in a system of variables $w_k = (w_{(k,0)}, \dots, w_{(k,n)})$, where

$$w_{(k,j)} = \left(w_{k,j,\nu} : \nu \in \mathbb{N}^{n+1},\ |\nu| = kd_j\right), \tag{14}$$

is a system of indeterminate coefficients of the polynomial

$$F_{k,j}(w_{(k,j)}, x) = \sum_{\substack{\nu \in \mathbb{N}^{n+1} \\ |\nu| = kd_j}} w_{k,j,\nu} x^\nu, \quad j = 0, \dots, n.$$

In fact $w_{(k,j)}$ is a system of

$$N_{kd_j} := \binom{kd_j + n}{n} \tag{15}$$

variables. From Fact 3.2 we have that $\mathrm{Res}_{kd_0,\dots,kd_n}$ is homogeneous in any system of variables $w_{(k,j)}$ of degree

$$e_{k,j} = k^n d_0 \cdots d_{j-1} d_{j+1} \cdots d_n, \quad j = 0, \dots, n.$$

The polynomial $D_{kd_0,\dots,kd_n}$ is also homogeneous in any system of variables $w_{(k,j)}$. Let $s_{k,j}$ be the degree of $D_{kd_0,\dots,kd_n}$ with respect to $w_{(k,j)}$, $j = 0, \dots, n$. Obviously

$$s_{k,0} + \cdots + s_{k,n} = \binom{k(d_0 + \cdots + d_n)}{n}. \tag{16}$$

Let

$$\mathscr{I}_k = \{\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathbb{N}^{N_{kd_0}} \times \dots \times \mathbb{N}^{N_{kd_n}} : |\eta^{(j)}| = s_{k,j} \quad \text{for } j = 0, \dots, n\}.$$

Then $D_{kd_0,\dots,kd_n}$ one can write

$$D_{kd_0,\dots,kd_n} = \sum_{\eta \in \mathscr{I}_k} C_\eta J_\eta, \tag{17}$$

where $C_\eta \in \mathbb{Z}$ for $\eta \in \mathscr{I}_k$ and

$$J_\eta = w_{(k,0)}^{\eta^{(0)}} \cdots w_{(k,n)}^{\eta^{(n)}} \quad \text{for } \eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathscr{I}_k. \tag{18}$$

Since

$$f_j^k = \sum_{\substack{\nu \in \mathbb{N}^{n+1} \\ |\nu| = kd_j}} x^\nu \sum_{\substack{\nu^1, \dots, \nu^k \in \mathbb{N}^{n+1} \\ \nu^1 + \dots + \nu^k = \nu \\ |\nu^1| = \dots = |\nu^k| = d_j}} u_{j,\nu^1} \cdots u_{j,\nu^k}, \quad j = 0, \dots, n,$$

then we may define a mapping

$$W_k = (W_{(k,0)}, \dots, W_{(k,n)}) : \mathbb{C}^{N_{d_0}} \times \dots \times \mathbb{C}^{N_{d_n}} \to \mathbb{C}^{N_{kd_0}} \times \dots \times \mathbb{C}^{N_{kd_n}},$$

by $W_{(k,j)} = (W_{k,j,\nu} : \nu \in \mathbb{N}^{n+1},\ |\nu| = kd_j)$ for $j = 0, \dots, n$, and

$$W_{k,j,\nu}(u_{(j)}) = \sum_{\substack{\nu^1, \dots, \nu^k \in \mathbb{N}^{n+1} \\ \nu^1 + \dots + \nu^k = \nu \\ |\nu^1| = \dots = |\nu^k| = d_j}} u_{j,\nu^1} \cdots u_{j,\nu^k} \quad \text{for } \nu \in \mathbb{N}^{n+1},\ |\nu| = kd_j.$$

In other words, $W_{(k,j)}$ is a system of coefficients of f_j^k as a polynomial in x. So for any positive integer k we may define

$$R_k = \mathrm{Res}_{kd_0,\dots,kd_n}(f_0^k, \dots, f_n^k),$$
$$D_k = D_{kd_0,\dots,kd_n}(f_0^k, \dots, f_n^k).$$

More precisely,

$$R_k = \mathrm{Res}_{kd_0,\dots,kd_n} \circ W_k\,,$$
$$D_k = D_{kd_0,\dots,kd_n} \circ W_k\,.$$

Then from (17) and (18) we have

$$D_k = \sum_{\eta = (\eta^{(0)}, \dots, \eta^{(n)}) \in \mathscr{I}_k} C_\eta W_{(k,0)}^{\eta^{(0)}} \cdots W_{(k,n)}^{\eta^{(n)}}. \tag{19}$$

From Fact 3.4 we have

Fact 3.7. *For any positive integer k we have $D_k \neq 0$.*

From [2, Proposition 1.3 in Chapter 13] and [1, Theorem 3.2], we immediately obtain

Fact 3.8. *For any positive integer k we have*

$$\operatorname{Res}_{kd_0,\dots,kd_n}(f_0^k,\dots,f_n^k) = \operatorname{Res}_{d_0,\dots,d_n}(f_0,\dots,f_n)^{k^{n+1}}.$$

Recall that $d_* = \max\{d_0,\dots,d_n\}$. Put

$$N_{*,k} = \binom{kd_*+n}{n}, \quad N_k^* = \binom{(n+1)kd_*}{n}, \quad k \in \mathbb{Z},\ k>0.$$

Lemma 3.9. $L_1(D_k) \leqslant (N_{*,1})^{kN_k^*}\, L_1(D_{kd_0,\dots,kd_n})$.

Proof. Indeed, for any $j=0,\dots,n$ and any $\nu \in \mathbb{N}^{n+1}$, $|\nu| = kd_j$ the polynomial $W_{k,j,\nu}$ consists of at most $(N_{*,1})^k$ monomials with coefficients equal to 1, i.e., $(N_{*,1})^k$ is not smaller that

$$\#\{(\nu^1,\dots,\nu^k) \in \left(\mathbb{N}^{n+1}\right)^k : \nu^1+\cdots+\nu^k = \nu,\ |\nu^1| = \cdots = |\nu^k| = d_j\}$$

for $j=0,\dots,n$. So from (19) we easily see that

$$L_1(D_k) \leqslant \sum_{\eta=(\eta^{(0)},\dots,\eta^{(n)})\in\mathscr{I}_k} |C_\eta|\,(N_{*,1})^{k|\eta^{(0)}|}\cdots(N_{*,1})^{k|\eta^{(n)}|}.$$

Then (16) easily gives the assertion. □

3.3. **Height of a multipolynomial resultant.** From Lemmas 2.2, 3.6 and 3.9 and Fact 3.8 we have

Lemma 3.10. *For any $k \in \mathbb{Z}$, $k>0$ we have*

$$M(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{N_k^*/k^n}\,(N_{*,k})^{N_k^*/k^{n+1}}, \tag{20}$$

$$H(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n-n-1}\,M(\operatorname{Res}_{d_0,\dots,d_n}), \tag{21}$$

$$L_1(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n}\,M(\operatorname{Res}_{d_0,\dots,d_n}). \tag{22}$$

Proof. Let $e_j = d_0\cdots d_{j-1}d_{j+1}\cdots d_n$ for $j=0,\dots,n$. By Lemma 2.2 and (6) we obtain

$$H(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant \left(\prod_{j=0}^n \left(N_{d_j}\right)^{e_j-1}\right) M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}},$$

$$L_1(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant \left(\prod_{j=0}^n \left(N_{d_j}\right)^{e_j}\right) M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}}.$$

Since $e_0+\cdots+e_n \leqslant (n+1)d_*^n$, then from the above we have

$$H(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n-n-1}\,M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}},$$

$$L_1(\operatorname{Res}_{d_0,\dots,d_n}) \leqslant (N_{*,1})^{(n+1)d_*^n}\,M(\operatorname{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}}.$$

This, together with Fact 3.8, gives (21) and (22).

From Fact 3.8 we also have $M(\mathrm{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}} = M(R_k)^{1/k^{n+1}}$, and since $M(R_k) \leqslant M(D_k)$ (by (7) and Facts 3.5 and 3.7), so (9) gives

$$M(\mathrm{Res}_{d_0,\dots,d_n}^{k^{n+1}})^{1/k^{n+1}} \leqslant L_2(D_k)^{1/k^{n+1}}. \tag{23}$$

By Lemma 3.9 we have

$$L_1(D_k) \leqslant (N_{*,1})^{kN_k^*} L_1(D_{kd_0,\dots,kd_n}). \tag{24}$$

Since

$$N_{kd_j} \leqslant N_{*,k}, \quad \text{for } j = 0,\dots,n,$$
$$N_{kd_0,\dots,kd_n} \leqslant N_k^*,$$

so, from Lemma 3.6 we obtain

$$L_1(D_{kd_0,\dots,kd_n}) \leqslant (N_{*,k})^{N_k^*} \quad \text{for } k > 0.$$

Since $L_2(D_k) \leqslant L_1(D_k)$ then (23) and (24) gives (20). □

In general $N_k^* \leqslant (n+1)!(kd_*)^n$. It turns out that asymptotically this number has better properties.

Lemma 3.11.

$$\lim_{k\to\infty} \frac{N_k^*}{k^n} = \frac{(n+1)^n d_*^n}{n!} < \frac{e^{n+1}}{\sqrt{2\pi n}} d_*^n.$$

Proof. Indeed,

$$\frac{N_k^*}{k^n} = \frac{\prod_{j=1}^n [(n+1)kd_* - n + j]}{n!k^n},$$

so,

$$\lim_{k\to\infty} \frac{N_k^*}{k^n} = \frac{(n+1)^n d_*^n}{n!} = \left(\frac{n+1}{n}\right)^n \frac{n^n}{n!} d_*^n < e\frac{n^n}{n!} d_*^n.$$

Since from Stirling formula,

$$\frac{n^n}{n!} \leqslant \frac{e^{n-1/(12n+1)}}{\sqrt{2\pi n}},$$

then we obtain the assertion. □

Lemmas 3.10 and 3.11 gives the main result of this paper.

Theorem 3.12. *Let* $d_* = \max\{d_0,\dots,d_n\}$ *and* $K_n = e^{n+1}/\sqrt{2\pi n}$, $n > 0$. *Then*

$$M(\mathrm{Res}_{d_0,\dots,d_n}) \leqslant (d_*+1)^{nK_n d_*^n}, \tag{25}$$

$$H(\mathrm{Res}_{d_0,\dots,d_n}) \leqslant (d_*+1)^{n(K_n+n+1)d_*^n - n(n+1)}, \tag{26}$$

$$L_1(\mathrm{Res}_{d_0,\dots,d_n}) \leqslant (d_*+1)^{n(K_n+n+1)d_*^n}. \tag{27}$$

Moreover, if $n \geqslant 2$ and $d_* \geqslant 4$, then

$$
\begin{aligned}
M(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*)^{nK_n d_*^n},\\
H(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*)^{n(K_n+n+1)d_*^n-n(n+1)},\\
L_1(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (d_*)^{n(K_n+n+1)d_*^n}.
\end{aligned} \tag{28}
$$

Proof. From Lemma 3.10 for nay $k \in \mathbb{Z}$, $k > 0$ we have

$$
\begin{aligned}
M(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (N_{*,1})^{N_k^*/k^n} (N_{*,k})^{N_k^*/k^{n+1}},\\
H(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (N_{*,1})^{(n+1)d_*^n-n-1} (N_{*,1})^{N_k^*/k^n} (N_{*,k})^{N_k^*/k^{n+1}},\\
L_1(\mathrm{Res}_{d_0,\ldots,d_n}) &\leqslant (N_{*,1})^{(n+1)d_*^n} (N_{*,1})^{N_k^*/k^n} (N_{*,k})^{N_k^*/k^{n+1}}.
\end{aligned}
$$

Since $1 \leqslant N_{*,k} \leqslant (kd_*+1)^n$, then

$$
\lim_{k\to\infty} (N_{*,k})^{1/k} = 1, \tag{29}
$$

so passing to the limit as $k \to \infty$ in the above inequalities, by Lemma 3.11, we obtain (25), (26) and (27).

Since for $n \geqslant 2$ and $d_* \geqslant 4$ we have $N_{*,1} \leqslant d_*^n$ then we obtain the second part of the assertion (28). □

Remark 3.13. *The estimation* (27) *of* $L_1(\mathrm{Res}_{d_0,\ldots,d_n})$ *is not a direct consequence of the estimation* (26) *of the height* $H(\mathrm{Res}_{d_0,\ldots,d_n})$ *because the number of coefficients of* $\mathrm{Res}_{d_0,\ldots,d_n}$ *can be bigger than* $(d_*+1)^{n(n+1)}$. *The number of coefficients of the resultant can be estimated by*

$$
E_{d_0,\ldots,d_n} := \prod_{j=0}^{n} \binom{\binom{d_j+n}{n} + d_0\cdots d_{j-1}d_{j+1}\cdots d_n}{d_0\cdots d_{j-1}d_{j+1}\cdots d_n} \leqslant (d_*+1)^{n(n+1)d_*^n}.
$$

4. Gelfond-Mahler type inequalities for homogeneous polynomials

As a corollaries from Lemma 2.2 we obtain the following Gelfond-Mahler type theorems.

Corollary 4.1. *Let $f \in \mathbb{Z}[u_{(m,0)},\ldots,u_{(m,n)}]$ be a nonzero polynomial such that f is homogeneous of degree $s_j > 0$ as a polynomial in each system of variables $u_{(m,j)}$. Then for any polynomials $f_1,\ldots,f_k \in \mathbb{Z}[u_{(m,0)},\ldots,u_{(m,n)}]$ such that $f = f_1\cdots f_k$ we have*

$$
\begin{aligned}
H(f_1)\cdots H(f_k) &\leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j-1}\right) M(f)\\
&\leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j-1}\right)\left(\prod_{j=0}^{n} \sqrt{N_{m,d_j}+1}^{\,s_j}\right) H(f)
\end{aligned} \tag{30}
$$

and

$$L_1(f_1)\cdots L_1(f_k) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j}\right) M(f) \leqslant \left(\prod_{j=0}^{n} N_{m,d_j}^{s_j}\right) L_1(f). \tag{31}$$

Proof. The left hand inequalities in (30) and (31) immediately follows from Lemma 2.2, because $M(f_1)\cdots M(f_k) = M(f)$ from (6). Since the polynomial f is homogeneous with respect to $u_{(m,j)}$ of degree s_j, $j = 0, \ldots, n$, then from (9) we have

$$M(f) \leqslant \left(\prod_{j=0}^{n} \sqrt{\binom{s_j + N_{m,d_j}}{N_{m,d_j}}}\right) H(f) \leqslant \left(\prod_{j=0}^{n} \sqrt{N_{m,d_j}+1}^{s_j}\right) H(f).$$

This gives the right hand inequalities in (30) and (31) and ends the proof. □

Applying Corollary 4.1 for $n = 0$, $d_0 = 1$ and $m = N - 1$ and a homogenisation $f^*(x_0, \ldots, x_m) := x_0^{\deg f} f(x_1/x_0 \ldots, x_m/x_0)$ of a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_m]$ we obtain the following corollary.

Corollary 4.2. *Let $f \in \mathbb{Z}[x_1, \ldots, x_m]$ be a nonzero polynomial of degree $s > 0$. Then for any polynomials $f_1, \ldots, f_k \in \mathbb{Z}[x_1, \ldots, x_m]$ such that $f = f_1 \cdots f_k$ we have*

$$H(f_1)\cdots H(f_k) \leqslant (N+1)^{s-1} M(f^*) \leqslant (N+1)^{s-1}\sqrt{N+2}^{\,s} H(f)$$

and

$$L_1(f_1)\cdots L_1(f_k) \leqslant (N+1)^{s} M(f^*) \leqslant (N+1)^{s} L_1(f).$$

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(Aleksandra Gala-Jaskórzyńska) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

E-mail address: agjaskorzynska@math.uni.lodz.pl

(Krzysztof Kurdyka) Laboratoire de Mathematiques (LAMA) Universié Savoie Mont Blanc, UMR-5127 de CNRS 73-376 Le Bourget-du-Lac cedex FRANCE

E-mail address: Krzysztof.Kurdyka@univ-savoie.fr

(Katarzyna Rudnicka) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

E-mail address: kkuta@math.uni.lodz.pl

(Stanisław Spodzieja) Faculty of Mathematics and Computer Science, University of Łódź, S. Banacha 22, 90-238 Łódź, POLAND

E-mail address: stanislaw.spodzieja@wmii.uni.lodz.pl

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CONTACT EXPONENT AND THE MILNOR NUMBER OF PLANE CURVE SINGULARITIES

EVELIA ROSA GARCÍA BARROSO AND ARKADIUSZ PŁOSKI

Abstract. We investigate properties of the contact exponent (in the sense of Hironaka [Hi]) of plane algebroid curve singularities over algebraically closed fields of arbitrary characteristic. We prove that the contact exponent is an equisingularity invariant and give a new proof of the stability of the maximal contact. Then we prove a bound for the Milnor number and determine the equisingularity class of algebroid curves for which this bound is attained. We do not use the method of Newton's diagrams. Our tool is the logarithmic distance developed in [GB-P1].

Introduction

Let C be a plane algebroid curve of multiplicity $m(C)$ defined over an algebraically closed field K. To calculate the number of infinitely near $m(C)$-fold points, Hironaka [Hi] (see also [B-K] or [T2]) introduced the concept of contact exponent $d(C)$ and study its properties using Newton's diagrams.

In this note we prove an explicit formula for a generalization of contact exponent (Section 2, Theorem 2.3) using the logarithmic distance on the set of branches. Then we give a new proof of the stability of maximal contact (Section 3, Theorem 3.7) without resorting to Newton's diagrams. In Section 4 we define the Milnor number $\mu(C)$ in the case of arbitrary characteristic (see [M-W] and [GB-P2]), prove the bound $\mu(C) \geq (d(C)m(C)-1)(m(C)-1)$ and characterize the singularities for which the bound is attained. In Section 5 we reprove the formulae for the contact exponents of higher order (see [LJ] and [C]). Section 6 is devoted to the relation between polar invariants and the contact exponent in characteristic zero.

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1. Preliminaries

Let $K[[x,y]]$ be the ring of formal power series with coefficients in an algebraically closed field K of arbitrary characteristic. For any non-zero power series $f = f(x,y) = \sum_{i,j} c_{ij}x^iy^j \in K[[x,y]]$ we define its *order* as $\operatorname{ord} f = \inf\{i+j \ : \ c_{ij} \neq 0\}$ and its *initial form* as $\operatorname{in} f = \sum_{i+j=n} c_{ij}x^iy^j$, where $n = \operatorname{ord} f$. We let $(f,g)_0 = \dim_K K[[x,y]]/(f,g)$, and call $(f,g)_0$ the *intersection number* of f and g, where (f,g) denotes the ideal of $K[[x,y]]$ generated by f and g.

Let f be a nonzero power series without constant term. An *algebroid curve* $C : \{f = 0\}$ is defined to be the ideal generated by f in $K[[x,y]]$. The *multiplicity* of C is $m(C) = \operatorname{ord} f$. Let $\mathbb{P}^1(K)$ denotes the projective line over K. The *tangent cone* of C is by definition $\operatorname{cone}(C) = \{(a:b) \in \mathbb{P}^1(K) \ : \ \operatorname{in} f(a,b) = 0\}$.

The curve $C : \{f = 0\}$ is *reduced* (resp. *irreducible*) if the power series f has no multiple factors (resp. is irreducible). Irreducible curves are called *branches*. If $\sharp\operatorname{cone}(C) = 1$ then the curve $C : \{f = 0\}$ is called *unitangent*. Any irreducible curve is unitangent. For $C : \{f = 0\}$ and $D : \{g = 0\}$ we put $(C,D)_0 = (f,g)_0$. Then $(C,D)_0 \geq m(C)m(D)$, with equality if and only if the tangent cones of C and D are disjoint.

For any sequence $C_i : \{f_i = 0 \ : \ 1 \leq i \leq k\}$ of curves we put $C = \bigcup_{i=1}^k C_i : \{f_1 \cdots f_k = 0\}$. If C_i are irreducible and $C_i \neq C_j$ for $i \neq j$ then we call C_i the *irreducible components* of C.

Consider an irreducible power series $f \in K[[x,y]]$. The set

$$\Gamma(C) = \Gamma(f) := \{(f,g)_0 \ : \ g \in K[[x,y]], \ \ g \not\equiv 0 \ (\operatorname{mod} f)\}$$

is the *semigroup* associated with $C : \{f = 0\}$. Note that $\min(\Gamma(C)\backslash\{0\}) = m(C)$. It is well-known that $\gcd(\Gamma(C)) = 1$.

The branch C is *smooth* (that is its multiplicity equals 1) if and only if $\Gamma(C) = \mathbb{N}$.

Two branches $C : \{f = 0\}$ and $D : \{g = 0\}$ are *equisingular* if $\Gamma(C) = \Gamma(D)$.

Two reduced curves $C : \{f = 0\}$ and $D : \{g = 0\}$ are equisingular if and only if f and g have the same number r of irreducible factors and there are factorizations $f = f_1 \cdots f_r$ and $g = g_1 \cdots g_r$ such that

(1) the branches $C_i : \{f_i = 0\}$ and $D_i : \{g_i = 0\}$ are equisingular for $i \in \{1, \ldots, r\}$,
and
(2) $(C_i, C_j)_0 = (D_i, D_j)_0$ for any $i, j \in \{1, \ldots, r\}$.

A function $C \mapsto I(C)$ defined on the set of all reduced curves is an *equisingularity invariant* if $I(C) = I(D)$ for equisingular curves C and D. Note that the multiplicity $m(C)$, the number of branches $r(C)$ and the number of tangents $t(C)$ (which is the cardinality of the cone (C)) of the reduced curve C are equisingularity invariants.

For any reduced curve $C : \{f = 0\}$ we put $\mathcal{O}_C = K[[x, y]]/(f)$ and $\overline{\mathcal{O}}_C$ its integral closure. Let $\mathcal{C} = \overline{\mathcal{O}}_C : \mathcal{O}_C$ be the *conductor* of $\overline{\mathcal{O}}_C$ in $\mathcal{O}_C$. The number $c(C) = \dim_K \overline{\mathcal{O}}_C/\mathcal{C}$ is the *degree of the conductor*. If C is a branch then $c(C)$ equals to the smallest element of $\Gamma(C)$ such that $c(C) + N \in \Gamma(C)$ for all $N \in \mathbb{N}$ (see [C, Chapter IV]).

Suppose that C is a branch. Let $(v_0, v_1, \ldots, v_g)$ be the *minimal system of generators* of $\Gamma(C)$ defined by the following conditions:

(3) $v_0 = \min(\Gamma(C)\backslash\{0\}) = m(C)$.
(4) $v_k = \min(\Gamma(C)\backslash\mathbb{N}v_0 + \cdots + \mathbb{N}v_{k-1})$, for $k \in \{1, \ldots, g\}$.
(5) $\Gamma(C) = \mathbb{N}v_0 + \cdots + \mathbb{N}v_g$.

In what follows we write $\Gamma(C) = \langle v_0, v_1, \ldots, v_g\rangle$ when $v_0 < v_1 < \cdots < v_g$ is the increasing sequence of minimal system of generators of $\Gamma(C)$.

Since $\gcd(\Gamma(C)) = 1$ the sequence $v_0, \ldots, v_g$ is well-defined. Let $e_k := \gcd(v_0, \ldots, v_k)$ for $0 \leq k \leq g$. We define the *Zariski pairs* $(m_k, n_k) = \left(\frac{v_k}{e_k}, \frac{e_{k-1}}{e_k}\right)$ for $1 \leq k \leq g$. One has $c(C) = \sum_{k=1}^{g} (n_k - 1)\, v_k - v_0 + 1$ (see[GB-P1, Corollary 3.5]).

If K is a field of charateristic zero the Zariski pairs determine the *Puiseux pairs* and vice versa (see [T2, pp. 19, 47]).

If $\Gamma(C) = \langle v_0, v_1, \ldots, v_g\rangle$ then the sequence $(v_i)_i$ is *strongly increasing*, that is $n_{i-1}v_{i-1} < v_i$ for $i \in \{2, \ldots, g\}$.

Let $C : \{f = 0\}$ be a reduced unitangent curve of multiplicity n. Let us consider two possible cases:

(i) $f = c(y - ax)^n +$ higher order terms, where $a, c \in K$, $c \neq 0$ and
(ii) $f = cx^n +$ higher order terms, $c \in K\backslash\{0\}$.

We associate with C a power series $f_1 = f_1(x_1, y_1) \in K[[x_1, y_1]]$ by putting $f_1(x_1, y_1) = x_1^{-n} f(x_1, ax_1 + x_1y_1)$ in the case (i) and $f_1(x_1, y_1) = y_1^{-n} f(x_1y_1, y_1)$ otherwise. The *strict quadratic transform* of $C : \{f = 0\}$ is the curve $\widehat{C} : \{f_1 = 0\}$.

Obviously $m(\widehat{C}) \leq m(C)$. If $C = \bigcup_{i=1}^{k} C_i$ is a unitangent curve then C_i are unitangent and $\widehat{C} = \bigcup_{i=1}^{k} \widehat{C}_i$.

The following lemma is a particular case of a theorem due to Angermüller [Ang, Lemma II.2.1].

Lemma 1.1. *Let C be a singular branch. Then the strict quadratic transform $\widehat{C}$ of C is also a plane branch. If $\Gamma(C) = \langle v_0, \ldots, v_g\rangle$ then*

- $\Gamma(\widehat{C}) = \langle v_0, v_1 - v_0, \ldots\rangle$ *if* $v_0 < v_1 - v_0$
  or
- $\min(\Gamma(\widehat{C})\backslash\{0\}) = v_1 - v_0$ *if* $v_1 - v_0 < v_0$.

2. Logarithmic distance

A *log-distance* δ associates with any two branches C, D a number $\delta(C,D) \in \mathbb{R}_+ \cup \{+\infty\}$ such that for any branches C, D and E we have:

(δ_1) $\delta(C,D) = \infty$ if and only if $C = D$,
(δ_2) $\delta(C,D) = \delta(D,C)$,
(δ_3) $\delta(C,D) \geq \inf\{\delta(C,E), \delta(E,D)\}$.

Note that if $\delta(C,E) \neq \delta(E,D)$ then $\delta(C,D) = \inf\{\delta(C,E), \delta(E,D)\}$.

If C and D are reduced curves with irreducible components C_i and D_j then we set $\delta(C,D) := \inf_{i,j}\{\delta(C_i, D_j)\}$.

If δ is a log-distance then $\Delta := \frac{1}{\delta}$ (by convention $\frac{1}{+\infty} = 0$) is an *ultrametric* (see [GB-GP-PP, Definition 41]) on the set of branches and vice versa: if Δ is an ultrametric then $\frac{1}{\Delta}$ is a log-distance.

Examples 2.1.

(1) *The order of contact of branches* $d(C,D) = \frac{(C,D)_0}{m(C)m(D)}$ *is a log-distance (see* [GB-P1, Corollary 2.9]*).*
(2) *The minimum number of quadratic transformations* $\gamma(C,D)$ *necessary to separate* C *from* D *is a log-distance (see* [W, Theorem 3]*).*

Let δ be a log-distance.

Lemma 2.2. *If* C *has* $r > 1$ *branches* C_i *and* D *is any branch then* $\delta(C,D) \leq \inf_{i,j}\{\delta(C_i, C_j)\}$.

Proof. Let i_0, j_0 be such that $\inf_{i,j}\{\delta(C_i,C_j)\} = \delta(C_{i_0}, C_{j_0})$. Then $\delta(C,D) = \inf_{1\leq i\leq r}\{\delta(C_i,D)\} \leq \inf\{\delta(C_{i_0},D), \delta(C_{j_0},D)\}$ and using ($\delta 3$) we get $\delta(C,D) \leq \delta(C_{i_0}, C_{j_0})$, which proves the lemma. □

Let C be a reduced curve. For every non-empty family of branches $\mathcal{B}$ we put

$$\delta(C,\mathcal{B}) := \sup\{\delta(C,W) \ : \ W \in \mathcal{B}\}.$$

Note that $\delta(C,\mathcal{B}) = +\infty$ if $C \in \mathcal{B}$. In what follows we assume the following condition

(*) for any branch C there exists $W_0 \in \mathcal{B}$ such that $\delta(C,\mathcal{B}) = \delta(C,W_0)$,

we say that W_0 has *maximal* δ*-contact* with C.

We will prove the following

Theorem 2.3. *Let* C *be a reduced curve with* $r > 1$ *branches* C_i *and let* $\mathcal{B}$ *be a family of branches such that the condition (*) holds.*

Then

$$\delta(C,\mathcal{B}) = \inf\{\inf_i\{\delta(C_i,\mathcal{B})\}, \inf_{i,j}\{\delta(C_i,C_j)\}\}.$$

Moreover, there exists $i_0 \in \{1,\dots,r\}$ such that if a branch $W \in \mathcal{B}$ has maximal δ-contact with C_{i_0} then it has maximal δ-contact with C.

Proof. Set $\delta^*(C,\mathcal{B}) = \inf\{\inf_i \delta(C_i,\mathcal{B}), \inf_{i,j}\delta(C_i,C_j)\}$.

The inequality $\delta(C,\mathcal{B}) \le \delta^*(C,\mathcal{B})$ follows from Lemma 2.2 and from the definition of $\delta(C_i,\mathcal{B})$. Thus to prove the result let us consider two cases:

First case: $\inf_i\{\delta(C_i,\mathcal{B})\} \le \inf_{i,j}\{\delta(C_i,C_j)\}$.

Let $i_0 \in \{1,\dots,r\}$ be such that $\delta(C_{i_0},\mathcal{B}) = \inf_i\{\delta(C_i,\mathcal{B})\}$. Then, we have

$$\delta(C_{i_0},\mathcal{B}) = \delta^*(C,\mathcal{B}). \tag{1}$$

Let $W \in \mathcal{B}$ such that $\delta(C_{i_0},W) = \delta(C_{i_0},\mathcal{B})$. We claim that

$$\delta(C_{i_0},W) \le \delta(C_i,W) \text{ for all } i \in \{1,\dots,r\}. \tag{2}$$

To obtain a contradiction suppose that (2) does not hold. Thus there is $i_1 \in \{1,\dots,r\}$ such that

$$\delta(C_{i_1},W) < \delta(C_{i_0},W). \tag{3}$$

Applying Property (δ_3) to the branches C_{i_0}, C_{i_1} and W we get

$$\delta(C_{i_1},W) = \delta(C_{i_0},C_{i_1}). \tag{4}$$

On the other hand, in the case under consideration we have

$$\delta(C_{i_0},\mathcal{B}) = \inf_i\{\delta(C_i,\mathcal{B})\} \le \delta(C_{i_0},C_{i_1}). \tag{5}$$

Therefore by (5), (4) and (3) we get $\delta(C_{i_0},\mathcal{B}) \le \delta(C_{i_1},W) < \delta(C_{i_0},W)$, which contradicts the definition of $\delta(C_{i_0},\mathcal{B})$.

Now, using (2) and (1), we compute

$$\delta(C,W) = \inf\{\delta(C_{i_0},W), \inf_{i\neq i_0}(\delta(C_i,W))\} = \delta(C_{i_0},W) = \delta(C_{i_0},\mathcal{B}) = \delta^*(C,\mathcal{B}),$$

which proves the theorem in the first case.

Second case: $\inf_i\{\delta(C_i,\mathcal{B})\} > \inf_{i,j}\{\delta(C_i,C_j)\}$.

Let i_0, j_0 be such that $\delta(C_{i_0},C_{j_0}) = \inf_{i,j}\delta(C_i,C_j) = \delta^*(C,\mathcal{B})$.

Let $W \in \mathcal{B}$ such that $\delta(C_{i_0},W) = \delta(C_{i_0},\mathcal{B})$. We claim that

$$\delta(C_{i_0},C_{j_0}) \le \delta(C_i,W) \text{ for all } i \in \{1,\dots,r\} \text{ with equality for } i = j_0. \tag{6}$$

First observe that in the case under consideration we have

$$\delta(C_{i_0},C_{j_0}) < \delta(C_{i_0},\mathcal{B}) = \delta(C_{i_0},W). \tag{7}$$

Fix $i \in \{1,\dots,r\}$. If $\delta(C_{i_0},W) \le \delta(C_i,W)$ then (6) follows from (7). If $\delta(C_i,W) < \delta(C_{i_0},W)$ then by Property (δ_3) applied to the branches C_i, C_{i_0} and W we get $\delta(C_i,W) = \delta(C_i,C_{i_0}) \ge \inf_{i,j}\{\delta(C_i,C_j)\} = \delta(C_{i_0},C_{j_0})$. In particular for $i = j_0$, $\delta(C_{j_0},W) = \delta(C_{j_0},C_{i_0}) = \delta(C_{i_0},C_{j_0})$.

Now by the definition of $\delta(C,W)$ and inequalities (6) and (7) we get:

$$\begin{aligned}\delta(C,W) &= \inf\{\delta(C_{i_0},W),\delta(C_{j_0},W),\inf_{i\neq i_0,j_0}\delta(C_i,W)\}\\ &= \delta(C_{j_0},W)=\delta(C_{i_0},C_{j_0})=\delta^*(C,\mathcal{B}),\end{aligned}$$

which proves the theorem in the second case. □

Proposition 2.4. *Let C and D be two branches. Then*

(1) *If there exists a branch of $\mathcal{B}$ which has maximal δ-contact with C and D then $\delta(C,D)\geq\inf\{\delta(C,\mathcal{B}),\delta(D,\mathcal{B})\}$ with equality if $\delta(C,\mathcal{B})\neq\delta(D,\mathcal{B})$.*
(2) *If there does not exist such a branch and U has maximal δ-contact with C and V has maximal δ-contact with D then $\delta(C,D)=\delta(U,V)<\inf\{\delta(C,\mathcal{B}),\delta(D,\mathcal{B})\}$.*

Proof. (see [GB-L-P, Proposition 2.2] for $\delta=d$). If there exists a branch $W\in\mathcal{B}$ such that $\delta(W,C)=\delta(C,\mathcal{B})$ and $\delta(W,D)=\delta(D,\mathcal{B})$ then we get the first part of the proposition by using Property (δ3) to the branches C, D and W. In order to check the second part suppose that such a branch does not exist. Let $U,V\in\mathcal{B}$ such that $\delta(U,C)=\delta(C,\mathcal{B})$ and $\delta(V,D)=\delta(D,\mathcal{B})$. By hypothesis $\delta(C,V)<\delta(C,\mathcal{B})=\delta(C,U)$ and $\delta(D,U)<\delta(D,\mathcal{B})=\delta(D,V)$. According to ($\delta$3) we get $\delta(U,V)=\inf\{\delta(C,V),\delta(C,U)\}=\delta(C,V)$ and $\delta(U,V)=\inf\{\delta(D,U),\delta(D,V)\}=\delta(D,U)$ thus

$$\delta(C,V)=\delta(D,U)=\delta(U,V). \tag{8}$$

Without loss of generality we can suppose that $\delta(C,\mathcal{B})\leq\delta(D,\mathcal{B})$. Since $\delta(C,V)<\delta(C,\mathcal{B})$ so $\delta(C,V)<\delta(D,\mathcal{B})=\delta(D,V)$ and using (δ3) we get

$$\delta(C,D)=\inf\{\delta(C,V),\delta(D,V)\}=\delta(C,V). \tag{9}$$

From (8) and (9) it follows that $\delta(C,D)=\delta(U,V)$. Moreover $\delta(C,D)<\inf\{\delta(C,\mathcal{B}),\delta(D,\mathcal{B})\}$ and we are done. □

Proposition 2.5. *Let C be a reduced curve with $r>1$ branches C_i and let D be a branch. Suppose that $\delta(C,D)<\inf\{\sigma,\inf_{i,j}\{\delta(C_i,C_j)\}\}$, where σ is a real number. Then $\delta(C_i,D)<\sigma$, for $i\in\{1,\dots,r\}$.*

Proof. By definition we have $\delta(C,D)=\inf_{i=1}^r\{\delta(C_i,D)\}$. Thus there exists $i_0\in\{1,\dots,r\}$ such that $\delta(C,D)=\delta(C_{i_0},D)$. Fix $j_0\in\{1,\dots,r\}$. By hypothesis $\delta(C_{i_0},D)<\delta(C_{i_0},C_{j_0})$ and after (δ_3) we have $\delta(C_{i_0},D)=\delta(C_{j_0},D)<\delta(C_{i_0},C_{j_0})$. Now $\delta(C_{j_0},D)=\delta(C_{i_0},D)=\delta(C,D)<\sigma$ and we are done since $j_0\in\{1,\dots,r\}$ is arbitrary. □

Corollary 2.6. *Let C be a reduced curve with $r>1$ branches C_i and let $\mathcal{B}$ be a family of branches such that the condition (*) holds. If $\delta(C,W)<\delta(C,\mathcal{B})$ for a branch $W\in\mathcal{B}$ then $\delta(C_i,W)<\delta(C_i,\mathcal{B})$, for $i\in\{1,\dots,r\}$.*

3. The contact exponent

Recall that $d(C,D)=\frac{(C,D)_0}{m(C)m(D)}$ is a log-distance (see Example 2.1 (1)).

If C and D are reduced curves with irreducible components C_i and D_j then we set $d(C,D)=\inf_{i,j}\{d(C_i,D_j)\}$.

Lemma 3.1. *If C has $r>1$ branches C_i and D is any branch then*

(1) $d(C,D)\leq\inf_{i,j}\{d(C_i,C_j)\}$,
(2) $d(C,D)\leq\frac{(C,D)_0}{m(C)m(D)}$ *with equality if* $d(C,D)<\inf_{i,j}\{d(C_i,C_j)\}$.

Proof. The first part of the lemma follows from Lemma 2.2, for $\delta=d$. In order to check the second part let us observe that

$$\begin{aligned}(C,D)_0 &= \sum_{i=1}^{r}(C_i,D)_0=\sum_{i=1}^{r}d(C_i,D)m(C_i)m(D)\geq\sum_{i=1}^{r}d(C,D)m(C_i)m(D)\\ &= d(C,D)m(C)m(D),\end{aligned}$$

so $d(C,D)\leq\frac{(C,D)_0}{m(C)m(D)}$ with equality if and only if $d(C,D)=d(C_i,D)$ for all $i\in\{1,\dots,r\}$.

Suppose that $d(C,D)<\inf_{i,j}\{d(C_i,C_j)\}$. By definition there is $i_0\in\{1,\dots,r\}$ such that $d(C,D)=d(C_{i_0},D)$, so $d(C_{i_0},D)<d(C_{i_0},C_j)$ for all $j\in\{1,\dots,r\}$. Applying $(\delta 3)$ $(\delta=d)$ to C_{i_0}, D and C_j we get

$$d(C_j,D)=\inf\{d(C_{i_0},D),d(C_j,C_{i_0})\}=d(C_{i_0},D)=d(C,D)\quad\text{for all } j,$$

so $d(C,D)=\frac{(C,D)_0}{m(C)m(D)}$. □

Now we put for any reduced curve C:

$$d(C):=\sup\{d(C,W)\ :\ W\ \text{runs over all smooth branches}\}$$

and call $d(C)$ the *contact exponent* of C (see [Hi, Definition 1.5] where the term *characteristic exponent* is used). We say that a smooth branch W has *maximal contact* with C if $d(C,W)=d(C)$.

Observe that $d(C)=+\infty$ if C is a smooth branch.

Lemma 3.2. *Let C be a singular branch with $\Gamma(C)=\langle v_0,v_1,\dots,v_g\rangle$. Then there exists a smooth branch W_0 such that $(C,W_0)_0=v_1$. Moreover, $d(C)=\frac{v_1}{v_0}$ and W_0 has maximal contact with C.*

Proof. See [GB-P1, Proposition 3.6] or [Ang, Folgerung II.1.1] for the first part of the lemma. To check the second part, let W be a smooth branch. We have $d(C,W_0)=\frac{v_1}{v_0}\notin\mathbb{N}$ and $d(W,W_0)=(W,W_0)_0\in\mathbb{N}$. Therefore $d(C,W_0)\neq d(W,W_0)$ and $d(C,W)=\inf\{d(C,W_0),d(W,W_0)\}\leq d(C,W_0)$. □

Proposition 3.3. *Let C be a reduced curve with $r > 1$ branches C_i. Then*

$$d(C) = \inf\{\inf_i\{d(C_i)\}, \inf_{i,j}\{d(C_i, C_j)\}\}.$$

Moreover, there exists $i_0 \in \{1, \dots, r\}$ such that if a smooth branch W has maximal contact with the branch C_{i_0} then it has maximal contact with the curve C.

Proof. Use Theorem 2.3 when $\delta = d$ and $\mathcal{B}$ is the family of smooth branches. □

Corollary 3.4. *The contact exponent of a reduced curve is an equisingularity invariant.*

Proof. It is a consequence of Lemma 3.2 and Proposition 3.3. □

Corollary 3.5. *Let C be a reduced curve with $r \geq 1$ branches. Then $d(C)$ equals ∞ or a rational number greater than or equal to 1. There exists a smooth curve W that has maximal contact with C. Moreover,*

(1) *$d(C) = +\infty$ if and only if C is a smooth branch.*
(2) *$d(C) = 1$ if and only if C has at least two tangents.*
(3) *$d(C) < \inf_{i=1}^{r}\{d(C_i)\}$ if and only if $d(C)$ is an integer.*

Proof. The first and second properties follow from Lemma 3.2 and Proposition 3.3.

To check the third part suppose that $d(C) \in \mathbb{N}$. Then $d(C) \neq \inf_i\{d(C_i)\}$ and by Proposition 3.3 we get the inequality $d(C) < \inf_i\{d(C_i)\}$.

Suppose now that $d(C) < \inf_i\{d(C_i)\}$. We have to check that $d(C) \in \mathbb{N}$. By Proposition 3.3 we get $d(C) = \inf_{i,j}\{d(C_i, C_j)\} = d(C_{i_0}, C_{j_0})$ for some i_0, j_0. By hypothesis $d(C) = d(C_{i_0}, C_{j_0}) < \inf\{d(C_{i_0}), d(C_{j_0})\}$. Hence by Proposition 2.4 ($\delta = d$) there is not a branch with maximal contact with C_{i_0} and C_{j_0} and $d(C) = d(C_{i_0}, C_{j_0}) = d(U, V)$ for some smooth branches U, V, and we conclude that $d(C) \in \mathbb{N}$. □

Lemma 3.6. *Let C and D be two branches with common tangent. Suppose that $m(C) = m(\widehat{C})$ and $m(D) = m(\widehat{D})$. Then*

$$d(C, D) = d(\widehat{C}, \widehat{D}) + 1.$$

Proof. It is a consequence of Max Noether's theorem, which states $(C, D)_0 = m(C)m(D) + (\widehat{C}, \widehat{D})_0$. □

Theorem 3.7 (Hironaka)**.** *Let $\widehat{C}$ be the strict quadratic transformation of a reduced singular unitangent curve C. Then*

(i) *if $d(C) < 2$ then $m(\widehat{C}) < m(C)$,*
(ii) *if $d(C) \geq 2$ then $m(\widehat{C}) = m(C)$ and $d(\widehat{C}) = d(C) - 1$,*
(iii) *if $d(C) \geq 2$ and W is a smooth curve tangent to C then $d(C, W) = d(\widehat{C}, \widehat{W}) + 1$. If W has maximal contact with C then $\widehat{W}$ has maximal contact with $\widehat{C}$.*

Proof. Firstly consider the case when C is a singular branch. Let $\Gamma(C) = \langle v_0, v_1, \dots, v_g \rangle$. Let us prove (i). By Lemma 3.2 $d(C) = \frac{v_1}{v_0}$ so $d(C) < 2$ if and only if $v_1 - v_0 < v_0$. By the second part of Lemma 1.1 we have $m(\widehat{C}) = \min(\Gamma(\widehat{C})\backslash\{0\}) = v_1 - v_0 < v_0 = \min(\Gamma(C)\backslash\{0\}) = m(C)$.

Now we will prove (ii) when C is irreducible. Assume that $d(C) \geq 2$ (in fact $d(C) > 2$ since $d(C) \notin \mathbb{N}$). The condition $d(C) \geq 2$ means $v_0 < v_1 - v_0$ and by the first part of Lemma 1.1 we get $\Gamma(\widehat{C}) = \langle v_0, v_1 - v_0, \cdots \rangle$. Consequently $m(\widehat{C}) = v_0 = m(C)$ and $d(\widehat{C}) = \frac{v_1 - v_0}{v_0} = d(C) - 1$.

Now let $C = \bigcup_{i=1}^{r} C_i$, $r > 1$ with irreducible C_i and let us prove (i) and (ii) in this case.
Assume that $d(C) < 2$. We claim that there exists $i_0 \in \{1, \dots, r\}$ such that $d(C) = d(C_{i_0})$. Suppose that such i_0 does not exist. Then $d(C) \neq d(C_i)$ for any $i \in \{1, \dots, r\}$ and by Proposition 3.3 $d(C) = \inf_{i,j}\{d(C_i, C_j)\} = d(C_{i_0}, C_{j_0})$ for some $i_0, j_0 \in \{1, \dots, r\}$. We claim that $d(C_{i_0}) < 2$ or $d(C_{j_0}) < 2$. In the contrary case, we had $d(C_{i_0}) \geq 2$ and $d(C_{j_0}) \geq 2$ and we would get $m(\widehat{C}_{i_0}) = m(C_{i_0})$ and $m(\widehat{C}_{j_0}) = m(C_{j_0})$, which implies by Lemma 3.6 $d(C_{i_0}, C_{j_0}) = d(\widehat{C}_{i_0}, \widehat{C}_{j_0}) + 1 \geq 2$. This is a contradiction since $d(C_{i_0}, C_{j_0}) = d(C) < 2$.
If $d(C_{i_0}) = d(C) < 2$ then by the irreducibility case, $m(\widehat{C}_{i_0}) < m(C_{i_0})$ and $m(C) - m(\widehat{C}) = \sum_{i=1}^{r}(m(C_i) - m(\widehat{C}_i)) \geq m(C_{i_0}) - m(\widehat{C}_{i_0}) > 0$.

Suppose now that $d(C) \geq 2$. We have

$$\inf\{d(C_i)\} \geq \inf\{\inf(d(C_i)), \inf(d(C_i, C_j))\} = d(C) \geq 2.$$

Thus $d(C_i) \geq 2$ for $i \in \{1, \dots, r\}$ and by the first part of the proof $m(\widehat{C}_i) = m(C_i)$ and $d(\widehat{C}_i) = d(C_i) - 1$. Hence $m(\widehat{C}) = m(C)$. Moreover, by Lemma 3.6, $d(C_i, C_j) = d(\widehat{C}_i, \widehat{C}_j) + 1$ and $d(C) = \inf\{\inf(d(C_i)), \inf(d(C_i, C_j))\} = \inf\{\inf(d(\widehat{C}_i)), \inf(d(\widehat{C}_i, \widehat{C}_j))\} + 1 = d(\widehat{C}) + 1$.

To finish let us prove (iii). By Lemma 3.6 $d(C_i, W) = d(\widehat{C}_i, \widehat{W}) + 1$ for $i \in \{1, \dots, r\}$ and $d(C, W) = \inf\{d(C_i, W)\} = \inf\{d(\widehat{C}_i, W)\} + 1 = d(\widehat{C}, W) + 1$. Suppose that W has maximal contact with C. Then $d(C) = d(C, W) = d(\widehat{C}, \widehat{W}) + 1 \leq d(\widehat{C}) + 1 = d(C)$, where the last equality is a consequence of statement (ii) of the theorem. This implies $d(\widehat{C}, \widehat{W}) = d(\widehat{C})$. Thus $\widehat{W}$ has maximal contact with $\widehat{C}$. □

Lemma 3.8. *Let C be a reduced curve with $r > 1$ branches and W a smooth branch. If $d(C, W) \notin \mathbb{N}$ then $d(C, W) = d(C)$.*

Proof. The lemma is obvious if C is a branch. In the general case $d(C, W) = \inf_i\{d(C_i, W)\} = d(C_{i_0}, W)$ for some $i_0 \in \{1, \dots, r\}$. If $d(C, W) \notin \mathbb{N}$ then $d(C_{i_0}, W) \notin \mathbb{N}$ and $d(C_{i_0}, W) = d(C_{i_0})$ since C_{i_0} is a branch. Consequently, we get $d(C, W) = d(C_{i_0})$ which implies, by Proposition 3.3, $d(C) = d(C_{i_0}) = d(C, W)$. □

Now we give a characterization of smooth curves which does not have maximal contact with a reduced curve.

Proposition 3.9. *Let C be a reduced curve with $r > 1$ branches. A smooth branch W does not have maximal contact with C if and only if $(C,W)_0 < d(C)m(C)$. Moreover, in this case $(C,W)_0 \equiv 0 \pmod{m(C)}$.*

Proof. Let us suppose that W is a smooth branch which does not have maximal contact with C. We will check that $(C,W)_0 < d(C)m(C)$ and $\frac{(C,W)_0}{m(C)} \in \mathbb{N}$. By Proposition 3.3 we get $d(C,W) < \inf_{i,j}\{d(C_i,C_j)\}$ since $d(C,W) < d(C)$. According to the second part of Lemma 3.1 we can write $d(C,W) = \frac{(C,W)_0}{m(C)}$, thus $(C,W)_0 = d(C,W)m(C) < d(C)m(C)$. We claim $\frac{(C,W)_0}{m(C)} = d(C,W)$ is an integer. Indeed, by Lemma 3.8 we get $d(C,W) = d(C)$, which is a contradiction.

Now suppose that $(C,W)_0 < d(C)m(C)$. By the second part of Lemma 3.1 we get $d(C,W) \leq \frac{(C,W)_0}{m(C)}$ and consequently $d(C,W) < d(C)$, which means that W does not have maximal contact with C. □

4. Milnor number and Hironaka contact exponent

Let C be a reduced curve. We define the *Milnor number* $\mu(C)$ of C by the formula $\mu(C) = c(C) - r(C) + 1$, where $c(C)$ is the degree of the conductor of the local ring of C and $r(C)$ is the number of branches (see Preliminaries).

If $C : \{f = 0\}$ then $\mu(C) = \dim_K K[[x,y]]/\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$ provided that K is of characteristic zero (see [GB-P2]).

Lemma 4.1. *Let $C = \bigcup_{i=1}^r C_i$, where $r \geq 1$ and C_i are irreducible. Then*

(1) $\mu(C) + r - 1 = \sum_{i=1}^r \mu(C_i) + 2\sum_{1\leq i<j\leq r}(C_i,C_j)_0$,
(2) *if C is a branch then $\mu(C)$ equals the conductor of the semigroup $\Gamma(C)$,*
(3) $\mu(C) \geq 0$ *with equality if and only if C is a smooth branch.*

Proof. See [GB-P2, Proposition 2.1]. □

Proposition 4.2. *Let $C = \bigcup_{i=1}^r C_i$ be a singular reduced curve with r branches C_i. Then $\mu(C) \geq (d(C)m(C) - 1)(m(C) - 1)$ with equality if and only if the following two conditions are satisfied:*

(e_1) $d(C_i,C_j) = d(C)$ *for all $i \neq j$,*
(e_2) *if the branch C_i is singular then C_i has exactly one Zariski pair and $d(C_i) = d(C)$.*

Proof. First let us suppose that C is a branch with $\Gamma(f) = \langle v_0, v_1, \ldots, v_g\rangle$. Let $n_0 = 1$. Since $n_{i-1}v_{i-1} \leq v_i$ for $i \in \{1,\ldots,g\}$ we have $n_0 n_1 \cdots n_{k-1} v_1 \leq v_k$ for

$k \in \{1, \dots, g\}$. We get

$$\begin{aligned} c(C) &= \sum_{k=1}^{g}(n_k - 1)v_k - v_0 + 1 \geq \sum_{k=1}^{g}((n_k - 1)n_{k-1} \cdots n_1 n_0)v_1 - v_0 + 1 \\ &= (n_g \cdots n_1 n_0 - n_0)v_1 - (v_0 - 1) = (v_0 - 1)v_1 - (v_0 - 1) \\ &= (v_0 - 1)(v_1 - 1). \end{aligned}$$

Moreover, $c(C) = (v_0 - 1)(v_1 - 1)$ if and only if $\Gamma(C) = \langle v_0, v_1 \rangle$.

Now suppose that the curve C has $r > 1$ branches C_i and let $\overline{m}_i = m(C_i)$ for $i \in \{1, \dots, r\}$. From Proposition 3.3 we get $d(C_i) \geq d(C)$ and $d(C_i, C_j) \geq d(C)$ for all $i, j \in \{1, \dots, r\}$. By the first part of the proof $\mu(C_i) \geq (d(C_i)\overline{m}_i - 1)(\overline{m}_i - 1)$ for the singular branches with equality if and only if C_i is a singular branch satisfying condition (e_2).

Let $I := \{i \ : \ C_i \text{ is singular}\}$. Now we get

$$\begin{aligned} \mu(C) + r - 1 &= \sum_{i=1}^{r}\mu(C_i) + 2\sum_{1 \leq i < j \leq r}(C_i, C_j)_0 \\ &= \sum_{i=1}^{r}\mu(C_i) + 2\sum_{1 \leq i < j \leq r} d(C_i, C_j)\overline{m}_i\overline{m}_j \\ &\geq \sum_{i \in I}(d(C_i)\overline{m}_i - 1)(\overline{m}_i - 1) + 2\sum_{1 \leq i < j \leq r} d(C_i, C_j)\overline{m}_i\overline{m}_j \\ &\geq \sum_{i=1}^{r}(d(C)\overline{m}_i - 1)(\overline{m}_i - 1) + 2\sum_{1 \leq i < j \leq r} d(C)\overline{m}_i\overline{m}_j \\ &= d(C)(m(C)^2 - m(C)) - m(C) + r \end{aligned}$$

with equality if and only if the conditions (e_1) and (e_2) are satisfied. □

Lemma 4.3. *Let C be a unitangent singular curve. We have:*

(1) $d(C) \geq 1 + \frac{1}{m(C)}$. *Moreover* $d(C) = 1 + \frac{1}{m(C)}$ *if and only if C is a branch with semigroup* $\langle m(C), m(C) + 1 \rangle$.
(2) $\mu(C) \geq m(C)(m(C) - 1)$ *with equality if and only if* $d(C) = 1 + \frac{1}{m(C)}$.

Proof. Let $\{C_i\}_i$ be the set of branches of C. To check the first part of the lemma we may assume that $d(C)$ is not an integer. Then by Proposition 3.3 and the third part of Corollary 3.5 there is an i_0 such that $d(C) = d(C_{i_0})$. The contact exponent $d(C_{i_0})$ is a fraction with the denominator less than or equal to $m(C_{i_0})$. Therefore we get $d(C) = d(C_{i_0}) \geq 1 + \frac{1}{m(C_{i_0})} \geq 1 + \frac{1}{m(C)}$ and the equality $d(C) = 1 + \frac{1}{m(C)}$ implies $m(C_{i_0}) = m(C)$ and consequently $C_{i_0} = C$. Moreover the semigroup of C is $\langle m(C), m(C) + 1 \rangle$ since $m(C)$ and $m(C) + 1$ are coprime.

In order to prove the second part we get, by Proposition 4.2 and the first part of this lemma,

$$\begin{aligned}\mu(C) &\geq (d(C)m(C)-1)(m(C)-1)\\ &\geq \left(\left(1+\frac{1}{m(C)}\right)m(C)-1\right)(m(C)-1) = m(C)(m(C)-1).\end{aligned}$$

If $\mu(C) = m(C)(m(C)-1)$ then from the above calculation it follows that $d(C) = 1+\frac{1}{m(C)}$.

On the other hand if $d(C) = 1+\frac{1}{m(C)}$ then by the first part of this lemma C is a branch of semigroup $\langle m(C), m(C)+1\rangle$. According to Proposition 4.2 $\mu(C) = (d(C)m(C)-1)(m(C)-1) = m(C)(m(C)-1)$. □

If $\mu(C) = (d(C)m(C)-1)(m(C)-1)$ then the pair $(m(C), d(C))$ determines the equisingularity class of C. More specifically, we have:

Proposition 4.4. *Let C be a reduced singular curve. Then $\mu(C) = (d(C)m(C)-1)(m(C)-1)$ if and only if one of the following three conditions holds*

(1) *$d(C) \in \mathbb{N}$. All branches of C are smooth and intersect pairwise with multiplicity $d(C)$.*
(2) *$d(C) \notin \mathbb{N}$ and $m(C)d(C) \in \mathbb{N}$. The curve C has $r = \gcd(m(C), m(C)d(C))$ branches, each with semigroup generated by $\left(\frac{m(C)}{r}, \frac{m(C)d(C)}{r}\right)$, intersecting pairwise with multiplicity $\frac{m(C)^2 d(C)}{r^2}$.*
(3) *$m(C)d(C) \notin \mathbb{N}$. There is a smooth curve L such that $C = L \cup C'$, where C' is a curve of type (2) with $d(C') = d(C)$ and $m(C') = m(C)-1$. The branch L has maximal contact with any branch of C'.*

Proof. If one of conditions (1), (2) or (3) is satisfied then a direct calculation shows that $\mu(C) = (d(C)m(C)-1)(m(C)-1)$.

Suppose that $C = \bigcup_{i=1}^r C_i$ satisfy the equality $\mu(C) = (d(C)m(C)-1)(m(C)-1)$. By Proposition 4.2 the conditions (e_1) and (e_2) are satisfied. Let us consider three cases:

Case 1: All branches C_i are smooth. Then C is of type (1) by (e_1).

Case 2: All branches C_i are singular. Then the branches $\{C_i\}_i$ have the same semigroup $\langle v_0, v_1\rangle$ and according to (e_2) $d(C_i) = d(C)$ for all $i \in \{1, \ldots, r\}$. Clearly, we have $m(C) = \sum_{i=1}^r m(C_i) = rv_0$ and $m(C)d(C) = m(C)d(C_i) = rv_1$. Thus $m(C)d(C) \in \mathbb{N}$, $r = \gcd(m(C), m(C)d(C))$ and it is easy to see that C is of type (2).

Case 3: Neither Case 1 nor Case 2 holds, thus $r > 1$. We may assume that C_1 is smooth and C_2 is singular. If $r > 2$ then all branches C_i for $i \geq 3$ are singular. In fact, we have by (e_1): $d(C_1, C_i) = d(C_2, C_i) = d(C_1, C_2) = d(C)$ and by (e_2):

$d(C) = d(C_2) \notin \mathbb{N}$. Thus $d(C_1, C_i) \notin \mathbb{N}$ and C_i are singular for all $i \geq 3$. Let $L := C_1$ and $C' := \bigcup_{i=2}^{r}(C_i, 0)$. Then $C = L \cup C'$ and we check using Proposition 3.3 that C is of type (3). □

Corollary 4.5. *Let C_1, C_2 be two reduced singular curves such that $\mu(C_i) = (d(C_i)m(C_i) - 1)(m(C_i) - 1)$ for $i \in \{1, 2\}$. Then C_1 and C_2 are equisingular if and only if $(m(C_1), d(C_1)) = (m(C_2), d(C_2))$.*

Corollary 4.6. *Let C be a reduced singular curve. Suppose that $\mu(C) = (d(C)m(C) - 1)(m(C) - 1)$ and $m(C)d(C) \notin \mathbb{N}$. Then $(m(C) - 1)d(C) \in \mathbb{N}$.*

To compute $\mu(C)$ one can use Pham's formula.

Proposition 4.7 ([Ph])**.** *Let $C = \bigcup_i^t C_i$, where C_i are unitangent and the tangents to C_i and C_j are different for $i \neq j$. Then*

$$\mu(C) + t - 1 = m(C)(m(C) - 1) + \sum_{k=1}^{t} \mu(\widehat{C}_k).$$

Proof. We distinguish three cases.

Suppose that C is irreducible. Then $\mu(C) = m(C)(m(C) - 1) + \mu(\widehat{C})$ by the well-known formula $c(C) = m(C)(m(C) - 1) + c(\widehat{C})$ (see [Ang, Korollar II.1.8]).

Suppose now that C is unitangent and let $C = \bigcup_{i=1}^{r} C_i$, where C_i are irreducible and let $\overline{m}_i = m(C_i)$ for $i \in \{1, \ldots, r\}$. Then

$$\begin{aligned}
\mu(C) + r - 1 &= \sum_{i=1}^{r} \mu(C_i) + 2 \sum_{1 \leq i < j \leq r} (C_i, C_j)_0 \\
&= \sum_{i=1}^{t} \Big(\overline{m}_i(\overline{m}_i - 1) + \mu(\widehat{C_i}) \Big) + 2 \sum_{1 \leq i < j \leq r} \Big(\overline{m}_i \overline{m}_j + (\widehat{C_i}, \widehat{C_j})_0 \Big) \\
&= \sum_{i=1}^{r} \overline{m}_i(\overline{m}_i - 1) + 2 \sum_{1 \leq i < j \leq r} \overline{m}_i \overline{m}_j + \sum_{i=1}^{r} \mu(\widehat{C_i}) \\
&\qquad + 2 \sum_{1 \leq i < j \leq r} (\widehat{C_i}, \widehat{C_j})_0 \\
&= \sum_{i=1}^{r} \overline{m}_i(\overline{m}_i - 1) + 2 \sum_{1 \leq i < j \leq r} \overline{m}_i \overline{m}_j + \mu(\cup_{i=1}^{r} \widehat{C_i}) + r - 1 \\
&= \sum_{i=1}^{r} \overline{m}_i(\overline{m}_i - 1) + 2 \sum_{1 \leq i < j \leq r} \overline{m}_i \overline{m}_j + \mu(\widehat{\cup_{i=1}^{r} C_i}) \\
&= m(C)(m(C) - 1) + \mu(\widehat{C}) + r - 1.
\end{aligned}$$

Finally suppose that $C = \bigcup_{i=1}^{t} C_i$, where C_i are unitangent and the tangents to C_i and C_j are different for $i \neq j$. Put $\overline{m}_i = m(C_i)$ for $i \in \{1, \ldots, t\}$. Then

$$\begin{aligned}
\mu(C) + t - 1 &= \sum_{i=1}^{t} \mu(C_i) + 2 \sum_{1 \leq i < j \leq t} (C_i, C_j)_0 \\
&= \sum_{i=1}^{t} \Big(\overline{m}_i(\overline{m}_i - 1) + \mu(\widehat{C_i}) \Big) + 2 \sum_{1 \leq i < j \leq t} \overline{m}_i \overline{m}_j \\
&= \sum_{i=1}^{t} \mu(\widehat{C_i}) + \sum_{i=1}^{t} (\overline{m}_i)^2 + 2 \sum_{1 \leq i < j \leq t} \overline{m}_i \overline{m}_j - \sum_{i=1}^{t} \overline{m}_i \\
&= m(C)(m(C) - 1) + \sum_{i=1}^{t} \mu(\widehat{C_i}).
\end{aligned}$$

□

5. Contact exponents of higher order

Let $\mathcal{B}_k$ be the family of branches having at most $k-1$ Zariski pairs. If C is a reduced curve we put

$$d_k(C) := \sup\{d(C, W) \ : \ W \in \mathcal{B}_k\} = d(C, \mathcal{B}_k).$$

Observe that $d_1(C) = d(C)$.

A branch $D \in \mathcal{B}_k$ has k-maximal contact with C if $d(C, D) = d_k(C)$.

The concept of contact exponent of higher order was studied by Lejeune-Jalabert [LJ] and Campillo [C].

Lemma 5.1. *Let $C : \{f = 0\}$ be a singular branch with $\Gamma(C) = \langle v_0, v_1, \ldots, v_g \rangle$. There exist irreducible power series $f_0, \ldots, f_{g-1}$ such that* ord $f_{k-1} = \frac{v_0}{e_{k-1}}$ *and* $(f, f_{k-1})_0 = v_k$.

Proof. We may assume that $(f, x)_0 = \text{ord } f$. According to [GB-P1, Theorem 3.2] there exist distinguished polynomials $f_0, \ldots, f_{g-1}$ such that $(f_{k-1}, x)_0 = \frac{v_0}{e_{k-1}}$ and $(f, f_{k-1})_0 = v_k$. Consider the log-distances $d(f, x) = 1$, $d(f_{k-1}, x) = \frac{(f_{k-1}, x)_0}{\text{ord } f_{k-1}}$ and $d(f, f_k) = \frac{v_k}{v_0 \frac{v_0}{e_{k-1}}} = \frac{e_{k-1} v_k}{(v_0)^2}$. Since $d(f_{k-1}, x) = \frac{e_{k-1} v_k}{(v_0)^2} \geq \frac{e_0 v_1}{(v_0)^2} = \frac{v_1}{v_0} > 1$ we have $d(f_{k-1}, x) = d(f, x) = 1$, that is $(f_{k-1}, x)_0 = \text{ord } f_{k-1}$. □

Lemma 5.2. *Let $C : \{f = 0\}$ be a singular branch with $\Gamma(C) = \langle v_0, v_1, \ldots, v_g \rangle$. If E is a branch such that $d(C, E) > \frac{e_{k-1} v_k}{(v_0)^2}$ then E has at least k Zariski pairs.*

Proof. See [GB-P1, Theorem 5.2]. □

Proposition 5.3. *Let C be a branch with $\Gamma(C) = \langle v_0, \ldots, v_g \rangle$. Then $d_k(C) = \frac{e_{k-1} v_k}{(v_0)^2}$.*

Proof. By Lemma 5.1 there is $D_{k-1} \in \mathcal{B}_k$ such that ord $D_{k-1} = \frac{v_0}{e_{k-1}}$ and $(C, D_{k-1})_0 = v_k$. Then $d_k(C) \geq d(C, D_{k-1}) = \frac{(C,D_{k-1})_0}{\text{ord } C \text{ ord } D_{k-1}} = \frac{e_{k-1}v_k}{(v_0)^2}$. Suppose now that there is a branch $E \in \mathcal{B}_k$ such that $d(C, E) > \frac{e_{k-1}v_k}{(v_0)^2}$. Then $\frac{(C,E)_0}{v_0 \text{ord } E} > \frac{e_{k-1}v_k}{(v_0)^2}$, hence $\frac{(C,E)_0}{\text{ord } E} > \frac{e_{k-1}v_k}{v_0}$. By Lemma 5.2 we conclude that E has at least k Zariski pairs which is a contradiction (since $E \in \mathcal{B}_k$). □

Proposition 5.4. *Let C be a reduced curve with $r > 1$ branches C_i. Then*

$$d_k(C) = \inf\{\inf_i\{d_k(C_i)\}, \inf_{i,j}\{d(C_i, C_j)\}\}.$$

Proof. Use Theorem 2.3 when $\delta = d$ and $\mathcal{B}_k$ is the family of branches having at most $k-1$ Zariski pairs. □

6. Polar invariants and the contact exponent

Let K be a field of characteristic zero. Let C be a reduced plane singular curve and let $P(C)$ be a generic polar of C. Then $P(C)$ is a reduced germ of multiplicity $m(P(C)) = m(C) - 1$. Let $P(C) = \bigcup_{j=1}^{s} D_j$ be the decomposition of $P(C)$ into branches D_j.

We put $\mathcal{Q}(C) = \left\{ \frac{(C,D_j)_0}{m(D_j)} \ : \ j \in \{1, \ldots, s\} \right\}$ and call the elements of $\mathcal{Q}(C)$ the *polar invariants* of C. They are equisingularity invariants of C (see [T2], [Gw-P]). In particular if C is a branch then

$$\mathcal{Q}(C) := \{m(C)d_k(C)\}_{k=1}^{g}.$$

Let us consider the minimal polar invariant $\alpha(C) := \inf \mathcal{Q}(C)$.

Proposition 6.1. *For any singular reduced germ C we have $\alpha(C) = m(C)d(C)$.*

Proof.- See [L-M-P, Theorem 2.1 (iii)].

One could prove Proposition 6.1 by using Theorem 3.3 and the explicit formulae for the polar invariants given in [Gw-P, Theorem 1.3].

We say that C is an *Eggers singularity* if $\mathcal{Q}(C)$ has exactly one element.

Proposition 6.2. *Let C be a singular reduced curve. Then $\mu(C) = (d(C)m(C) - 1)(m(C) - 1)$ if and only if C is an Eggers singularity.*

Proof. By [T1] Proposition 1.2 we get

$$\begin{aligned} \mu(C) &= (C, P(C))_0 - m(C) + 1 = \sum_{j=1}^{s} (C, D_j)_0 - m(C) + 1 \\ &\geq \alpha(C)m(P(C)) - m(C) + 1 = \alpha(C)(m(C) - 1) - m(C) + 1 \\ &= (\alpha(C) - 1)(m(C) - 1) \end{aligned}$$

with equality if and only if C is an Eggers singularity. We use Proposition 6.1. □

Proposition 4.4 provides an explicit description of Eggers singularities.

Corollary 6.3. *([E, p. 16]) If C has exactly one polar invariant then C is equisingular to $y^n - x^m = 0$ or $y^n - yx^m = 0$, for some integers $1 < n < m$.*

Proof. We check that if $C : \{y^n - x^m = 0\}$ then $m(C) = n$, $d(C) = \frac{m}{n}$ and $\mu(C) = nm - n - m + 1$. On the other hand if $C : \{y^n - yx^m = 0\}$ then $m(C) = n$, $d(C) = \frac{m}{n-1}$ and $\mu(C) = nm - n + 1$. In both cases $\mu(C) = (d(C)m(C) - 1)(m(C) - 1)$, that is C is an Eggers singularity.

Now let C be an Eggers singularity. If $m(C)d(C) \in \mathbb{N}$ then C and $\{y^{m(C)} - x^{m(C)d(C)} = 0\}$ are equisingular by Corollary 4.5. Analogously, if $m(C)d(C) \notin \mathbb{N}$ then, by Corollary 4.6, $(m(C) - 1)d(C) \in \mathbb{N}$ and C is equisingular to $\{y^{m(C)} - yx^{(m(C)-1)d(C)} = 0\}$. □

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(Evelia Rosa García Barroso) Departamento de Matemáticas, Estadística e I.O. Sección de Matemáticas, Universidad de La Laguna Apartado de Correos 456, 38200 La Laguna, Tenerife, España

E-mail address: ergarcia@ull.es

(Arkadiusz Płoski) Department of Mathematics and Physics, Kielce University of Technology, Al. 1000 L PP7, 25-314 Kielce, Poland

E-mail address: matap@tu.kielce.pl

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A NON-CONTAINMENT EXAMPLE ON LINES AND A SMOOTH CURVE OF GENUS 10

MAREK JANASZ AND GRZEGORZ MALARA

Abstract. The containment problem between symbolic and ordinary powers of homogeneous ideals has stimulated a lot of interesting research recently. In the most basic case of points in $\mathbb{P}^2$ and powers $I^{(3)}$ and I^2, there is a number of non-containment results based on arrangements of lines. In a joint paper with Lampa-Baczyńska we discovered the first example of non-containment based on an arrangement of axes and a singular irreducible curve of high degree.

In the present note we show a similar example based on lines and a smooth curve of degree 6.

1. Introduction

For algebraic geometers the most intriguing thing is the question about relations between algebraic structures and the geometry that is represented by these objects. One of the famous question related to this investigations is connected to the so-called containment problem which can be formulated as follows.

Containment Problem. Assume that $I \subseteq \mathbb{K}[x_0, \ldots, x_N]$ is a homogeneous ideal in the ring of polynomials over a field $\mathbb{K}$. For which pairs of positive integers (m, r) there is a containment

$$I^{(m)} \subset I^r?$$

A break through in finding an answer to this problem appeared in the paper by Ein, Lazarsfeld and Smith [6], which gives almost optimal relations between numbers m and r. What they proved is that the inclusion $I^{(m)} \subset I^r$ always holds

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under the condition $m \geq Nr$. Since then the problem if the containment holds for m strictly smaller than Nr remained open. Right after [6], some conjectures about the relation between symbolic and ordinary powers of ideals appeared (see [1, Conjecture 8.4.2], or [12, Conjecture 4.1.1], or [2, Conjecture 1.1]). We can collect all these ideals in the following conjecture attributed to Harbourne.

Conjecture 1.1. *Let I be a proper homogeneous ideal such that bigheight of $I = e$. Then*

$$I^{(m)} \subseteq I^r$$

whenever $m \geq er - (e-1)$.

If I describes a smooth variety, then e is its codimension. In general, it can be thought of as the maximal codimension of an embedded component.

If we consider the ideal of points $I \subset \mathbb{K}[x, y, z]$ then the conjecture, for the first non-trivial case of $r = 2$, claims that $I^{(3)} \subseteq I^2$. As we know now, it is not true in general. The first counterexample to $I^{(3)} \subseteq I^2$ was given in [5] and it involves a special point-line configuration defined over the complex numbers. This configuration is known in the literature under many names: dual Hesse configuration, Fermat or Ceva arrangement. It is also denoted by $G(3, 3, 3)$ in the Shephard, Todd classification [18], so it belongs to the irreducible complex reflection group. In a short time a lot of other examples of ideals of points in $\mathbb{P}^2$ were found, for which Conjecture 1.1 fails. In the chronological order of appearing there are:

- A whole family of real counterexamples, coming from the so-called Böröczky configurations [3],
- A collection of counterexamples over the field of finite characteristic [13],
- Examples over the rational numbers [15].

After the appearance of the papers mentioned above, many mathematicians tried to find out a criterion to detect wherever a given configuration of points leads to a counterexample to Conjecture 1.1. The first guess about which property of the set of lines is important to be a counterexample, was the overall number of tripe points in relation to double points. Indeed, Fermat-type arrangements, which are defined over $\mathbb{C}$, have no double points, and for almost all members of this family the inclusion $I^{(3)} \subseteq I^2$ fails (see [13]). Similarly for Böröczky configurations for which the number of double points is the smallest possible and the number of triple points attains the highest possible value (see [9] and [10] for more information).

As the first counterexamples of configurations with relatively high number of double points were discovered (see [16]) another guess about what distinguish configurations of lines $\mathcal{A}$ to give a counterexample was a weak combinatorics, expressed in terms of the t-vector associated to $\mathcal{A}$. But that also occurred not to be essential, as shown in [8]. Nevertheless, all mentioned so far articles deal with the set of at least triple points configuration coming from intersection of lines and the element from $I^{(3)}$ not belonging to I^2 was always the product of equations of all lines. The last results in this subject [14] showed some surprising facts: the ideal of points I

which is the counterexample to Conjecture 1.1 can be generated by triple points, as well as, a big number of double points; the element from the set $I^{(3)} \setminus I^2$ can be generated by a polynomial of degree higher than 2, whose set of zeroes is an irrational curve.

In this paper we continue investigation in finding geometric and combinatorial properties for the set of points which detects wherever this set is generating an ideal giving the counterexample or not. The latest results shows a new direction of researches. Thus in this paper, instead of taking the points of given multiplicity from the configuration of lines, we consider the set of points in some special position. Namely, we start from one of the real reflection arrangement, known as $A(31, 3)$, and we take some orbits of points. What we get can be formulated as the following theorem

Main Theorem. *The ideal I defined as the intersection*

$$I = \bigcap_{P_i \in \mathcal{P}} I(P_i),$$

of 48 *points from the Table 2 is an example of failure for the containment $I^{(3)} \subseteq I^2$. Moreover, there exists an element $F \in I^{(3)} \setminus I^2$ which consists of* 13 *lines and an irreducible curve of degree* 6.

2. Preliminaries

We begin this section with recalling some basic facts and definitions concerning containment problem (see [19] for a detailed survey).

Take any homogeneous ideal $I \subset \mathbb{K}[x_0, \ldots, x_N]$. The m-th symbolic powers of ideal I is defined as

Definition 2.1. (Symbolic power) For $m \geq 1$, the m-th symbolic power of I is the ideal

$$I^{(m)} = \mathbb{K}[x_0, \ldots, x_N] \cap \left(\bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^m)_{\mathfrak{p}} \right),$$

where the intersection is taken over all associated primes of I.

Definition 2.2. Let $I, J \subset \mathbb{K}[x_0, \ldots, x_N]$ be ideals. The saturation of I with respect to J is

$$I : J^{\infty} = \bigcup_{j=0}^{\infty} I : (J^j).$$

From now on we assume that we work over complex projective plane $\mathbb{P}^2(\mathbb{C})$ and that the ideal I is the radical ideal of a finite number of points $P_1, \ldots, P_s$. Under our assumption about the ground field, the symbolic power can be translated into the language of geometry due to Nagata-Zariski theorem ([7, Theorem 3.14]). By

considering all homogeneous forms which vanish up to order m, this theorem tells us that we can think about symbolic power of I as the intersection

$$I^{(m)} = \bigcap_{i=1}^{s} I(P_i)^m,$$

where $I(P_i)$ denotes the ideal of point P_i.

Denote by $\mathcal{A}$ a finite set of lines $L_1, \ldots, L_r$ and by $t_i(\mathcal{A})$ the number of points from the set of all intersection points of L_j where exactly i lines from $\mathcal{A}$ intersect. Thus we obtain a t-vector, which we can associate to any configuration of lines, defined as follows $t(\mathcal{A}) = (t_2(\mathcal{A}), \ldots, t_r(\mathcal{A}))$.

The subject of our consideration is the configuration of points which comes from the line arrangement denoted by $A(31, 1)$ in the Grünbaum list [11]. This special configuration was described in details in [14], where the reader can find informations how to construct this configuration over rational numbers. Let us recall all information relevant for concerning this configuration here.

Put $e = \frac{\sqrt{3}}{2}$, then $A(31, 1)$ consists of 31 lines which equations are given by

$$x \pm aez = 0, \quad x \pm ey \pm bez = 0, \quad ex \pm y \pm cz = 0, \quad y \pm dz = 0, \quad z = 0,$$

where $a \in \{0, \frac{1}{2}, 1, 2\}$, $b \in \{0, 1, 2, 4\}$, $c \in \{0, 2\}$ and $d \in \{0, 1\}$. This configuration is indicated on Figure 2. Line arrangement $A(31, 3)$ is invariant under group

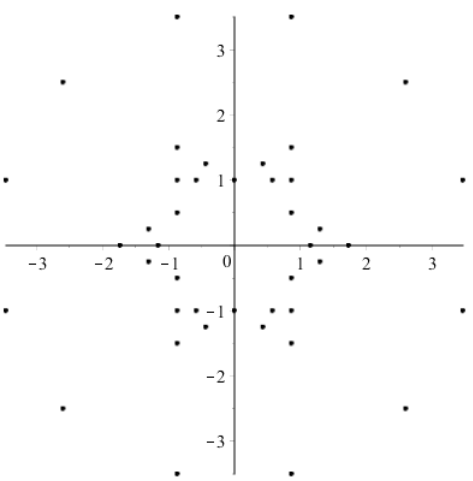

FIGURE 1. Points from set $\mathcal{P}$. Orbit of 6 points at infinity is not indicated.

action $H = Z_3 \times Z_2$, thus all 127 intersection points can be viewed as a groups of points which lies on orbits. It turns out that for these intersection points a different group can be applied, $G = Z_6 \times Z_2$, and from now on we work with this group

action. We can distinguish 16 orbits among all 127 points under the action of G. One of length 1, nine of length 6 and six of length 12. We choose six of the orbits (4 of length 6 and 2 of length 12) and denote their union as $\mathcal{P}$. All 48 chosen points with coordinates are collected in Table 2 and visualized on Figure 1.

number of orbits	length of orbit	points
1	6	$(e:3:-3)$, $(e:-3:3)$, $(-2e:0:3)$, $(2e:0:3)$, $(e:-3:-3)$, $(e:3:3)$
2	6	$(e:-2:2)$, $(e:1:2)$, $(0:-1:1)$, $(0:1:1)$, $(e:1:-2)$, $(e:-1:-2)$
3	12	$(e:5:4)$, $(-3e:1:4)$, $(3e:-1:4)$, $(e:5:-4)$, $(3e:1:-4)$, $(e:-5:4)$, $(e:-5:-4)$, $(3e:1:4)$, $(e:-2:2)$, $(e:2:2)$, $(e:2:-2)$, $(e:-2:-2)$
4	6	$(e:0:1)$, $(e:-3:2)$, $(e:3:2)$, $(e:3:-2)$, $(e:-3:-2)$, $(e:0:-1)$
5	12	$(3e:5:-2)$, $(3e:-5:2)$, $(3e:-5:-2)$, $(3e:5:2)$, $(2e:-1:1)$, $(2e:1:1)$, $(e:-7:2)$, $(e:7:2)$, $(e:-7:-2)$, $(e:7:-2)$, $(-2e:-1:1)$, $(-2e:1:1)$
6	6	$(1:-e:0)$, $(3:e:0)$, $(1:e:0)$, $(3:-e:0)$, $(1:0:0)$, $(0:e:0)$

With all preceding notation we can formulate the following theorem

Main Theorem. *The ideal I defined as the intersection*

$$I = \bigcap_{P_i \in \mathcal{P}} I(P_i),$$

of 48 points from the Table 2 is an example of failure for the containment $I^{(3)} \subseteq I^2$. Moreover, there exists an element $F \in I^{(3)} \setminus I^2$ which consists of 13 lines and irreducible curve of degree 6.

Proof. Define the lines

$$L_{1,2,3,4} : \ x \pm ey \pm ez, \quad L_{5,6,7,8} : 3x \pm ey \pm 2ez,$$

$$L_{9,10} : \ y \pm z, \quad L_{11,12} : \ 2x \pm ez, \quad L_{13} : \ z,$$

and put

$$f := x^6 - \tfrac{9}{2}x^4y^2 + 8x^2y^4 + \tfrac{1}{6}y^6 - \tfrac{103}{12}x^4z^2 - \tfrac{103}{6}x^2y^2z^2 - \tfrac{103}{12}y^4z^2 + \tfrac{269}{12}x^2z^4 + \tfrac{269}{12}y^2z^4 - 17z^6.$$

We claim that

$$F = f \cdot \prod_{i=1}^{13} L_i \in I^{(3)} \setminus I^2.$$

It is a straightforward but tedious calculations to check that $F \in I^{(3)}$. For the reader convenience we prepared Singular [4] script which can be downloaded from web page (see [17]). Polynomial F, together with all visible in affine part $z = 1$ points, are indicated on Figure 3.

It is highly non trivial to check that $F \notin I^2$, due to fact that ideal I^2 is generated by 28 polynomials with very big coefficients. As before, for reader convenience, it can be done by running our script.

In order to prove irreducibility of polynomial f, observe that the Jacobian ideal is generated by

$$\begin{aligned} J = \mathrm{jacob}(f) = \quad & \Big(x(6x^4 - 18x^2y^2 + 16y^4 - \tfrac{103}{3}x^2z^2 - \tfrac{103}{3}y^2z^2 + \tfrac{269}{6}z^4), \\ & y(-9x^4 + 32x^2y^2 + y^4 - \tfrac{103}{3}x^2z^2 - \tfrac{103}{3}y^2z^2 + \tfrac{269}{6}z^4), \\ & z(-\tfrac{103}{6}x^4 - \tfrac{103}{3}x^2y^2 - \tfrac{103}{6}y^4 + \tfrac{269}{3}x^2z^2 + \tfrac{269}{3}y^2z^2 - 102z^4)\Big) \\ & = (xf_1, yf_2, zf_3). \end{aligned}$$

What we can show [17] is that

$$\begin{aligned} x^8 &= \textstyle\sum_{i=1}^{3} c_i f_i, \\ y^8 &= \textstyle\sum_{j=1}^{3} c_j f_j, \\ z^8 &= \textstyle\sum_{k=1}^{3} c_k f_k, \end{aligned}$$

for some $c_i, c_j, c_k \in \mathbb{K}[x, y, z]$, and therefore $V(J : (x, y, z)^\infty) = \emptyset$, which basically means that curve defined by f is smooth and thus irreducible.

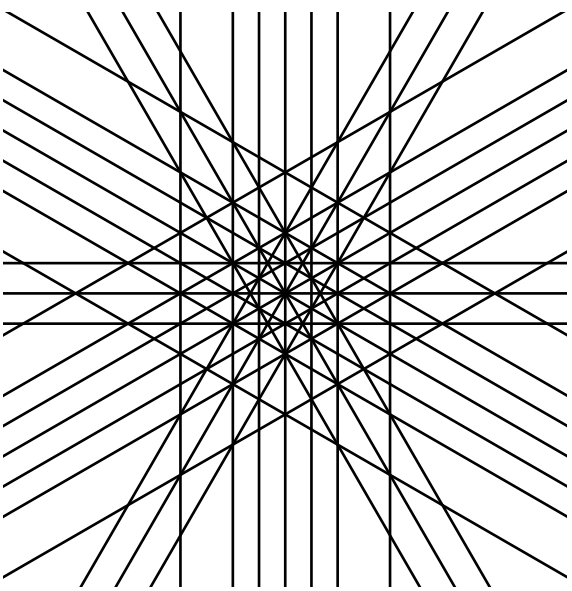

FIGURE 2. Configuration $A(31, 3)$. Line at infinity $z = 0$ is not shown.

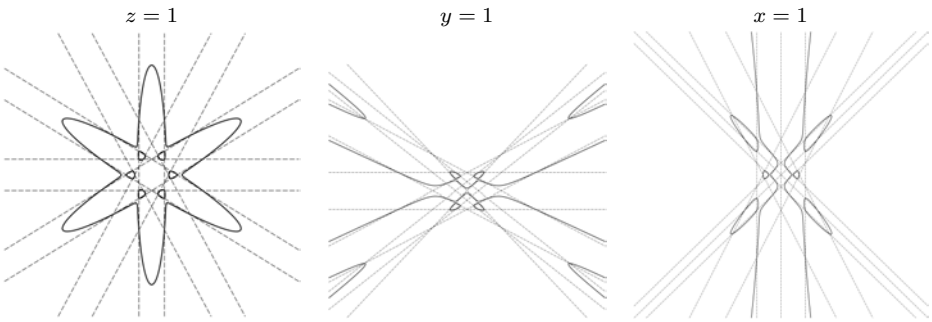

TABLE 1. The curve $f = 0$ with different affine projections.

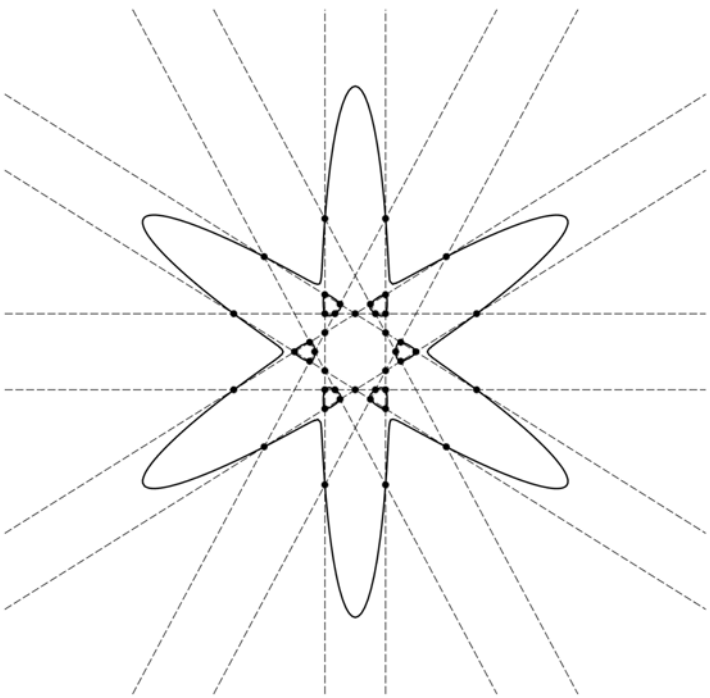

FIGURE 3. The affine part $z = 1$ of the curve $f = 0$ (solid curve), lines $L_i = 0$ (dashed lines) and points from the set $\mathcal{P}$.

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(Marek Janasz) Pedagogical University of Cracow, Department of Mathematics, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: `mjanasz@op.pl`

(Grzegorz Malara) Pedagogical University of Cracow, Department of Mathematics, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: `grzegorz.malara@up.krakow.pl`

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A NOTE ON DIVERGENCE-FREE POLYNOMIAL DERIVATIONS IN POSITIVE CHARACTERISTIC

PIOTR JĘDRZEJEWICZ

Abstract. In this paper we discuss an explicit form of divergence-free polynomial derivations in positive characteristic. It involves Jacobian derivations.

1. Introduction

Throughout this paper by a ring we mean a commutative ring with unity.

Let K be a ring. Recall that if d is a K-derivation of the polynomial algebra $K[x_1, \ldots, x_n]$ of the form $d = g_1 \frac{\partial}{\partial x_1} + \ldots + g_n \frac{\partial}{\partial x_n}$, where $g_1, \ldots, g_n \in K[x_1, \ldots, x_n]$, then a polynomial

$$d^* = \frac{\partial g_1}{\partial x_1} + \ldots + \frac{\partial g_n}{\partial x_n}$$

is called the divergence of d. The derivation d is called divergence-free if $d^* = 0$. See [2] and [3] for information on the properties of the divergence.

Given a polynomial $f \in K[x_1, \ldots, x_n]$, we denote by d^f_{ij} a Jacobian derivation of the form

$$d^f_{ij}(g) = \begin{vmatrix} \frac{\partial f}{\partial x_i} & \frac{\partial f}{\partial x_j} \\ \frac{\partial g}{\partial x_i} & \frac{\partial g}{\partial x_j} \end{vmatrix}$$

for $g \in K[x_1, \ldots, x_n]$. Of course, $(d^f_{ij})^* = 0$.

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In the case of positive characteristic, a divergence-free derivation in two variables is closely related to a single Jacobian derivation, see [1], Theorem 4.5. In the general case of n variables a divergence-free derivation is related to a sum of Jacobian ones (Theorem 3.1 below). This fact is well known (compare [4]). The aim of this paper is to give an elementary proof.

Note that the sum in Theorem 3.1 (ii) is different than the one in Theorem 8.3 of Nowicki's article [2]. There arises a natural question if, under the assumptions of Theorem 3.1, the derivation d can be presented in the form $d = \sum_{i=1}^{n-1} d_{i,i+1}^{f_i} + \delta'$, where $f_1, \ldots, f_{n-1} \in K[x_1, \ldots, x_n]$, and δ' is a K-derivation satisfying

$$\frac{\partial}{\partial x_1}(\delta'(x_1)) = \ldots = \frac{\partial}{\partial x_n}(\delta'(x_n)) = 0.$$

2. Preliminaries

Let K be a ring. Let r be a positive integer and $i \in \{1, \ldots, n\}$. For every polynomial $g \in K[x_1, \ldots, x_n]$ we denote by $g_{i,0}$, $g_{i,1}$, $\ldots$, $g_{i,r-1}$ the uniquely determined polynomials belonging to $K[x_1, \ldots, x_{i-1}, x_i^r, x_{i+1}, \ldots, x_n]$ satisfying the condition

$$g = g_{i,0} + g_{i,1}x_i + \ldots + g_{i,r-1}x_i^{r-1}.$$

The following easy observations will be used in the rest of the paper.

Lemma 2.1. *Every polynomial $g \in K[x_1, \ldots, x_n]$ can be uniquely presented in the form*

$$g = u + wx_i^{r-1},$$

where $u \in K[x_1, \ldots, x_n]$, $u_{i,r-1} = 0$ and $w \in K[x_1, \ldots, x_{i-1}, x_i^r, x_{i+1}, \ldots, x_n]$. In particular, if $g = 0$, then $u = 0$ and $w = 0$.

Lemma 2.2. *If $g \in K[x_1, \ldots, x_{i-1}, x_i^r, x_{i+1}, \ldots, x_n]$ and $j \in \{1, \ldots, n\}$, $j \neq i$, then*

$$\frac{\partial g}{\partial x_j} \in K[x_1, \ldots, x_{i-1}, x_i^r, x_{i+1}, \ldots, x_n].$$

In Lemmas 2.3 and 2.4 we assume that K is a ring of prime characteristic p.

Lemma 2.3. *Consider a polynomial $g \in K[x_1, \ldots, x_n]$. The polynomial g can be presented in the form*

$$g = \frac{\partial v}{\partial x_i}$$

for some $v \in K[x_1, \ldots, x_n]$ if and only if $g_{i,p-1} = 0$.

Lemma 2.4. *Consider a polynomial $g \in K[x_1, \ldots, x_n]$. The polynomial g belongs to $K[x_1, \ldots, x_{i-1}, x_i^p, x_{i+1}, \ldots, x_n]$ if and only if*

$$\frac{\partial g}{\partial x_i} = 0.$$

3. Poincaré-type Lemma

Let K be a ring of prime characteristic p. Consider the algebra of polynomials $K[x_1, \ldots, x_n]$. For $i = 1, \ldots, n$ put $z_i = \prod_{\substack{j=1,\ldots,n \\ j \neq i}} x_j$.

Theorem 3.1 (Poincaré-type Lemma)**.** *Let d be a K-derivation of $K[x_1, \ldots, x_n]$, where $n \geqslant 2$. The following conditions are equivalent:*

(i) $d^* = 0$,

(ii) $d = \sum_{1 \leqslant i < j \leqslant n} d_{ij}^{f_{ij}} + \delta$, *where* $f_{ij} \in K[x_1, \ldots, x_n]$ *for* $i < j$ *and*

$$\delta = h_1 z_1^{p-1} \frac{\partial}{\partial x_1} + h_2 z_2^{p-1} \frac{\partial}{\partial x_2} + \ldots + h_n z_n^{p-1} \frac{\partial}{\partial x_n}$$

for some $h_1, \ldots, h_n \in K[x_1^p, \ldots, x_n^p]$.

Note a version of the above theorem for $n = 1$.

Lemma 3.2. *If d is a K-derivation of $K[x]$, then the following conditions are equivalent:*

(i) $d^* = 0$,

(ii) $d = h \frac{\partial}{\partial x}$, *where* $h \in K[x^p]$.

Remark 3.3. It is convenient to express Theorem 3.1 in terms of polynomials $g_1, \ldots, g_n \in K[x_1, \ldots, x_n]$, where $d = g_1 \frac{\partial}{\partial x_1} + \ldots + g_n \frac{\partial}{\partial x_n}$.

The following conditions are equivalent:

(i) $\frac{\partial g_1}{\partial x_1} + \ldots + \frac{\partial g_n}{\partial x_n} = 0$,

(ii) there exist $f_{ij} \in K[x_1, \ldots, x_n]$ for $i < j$, and $h_1, \ldots, h_n \in K[x_1^p, \ldots, x_n^p]$, such that

$$\begin{aligned}
g_1 &= -\frac{\partial f_{12}}{\partial x_2} - \ldots - \frac{\partial f_{1n}}{\partial x_n} + h_1 z_1^{p-1}, \\
g_2 &= \frac{\partial f_{12}}{\partial x_1} - \frac{\partial f_{23}}{\partial x_3} - \ldots - \frac{\partial f_{2n}}{\partial x_n} + h_2 z_2^{p-1}, \\
&\vdots \\
g_{n-1} &= \frac{\partial f_{1,n-1}}{\partial x_1} + \ldots + \frac{\partial f_{n-2,n-1}}{\partial x_{n-2}} - \frac{\partial f_{n-1,n}}{\partial x_n} + h_{n-1} z_{n-1}^{p-1}, \\
g_n &= \frac{\partial f_{1n}}{\partial x_1} + \ldots + \frac{\partial f_{n-1,n}}{\partial x_{n-1}} + h_n z_n^{p-1}.
\end{aligned}$$

4. Proof

Proof. Obviously, (ii) implies (i).

(i) $\Rightarrow$ (ii) We proceed by induction.

Let $n \geqslant 1$. Consider an arbitrary K-derivation of $K[x_1, \ldots, x_n, x_{n+1}]$

$$d = g_1 \frac{\partial}{\partial x_1} + \ldots + g_n \frac{\partial}{\partial x_n} + g_{n+1} \frac{\partial}{\partial x_{n+1}},$$

where $g_1, \ldots, g_{n+1} \in K[x_1, \ldots, x_{n+1}]$.

For each $i \in \{2, \ldots, n, n+1\}$ we present g_i in the form $g_i = u_i + w_i x_1^{p-1}$, where $u_i \in K[x_1, \ldots, x_{n+1}]$, $(u_i)_{1,p-1} = 0$, $w_i \in K[x_1^p, x_2, \ldots, x_{n+1}]$ (Lemma 2.1). Moreover, we have $u_{n+1} = \dfrac{\partial v_1}{\partial x_1}$ for some $v_1 \in K[x_1, \ldots, x_{n+1}]$ (Lemma 2.3), so $g_{n+1} = \dfrac{\partial v_1}{\partial x_1} + w_{n+1} x_1^{p-1}$.

Now we compute the divergence:

$$\begin{aligned} d^* &= \frac{\partial g_1}{\partial x_1} + \ldots + \frac{\partial g_n}{\partial x_n} + \frac{\partial g_{n+1}}{\partial x_{n+1}} \\ &= \frac{\partial g_1}{\partial x_1} + \frac{\partial}{\partial x_2}\big(u_2 + w_2 x_1^{p-1}\big) + \ldots + \frac{\partial}{\partial x_n}\big(u_n + w_n x_1^{p-1}\big) + \\ &\quad \frac{\partial}{\partial x_{n+1}}\big(\frac{\partial v_1}{\partial x_1} + w_{n+1} x_1^{p-1}\big) \\ &= \frac{\partial g_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_n}{\partial x_n} + \frac{\partial^2 v_1}{\partial x_1 \partial x_{n+1}} + \\ &\quad \frac{\partial w_2}{\partial x_2} x_1^{p-1} + \ldots + \frac{\partial w_n}{\partial x_n} x_1^{p-1} + \frac{\partial w_{n+1}}{\partial x_{n+1}} x_1^{p-1}. \end{aligned}$$

We obtain:

$$d^* = F + G x_1^{p-1},$$

where

$$F = \frac{\partial}{\partial x_1}\big(g_1 + \frac{\partial v_1}{\partial x_{n+1}}\big) + \frac{\partial u_2}{\partial x_2} + \ldots + \frac{\partial u_n}{\partial x_n},$$

$F \in K[x_1, \ldots, x_{n+1}]$, $F_{1,p-1} = 0$, and

$$G = \frac{\partial w_2}{\partial x_2} + \ldots + \frac{\partial w_{n+1}}{\partial x_{n+1}},$$

$G \in K[x_1^p, x_2, \ldots, x_{n+1}]$.

By Lemma 2.1, if $d^* = 0$, then $F = 0$ and $G = 0$.

Now, let $n = 1$ and assume that $d^* = 0$. We have $g_2 = \dfrac{\partial v_1}{\partial x_1} + w_2 x_1^{p-1}$, where $v_1 \in K[x_1, x_2]$, $w_2 \in K[x_1^p, x_2]$. Moreover, $\dfrac{\partial}{\partial x_1}\big(g_1 + \dfrac{\partial v_1}{\partial x_2}\big) = F = 0$ and $\dfrac{\partial w_2}{\partial x_2} = G = 0$. Hence, $g_1 + \dfrac{\partial v_1}{\partial x_2} \in K[x_1^p, x_2]$ and $w_2 \in K[x_1^p, x_2^p]$. Then $g_1 + \dfrac{\partial v_1}{\partial x_2} = u_1 + w_1 x_2^{p-1}$ for some $u_1 \in K[x_1^p, x_2]$, $(u_1)_{2,p-1} = 0$, $w_1 \in K[x_1^p, x_2^p]$ (Lemma 2.1), so $u_1 = \dfrac{\partial v_2}{\partial x_2}$ for some $v_2 \in K[x_1^p, x_2]$ (Lemma 2.3). Finally,

$$g_1 = -\frac{\partial(v_1 - v_2)}{\partial x_2} + w_1 x_2^{p-1}, \quad g_2 = \frac{\partial(v_1 - v_2)}{\partial x_1} + w_2 x_1^{p-1},$$

where $w_1, w_2 \in K[x_1^p, x_2^p]$. We have shown that for $n = 2$ condition (i) implies (ii).

Assume the statement for some $n \geqslant 2$, assume that $d^* = 0$.

By the inductive assumption for $K_1[x_1, \ldots, x_n]$, where $K_1 = K[x_{n+1}]$, since $F = 0$, we have:

$$\begin{aligned}
g_1 + \frac{\partial v_1}{\partial x_{n+1}} &= -\frac{\partial f_{12}}{\partial x_2} - \ldots - \frac{\partial f_{1n}}{\partial x_n} + h_1 z_1^{p-1},\\
u_2 &= \frac{\partial f_{12}}{\partial x_1} - \frac{\partial f_{23}}{\partial x_3} - \ldots - \frac{\partial f_{2n}}{\partial x_n} + h_2 z_2^{p-1},\\
&\vdots\\
u_{n-1} &= \frac{\partial f_{1,n-1}}{\partial x_1} + \ldots + \frac{\partial f_{n-2,n-1}}{\partial x_{n-2}} - \frac{\partial f_{n-1,n}}{\partial x_n} + h_{n-1} z_{n-1}^{p-1},\\
u_n &= \frac{\partial f_{1n}}{\partial x_1} + \ldots + \frac{\partial f_{n-1,n}}{\partial x_{n-1}} + h_n z_n^{p-1},
\end{aligned}$$

where $f_{ij} \in K_1[x_1, \ldots, x_n]$, $h_i \in K_1[x_1^p, \ldots, x_n^p]$ and $z_i = \prod\limits_{\substack{j=1,\ldots,n\\ j\neq i}} x_j$.

Moreover, by the inductive assumption for $K_2[x_2, \ldots, x_{n+1}]$, where $K_2 = K[x_1^p]$, since $G = 0$, we have:

$$\begin{aligned}
w_2 &= -\frac{\partial s_{23}}{\partial x_3} - \ldots - \frac{\partial s_{2,n+1}}{\partial x_{n+1}} + a_2 y_2^{p-1},\\
w_3 &= \frac{\partial s_{23}}{\partial x_2} - \frac{\partial s_{34}}{\partial x_4} - \ldots - \frac{\partial s_{3,n+1}}{\partial x_{n+1}} + a_3 y_3^{p-1},\\
&\vdots\\
w_n &= \frac{\partial s_{2,n}}{\partial x_2} + \ldots + \frac{\partial s_{n-1,n}}{\partial x_{n-1}} - \frac{\partial s_{n,n+1}}{\partial x_{n+1}} + a_n y_n^{p-1},\\
w_{n+1} &= \frac{\partial s_{2,n+1}}{\partial x_2} + \ldots + \frac{\partial s_{n,n+1}}{\partial x_n} + a_{n+1} y_{n+1}^{p-1},
\end{aligned}$$

where $s_{ij} \in K_2[x_2, \ldots, x_{n+1}]$, $a_2, \ldots, a_{n+1} \in K_2[x_2^p, \ldots, x_{n+1}^p]$ and $y_i = \prod_{\substack{j=2,\ldots,n+1 \\ j\neq i}} x_j$ for $i = 2, \ldots, n+1$.

Now, we present each h_i, for $i = 1, \ldots, n$, in the form $h_i = b_i + c_i x_{n+1}^{p-1}$, where $b_i \in K[x_1^p, \ldots, x_n^p, x_{n+1}]$, $(b_i)_{n+1,p-1} = 0$, $c_i \in K[x_1^p, \ldots, x_n^p, x_{n+1}^p]$ (Lemma 2.1). Then $b_i = \dfrac{\partial q_i}{\partial x_{n+1}}$ for some $q_i \in K[x_1^p, \ldots, x_n^p, x_{n+1}]$, so $h_i = \dfrac{\partial q_i}{\partial x_{n+1}} + c_i x_{n+1}^{p-1}$ and $\dfrac{\partial q_i}{\partial x_j} = 0$ for $j = 1, \ldots, n$. Denote: $t_i = \prod_{\substack{j=1,\ldots,n+1 \\ j\neq i}} x_j$ for $i = 1, \ldots, n+1$. We obtain

$$
\begin{aligned}
g_1 &= -\frac{\partial f_{12}}{\partial x_2} - \ldots - \frac{\partial f_{1n}}{\partial x_n} - \frac{\partial (v_1 - q_1 x_2^{p-1} \ldots x_n^{p-1})}{\partial x_{n+1}} + c_1 t_1^{p-1}, \\
g_2 &= \frac{\partial f_{12}}{\partial x_1} - \frac{\partial (f_{23} + s_{23} x_1^{p-1})}{\partial x_3} - \ldots - \frac{\partial (f_{2n} + s_{2n} x_1^{p-1})}{\partial x_n} \\
&\quad - \frac{\partial (s_{2,n+1} x_1^{p-1} - q_2 x_1^{p-1} x_3^{p-1} \ldots x_n^{p-1})}{\partial x_{n+1}} + (c_2 + a_2) t_2^{p-1}, \\
&\vdots \\
g_n &= \frac{\partial f_{1n}}{\partial x_1} + \frac{\partial (f_{2n} + s_{2n} x_1^{p-1})}{\partial x_2} + \ldots + \frac{\partial (f_{n-1,n} + s_{n-1,n} x_1^{p-1})}{\partial x_{n-1}} \\
&\quad - \frac{\partial (s_{n,n+1} x_1^{p-1} - q_n x_1^{p-1} x_2^{p-1} \ldots x_{n-1}^{p-1})}{\partial x_{n+1}} + (c_n + a_n) t_n^{p-1}, \\
g_{n+1} &= \frac{\partial (v_1 - q_1 x_2^{p-1} \ldots x_n^{p-1})}{\partial x_1} + \frac{\partial (s_{2,n+1} x_1^{p-1} - q_2 x_1^{p-1} x_3^{p-1} \ldots x_n^{p-1})}{\partial x_2} + \ldots \\
&\quad + \frac{\partial (s_{n,n+1} x_1^{p-1} - q_n x_1^{p-1} x_2^{p-1} \ldots x_{n-1}^{p-1})}{\partial x_n} + a_{n+1} t_{n+1}^{p-1}.
\end{aligned}
$$

□

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Nicolaus Copernicus University, Faculty of Mathematics and Computer Science, ul. Chopina 12/18, 87-100 Toruń, Poland

E-mail address: pjedrzej@mat.umk.pl

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KNOTS OF IRREDUCIBLE CURVE SINGULARITIES

TADEUSZ KRASIŃSKI

Abstract. In the article the relation between irreducible curve plane singularities and knots is described. In these terms the topological classification of such singularities is given.

1. Introduction

Local theory of analytic (algebraic) curves in $\mathbb{C}^2$ i.e. the theory of plane curve singularities is closely related to the theory of knots. If $V = V(f)$, $f \in \mathbb{C}\{x, y\}$, $f \neq \text{const}$, is a local analytic curve described by the equation $f(x, y) = 0$ in a neighbourhood U of the point $0 \in \mathbb{C}^2$, then the intersection $V \cap \mathbb{S}^3_r$ of V with a small 3-dimensional sphere $\mathbb{S}^3_r := \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = r^2\}$ is homeomorphic (even bianalytic) to the unit circle $\mathbb{S}^1$ (if V is irreducible) or to a finite disjoint union of such unit circles (if V is reducible). So this intersection is a knot or a link in $\mathbb{S}^3_r$. Moreover, for all sufficiently small r the knot (link) does not depend on r and uniquely characterizes the topology of V in 4-dimensional ball which boundary is $\mathbb{S}^3_r$. It turns out that knots corresponding to irreducible curve singularities are of very special kind: torus knots of higher orders (also called cable knots). In the article we describe torus knots and relation between irreducible singularities and knots. Due to the form of parameterizations of curve singularities, it is easier to consider the boundary of polycylinders $\{(x, y) \in \mathbb{C}^2 : |x| \leq r, |y| \leq r'\}$ instead of spheres (these both are, of course, homeomorphic sets).

In Section 2 we shortly remember the basics of the knot theory. Section 3 is devoted to the torus knots of the first order. They correspond to the irreducible singularities with one characteristic pair, in particular to singularities $x^n - y^m = 0$, $n, m \in \mathbb{N}$, $\text{GCD}(n, m) = 1$. In Section 4 we will consider the torus knots of higher

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orders. Section 5 describes correspondence between irreducible curve singularities and torus knots. Section 6 is devoted to topological classification of ireducible curve singularities.

2. Basics of the knot theory

The basic sources of this theory are classic textbooks [CF], [R]. Denote $\mathbb{S}^1 := \{z \in \mathbb{C} : |z| = 1\} = \{e^{2\pi i\theta} : \theta \in [0,1]\}$ – the *unit circle* in $\mathbb{C}$, and $\mathbb{S}^3$ – the 3-*dimensional sphere* defined as the space $\mathbb{R}^3$ with one point ∞ added, i.e.

$$\mathbb{S}^3 := \mathbb{R}^3 \cup \{\infty\}$$

with the Aleksandrov topology. Recall, open sets in $\mathbb{S}^3$ are: open sets in $\mathbb{R}^3$ and complements of compact sets in $\mathbb{R}^3$ with the point ∞ added. *A knot* is the homeomorphic image of $\mathbb{S}^1$ in $\mathbb{S}^3$ i.e. each subset $W \subset \mathbb{S}^3$ such that $W = \Phi(\mathbb{S}^1)$, where $\Phi : \mathbb{S}^1 \to W$ is a homeomorphism. *A link* is a finite disjoint union of knots (see Fig. 1).

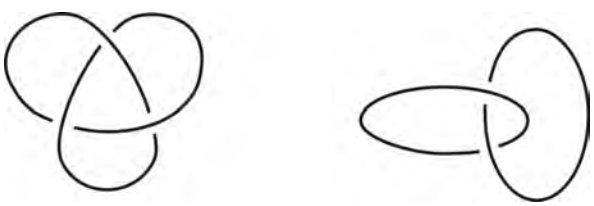

Fig. 1. Examples of a knot (the trefoil) and a link.

Two knots (links) W_1, W_2 are *equivalent* if there exists a homeomorphism $F : \mathbb{S}^3 \to \mathbb{S}^3$ such that $F(W_1) = W_2$. We say then W_1, W_2 have the same *type* and denote $W_1 \sim W_2$. In the sequel we will identify knots with their types. *Trivial knot* (or *unknot*) is the knot

$$\mathbb{S}^1 \ni e^{2\pi i\theta} \mapsto (\cos 2\pi\theta, \sin 2\pi\theta, 0) \in \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}.$$

Remark 2.1. *Studying types of knots in the sphere $\mathbb{S}^3$ is the same as in space $\mathbb{R}^3$ because two knots (links) in $\mathbb{S}^3 \setminus \{\infty\}$ are equivalent in $\mathbb{S}^3$ if and only if they are equivalent in $\mathbb{R}^3$.*

Since in the theory of singularities we will deal only with analytic knots i.e. homeomorphisms $\Phi : \mathbb{S}^1 \to \Phi(\mathbb{S}^1) \subset \mathbb{S}^3$ are analytic functions we will consider only *tame knots* that is knots equivalent to *polygonal knots*. It follows from the fact that each knot of class C^1 (in particular analytic) is equivalent to a polygonal knot

([CF], App. 1). See Figure 2.

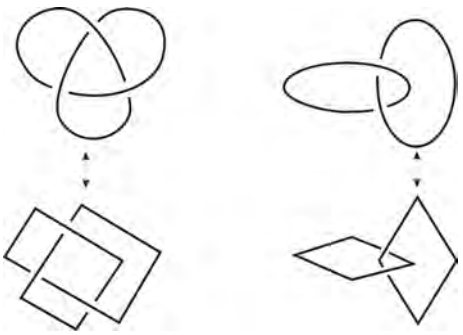

Fig. 2. A polygonal knot and link

A complete invariant of a knot W is its complement $\mathbb{S}^3 \setminus W$, treated as a topological space, because two knots are equivalent if and only if their complements are homeomorphic [GL]. Unforunately it is not true for links [R], p. 49. A weaker invariant of knots and links is the first homotopy group of their complements. We denote it by of $\pi(W)$ and call the *knot (link) group.* So

$$\pi(W) := \pi_1(\mathbb{R}^3 \setminus W; *),$$

where $*$ is an arbitrary point in $\mathbb{R}^3 \setminus W$. Since $\mathbb{R}^3 \setminus W$ is arc connected (remember W is equivalent to polygonal one) $\pi(W)$ does not depend (up to an isomorphism) on the choice of the point $*$. The knot group is not a complete invariant of W. There exist knots having isomorphic groups but not equivalent (see [CF],VIII, 4.8). There are general methods to calculate knot groups (e.g. Wirtinger method, see [CF]) by giving generators of $\pi(W)$ and relations between them. Since for knots related to curve singularities we will describe generators of $\pi(W)$ and relations between them directly, we don't present these methods. We will illustrate this with an example.

Example 2.2. *Two presentations of the knot group of the trefoil W:*

1.

$$\pi(W) = \mathcal{F}(x,y)/\left(xyxy^{-1}x^{-1}y^{-1}\right)$$

where x, y are loops in Figure 3(a) and $\mathcal{F}(x,y)$ is the free group generated by two elements x, y, and $\left(xyxy^{-1}x^{-1}y^{-1}\right)$ is the smallest normal subgroup in $\mathcal{F}(x,y)$ containing $xyxy^{-1}x^{-1}y^{-1}$,

2.

$$\pi(W) = \mathcal{F}(x,y)/\left(x^2y^{-3}\right)$$

where x, y are loops in Figure 3(b).

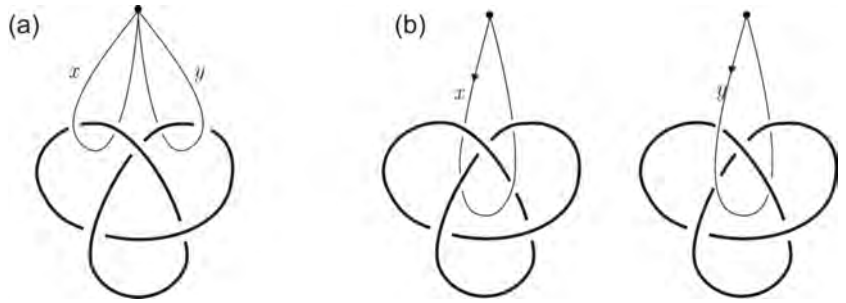

Fig. 3. Generators of the knot group of trefoil.

Although the knot group is not a complete invariant of a knot this is the case for knots (and links) associated to curve singularities. However it is difficult to decide on the basis of knowledge of generators and relations whether two given groups are or are not isomorphic. So we take a weaker invariant of knots to distinguish between torus knots – the Aleksander polynomial of a knot. Its definition is complicated and based on the formal differentation (the free calculus) in knot groups $\pi(W)$ ([CF]). Another approach one can find in [R], p. 206. We recall the first approach.

Let $\mathcal{F} = \mathcal{F}(x_1, \ldots, x_n)$ be the free group with n generators $x_1, \ldots, x_n$ and $G = \mathcal{F}(x_1, \ldots, x_n)/(r_1, \ldots, r_m)$ an arbitrary group (in general non-abelian) with generators $x_1, \ldots, x_n$ and relations $r_1, \ldots, r_m \in \mathcal{F}(x_1, \ldots, x_n)$ ($(r_1, \ldots, r_m)$ denotes the smallest normal subgroup in $\mathcal{F}$ containing $r_1, \ldots, r_m$). Adding trivial relations of the type $x_i x_i^{-1}$ we may assume that $m \geq n-1$. In the group ring $\mathbb{Z}[\mathcal{F}]$ we define a formal derivation; first on elements of $\mathcal{F} \subset \mathbb{Z}[\mathcal{F}]$, and next we extend it on the whole ring $\mathbb{Z}[\mathcal{F}]$ in an obvious way. Take any element $\boldsymbol{g} \in \mathcal{F}$. We may represent it in the following way

$$\boldsymbol{g} = x_{i_1}^{\varepsilon_1} \ldots x_{i_k}^{\varepsilon_k}, \qquad \varepsilon_j = \pm 1.$$

We define formal partial derivatives of g as follows

$$\frac{\partial \boldsymbol{g}}{\partial x_j} := \varepsilon_1 \delta_{ji_1} x_{i_1}^{(\varepsilon_1 - 1)/2} + x_{i_1}^{\varepsilon_1} \varepsilon_2 \delta_{ji_2} x_{i_2}^{(\varepsilon_2 - 1)/2} + \ldots + x_{i_1}^{\varepsilon_1} \ldots x_{i_{k-1}}^{\varepsilon_{k-1}} \varepsilon_k \delta_{ji_k} x_{i_2}^{(\varepsilon_k - 1)/2} \in \mathbb{Z}[\mathcal{F}].$$

In particular $\frac{\partial (x x^{-1})}{\partial x} = 1 - x x^{-1} = 0$, which proves the correctness of the definition of formal differentiation. For illustration, consider the following important examples.

Example 2.3. *In $\mathcal{F}(x, y)$ for $n, m \in \mathbb{N}$ we have:*

1. *If $\boldsymbol{g} = x^n$, then $\frac{\partial \boldsymbol{g}}{\partial x} = 1 + x + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1}$.*
2. *If $\boldsymbol{g} = y^{-m}$, then $\frac{\partial \boldsymbol{g}}{\partial y} = -y^{-1} - y^{-2} - \ldots - y^{-m} = -y^{-m} \frac{y^m - 1}{y - 1} = \frac{y^{-m} - 1}{y - 1}$.*

3. If $\boldsymbol{g} = x^n y^{-m}$, *then*

$$\frac{\partial \boldsymbol{g}}{\partial x} = 1 + x + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1},$$
$$\frac{\partial \boldsymbol{g}}{\partial y} = -x^n y^{-1} - x^n y^{-2} - \ldots - x^n y^{-m} = -x^n y^{-m} \frac{y^m - 1}{y - 1} = -\boldsymbol{g} \frac{y^m - 1}{y - 1}.$$

Next for the group $G = \mathcal{F}(x_1, \ldots, x_n) / (r_1, \ldots, r_m)$, $m \geq n - 1$, we define a matrix over $\mathbb{Z}[\mathcal{F}]$

$$M_G := \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \ldots & \frac{\partial r_1}{\partial x_n} \\ \ldots & \ldots & \ldots \\ \frac{\partial r_m}{\partial x_1} & \ldots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

Example 2.4. *Let* $G = \mathcal{F}(x, y) / \left(x^2 y^{-3}\right)$. *Then* M_G *is a matrix* 1×2.

$$M_G = \left[1 + x,\ x^2(-y^{-1} - y^{-2} - y^{-3})\right] = \left[\frac{x^2 - 1}{x - 1},\ -x^2 y^{-3} \frac{y^3 - 1}{y - 1}\right].$$

Because we will use minors of the matrix M_G and determinants have "good properties" in commutative rings, we abelianize the group G, i.e. we divide G by its commutator $[G : G] := (xyx^{-1}y^{-1} : x, y \in G) \subset G$. We obtain the abelian group

$$G' := G/[G : G].$$

Then the group ring $\mathbb{Z}[G']$ is a commutative ring.

In the case G is the knot group we have

Proposition 2.5. *If* $G = \pi(W)$ *is the knot group of a knot* W *then* $G' \cong \mathbb{Z}$.

This follows from some facts of algebraic topology. As is known, the abelianization of the first homotopy group $\pi_1(X)$ of a "good" topological space X (e.g. topological manifold, and this is the case for knot complement) is isomorphic to the first homology group of X, i.e. $\pi_1(X)' = \pi_1(X)/[\pi_1(X) : \pi_1(X)] \cong H_1(X, \mathbb{Z})$. In the case $X = \mathbb{R}^3 \setminus W$ is the complement of a knot it is easy to show that $H_1(\mathbb{R}^3 \setminus W, \mathbb{Z}) \cong \mathbb{Z}$. Its generator is any loop surrounding one thread of the knot. For instance each loop x and y in Figure 3(a) is a generator. In Figure 3(b) neither x nor y is such a generator. Hence in the case $G = \pi(W)$, choosing one generator in $G' \cong \mathbb{Z}$ we get the isomorphim $\mathbb{Z}[G'] \cong \mathbb{Z}[\mathbb{Z}]$. But the group ring $\mathbb{Z}[\mathbb{Z}]$ is isomorphic to the *ring of Laurent polynomials* $\mathbb{Z}[t, t^{-1}]$. It is easy to see that the ring $\mathbb{Z}[t, t^{-1}]$ has properties:

1. The only invertible elements in $\mathbb{Z}[t, t^{-1}]$ are powers $\pm t^n$, $n \in \mathbb{Z}$,
2. Each element $A(t) \in \mathbb{Z}[t, t^{-1}]$ has a unique representation in the form

$$A(t) = t^n \widetilde{A}(t), \text{ where } n \in \mathbb{Z} \text{ i } \widetilde{A}(t) \in \mathbb{Z}[t],\ \widetilde{A}(0) \neq 0,$$

After the extension of canonical homomorphisms $\mathcal{F} \to G \to G' \simeq \mathbb{Z}$ to homomorphisms of group rings $\mathbb{Z}[\mathcal{F}] \to \mathbb{Z}[G] \to \mathbb{Z}[G'] \simeq \mathbb{Z}[\mathbb{Z}] \simeq \mathbb{Z}[t, t^{-1}]$ and applying

this last sequence of homomorphisms to the elements of the matrix M_G we will get the matrix

$$M'_G := \begin{bmatrix} A_{11}(t) & \dots & A_{1n}(t) \\ \dots & \dots & \dots \\ A_{m1}(t) & \dots & A_{mn}(t) \end{bmatrix}$$

which elements are Laurent polynomials. We will call it the *Alexander matrix of* G. This matrix is important in the theory of knots by the following theorem (see [CF]).

Theorem 2.6. *The ideal* $E \subset \mathbb{Z}[t, t^{-1}]$ *generated by the* $(n-1)$ *minors of the matrix* M'_G *does not depend on the choice of the generators* $x_1, \dots, x_n$ *and the relations* $r_1, \dots, r_m$ *of group* G*, so it only depends on the group* G *.*

Let $M_1(t), \dots, M_k(t)$ be the minors of degree $n-1$ of the matrix M'_G. From the above the greatest common divisor of all M_i depends only, up to invertible elements, on the group G. We call it the *Alexander polynomial of* W and denote by A_W. Hence

$$A_W(t) = \mathrm{GCD}(M_1(t), \dots, M_k(t)) \in \mathbb{Z}[t, t^{-1}].$$

Because A_W is determined up to factors of the type $\pm t^n$, $n \in \mathbb{Z}$, we always choose its normalized form, i.e. one that is an ordinary polynomial in $\mathbb{Z}[t]$ with a non-zero constant term and the highest coefficient $\mathrm{inco}\, A_W$ positive. So, at the end

$$A_W(t) \in \mathbb{Z}[t], \quad A_W(0) \neq 0, \quad \mathrm{inco}\, A_W(t) > 0.$$

For instance the normalized form of the Laurent polynomial $t^{-2} + 2t^{-1} - 3t$ is $-1 - 2t + 3t^3$.

Immediately from theorem 2.6 we obtain

Proposition 2.7. *For knots* W_1 *and* W_2 *if* $\pi(W_1) \cong \pi(W_2)$ *then* $A_{W_1} = A_{W_2}$.

Example 2.8. *For the trefoil knot* W *we gave two presentations of its group (see Example 2.2):*

I presentation: $\pi(W) = \mathcal{F}(x, y) / \left(xyxy^{-1}x^{-1}y^{-1}\right)$ *. In this case*

$$M_{\pi(W)} = [1 + xy - xyxy^{-1}x^{-1},\ x - xyxy^{-1} - xyxy^{-1}x^{-1}y^{-1}]$$

Since a generator of $\pi(W)' = \pi(W)/[\pi(W) : \pi(W)]$ *is the abstract class* $[x]$ *(it can also be a class* $[y] = [x]$*), then denoting* $t = [x]$ *we have*

$$M'_{\pi(W)} = [1 + t^2 - t, t - t^2 - 1].$$

Hence

$$A_W(t) = t^2 - t + 1 = \frac{(t^6 - 1)(t - 1)}{(t^2 - 1)(t^3 - 1)}.$$

II presentation: $\pi(W) = \mathcal{F}(x,y)/\left(x^2y^{-3}\right)$. *We have*

$$M_{\pi(W)} = [1+x,\ x^2(-y^{-1}-y^{-2}-y^{-3})]$$

In this case neither $[x]$ *nor* $[y]$ *is a generator of* $\pi(W)'$. *If we take the loop* x *from the previous presentation (for distinction let us denote it by* $\widetilde{x}$ *and we take as before* $t = [\widetilde{x}]$*), then* $[x] = t^3$ *and* $[y] = t^2$. *Then*

$$M'_{\pi(W)} = [1+t^3,\ -t^4-t^2-1] = \left[\frac{t^6-1}{t^3-1},\ -\frac{t^6-1}{t^2-1}\right].$$

Hence

$$A_W(t) = \frac{(t^6-1)(t-1)}{(t^2-1)(t^3-1)} = t^2-t+1.$$

3. Torus knots of the first order

In this section, we will define a particular type of knots, the so-called torus knots. We will start with the simplest type of them - torus knots of the first order.

By T and $\mathbb{T}$ we will denote the *torus* and the *solid torus* in $\mathbb{C}^2$ defined by

$$\begin{aligned} T &:= \{(x,y)\in\mathbb{C}^2 : |x|=1,\ \ |y|=1\} \\ &= \{(x,y)\in\mathbb{C}^2 : x = e^{2\pi i\eta},\ y = e^{2\pi i\theta},\ 0\le\eta,\theta\le 1\},, \\ \mathbb{T} &:= \{(x,y)\in\mathbb{C}^2 : |x|=1,\ \ |y|\le 1\} \\ &= \{(x,y)\in\mathbb{C}^2 : x = e^{2\pi i\eta},\ y = re^{2\pi i\theta},\ 0\le\eta,\theta\le 1,\ 0\le r\le 1\}. \end{aligned}$$

We see that both T and $\mathbb{T}$ lie in the boundary ∂P of the policylinder $P = \{(x,y)\in\mathbb{C}^2 : |x|\le 1,\ |y|\le 1\}$. Because ∂P is homeomorphic to $\mathbb{S}^3 = \mathbb{R}^3\cup\{\infty\}$, any knot in T or $\mathbb{T}$ can be considered as a knot in $\mathbb{S}^3$. However, it depends on the chosen homeomorphism of ∂P on $\mathbb{S}^3$. For calculations and graphical presentation, we will choose such one that this homeomorphism will send T and $\mathbb{T}$ to the standard torus T^{st} and the standard solid torus $\mathbb{T}^{st}$ (see Fig. 4) defined parametrically in $\mathbb{R}^3$ as follows

$$T^{st}:\quad \begin{aligned} x_1 &= (2+\cos 2\pi\theta)\cos 2\pi\eta, \\ x_2 &= (2+\cos 2\pi\theta)\sin 2\pi\eta, \quad 0\le\eta,\theta\le 1 \\ x_3 &= \sin 2\pi\theta \end{aligned}$$

$$\mathbb{T}^{st}:\quad \begin{aligned} x_1 &= (2+r\cos 2\pi\theta)\cos 2\pi\eta, \\ x_2 &= (2+r\cos 2\pi\theta)\sin 2\pi\eta, \quad 0\le\eta,\theta\le 1,\ 0\le r\le 1 \\ x_3 &= r\sin 2\pi\theta \end{aligned}$$

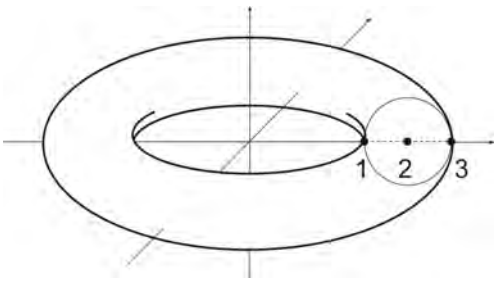

Fig. 4. The standard torus in $\mathbb{R}^3$

This homeomorphism

$$F : \partial P \to \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$$

we define separately on $(\partial P)_1$ and on $(\partial P)_2$, where $\partial P = (\partial P)_1 \cup (\partial P)_2$ and

$$\begin{aligned}
(\partial P)_1 &= \{(x,y) \in \mathbb{C}^2 : |x| = 1,\ \ |y| \leq 1\} \\
&= \{(x,y) \in \mathbb{C}^2 : x = e^{2\pi i\eta},\ y = re^{2\pi i\theta},\ 0 \leq \eta, \theta \leq 1,\ 0 \leq r \leq 1\}, \\
(\partial P)_2 &= \{(x,y) \in \mathbb{C}^2 : |x| \leq 1,\ \ |y| = 1\} \\
&= \{(x,y) \in \mathbb{C}^2 : x = re^{2\pi i\eta},\ y = e^{2\pi i\theta},\ 0 \leq \eta, \theta \leq 1,\ 0 \leq r \leq 1\},
\end{aligned}$$

On $(\partial P)_1$ we define F by formula

$$\begin{aligned}
F|_{(\partial P)_1}(e^{2\pi i\eta}, re^{2\pi i\theta}) := ((2 + r\cos 2\pi\theta)\cos 2\pi\eta, (2 + r\cos 2\pi\theta)\sin 2\pi\eta, r\sin 2\pi\theta)\,, \\
0 \leq \eta, \theta \leq 1,\ 0 \leq r \leq 1.
\end{aligned}$$

It transforms $(\partial P)_1 = \mathbb{T}$ on the standard solid torus $\mathbb{T}^{st}$ and at the same time T on T^{st}.

We define now F on $(\partial P)_2$. It has to transform $(\partial P)_2$ on the complement of solid torus $\mathbb{T}^{st}$ in $\mathbb{R}^3 \cup \{\infty\}$. It is easier to define the inverse homeomorphism

$$\left(F|_{(\partial P)_2}\right)^{-1} : \mathbb{R}^3 \cup \{\infty\} \setminus \mathrm{Int}(\mathbb{T}^{st}) \to (\partial P)_2.$$

$(\partial P)_2$ is homeomorphic to the cartesian product of a disc and a circle $\{x \in \mathbb{C} : |x| \leq 1\} \times \{y \in \mathbb{C} : |y| = 1\}$. As the Figure 5 suggests the complement of the solid torus is also homeomorphic to the cartesian product of a disc (a gray disc in the

Figure 5) and a circle.

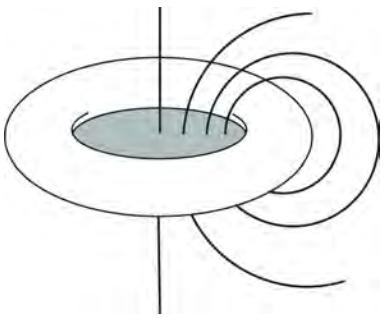

Fig. 5. The homeomorphism of $\mathbb{R}^3 \cup \{\infty\} \setminus \mathrm{Int}(\mathbb{T}^s t)$ on $(\partial P)_2$

The latter homeomorphism can be chosen so that $F|_{(\partial P)_2}$is identical to $F|_{(\partial P)_1}$ on the common part of their domains, i.e. on the torus T. This completes the construction of homeomorphism F.

For each knot W in ∂P we have the corresponding (by applying F) knot in $\mathbb{S}^3$. In particular to the knot

$$\mathbb{S}^1 \ni e^{2\pi i t} \mapsto (e^{2\pi i t}, 0) \in \partial P$$

corresponds the trivial knot in $\mathbb{S}^3$. If $W \subset T$ or $W \subset \mathbb{T}$, then we get the knot in $\mathbb{R}^3$ lying in T^{st} or in $\mathbb{T}^{st}$.

Now consider the simplest torus knots. Let $n, m \in \mathbb{N}$ be relatively prime i.e. $\mathrm{GCD}(n, m) = 1$. Then $\Phi : \mathbb{S}^1 \to \partial P$ defined by the formula

$$\Phi(e^{2\pi i t}) := (e^{2\pi i n t}, e^{2\pi i m t}), \qquad t \in [0, 1]$$

is one to one (except the ends) by the property of the exponential function, continuous, and thus it is a homeomorphism of the circle on its image, and thus defines a knot in ∂P. We denote it $T_{n,m}$ and the pair (n, m) call the *type* of this knot. Of course $T_{n,m} \subset T$. For each circle $O_\eta := \{(e^{2\pi i \eta}, e^{2\pi i \theta}) : \theta \in [0, 1]\}$, $\eta \in [0, 1]$, the common part $O_\eta \cap T_{n,m}$ consists of n points placed "symmetrically" and similarly for each circle $O_\theta := \{(e^{2\pi i \eta}, e^{2\pi i \theta}) : \eta \in [0, 1]\}$ $\theta \in [0, 1]$, the common part $O_\theta \cap T_{n,m}$ consists of m points placed also "symmetrically". Applying homeomorphism F to $T_{n,m}$ we get a knot in $\mathbb{S}^3$ lying in T^{st}. We denote it with the same symbol $T_{n,m}$ and we call it a *torus knot of the first order of the type* (n, m). Thus it is given in $\mathbb{S}^3$ by the formula

$$F \circ \Phi(e^{2\pi i t}) = ((2 + \cos 2\pi m t) \cos 2\pi n t, (2 + \cos 2\pi m t) \sin 2\pi n t, \sin 2\pi m t), \ t \in [0, 1].$$

The knot $T_{2,3}$ is presented in Figure 6.

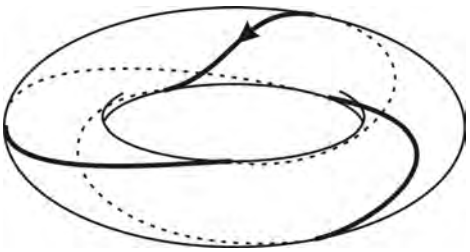

Fig. 6. The torus knot $T_2, 3$

Remark 3.1. *It can be shown that on the torus T there are no torus knots of $T_{n,m}$ for* $\mathrm{GCD}(n,m) > 1$ *(see* [R]*, p.19).*

Remark 3.2. *It is easy to prove that the torus knots $T_{n,m}$ for $n = 1$ or $m = 1$ are trivial.*

To describe the knot group of $T_{n,m}$ we recall some properties of T, in particular properties of the *universal covering* of the torus. This is the mapping

$$p : \mathbb{R}^2 \to T,$$
$$p(\eta, \theta) := (e^{2\pi i\eta}, e^{2\pi i\theta}), \quad (\eta, \theta) \in \mathbb{R}^2.$$

Notice that $p(\eta, \theta) = p(\eta', \theta')$ if and only if $(\eta, \theta) - (\eta', \theta') \in \mathbb{Z}^2$. The mapping p has the following known properties:

1. For each $z \in T$ there exists its neighbourhood U such that $p^{-1}(U)$ is a union $\bigcup V_i$ of open and disjoint sets V_i such that $p|_{V_i} : V_i \to U$ is a homeomorphism (this is the definition of a covering),

2. For each continuous curve (in short a curve) $\gamma : [0,1] \to T$ and arbitrary point $(\eta, \theta) \in p^{-1}(\gamma(0))$ there exists a unique continuous curve $\widehat{\gamma} : [0,1] \to \mathbb{R}^2$ such that $\widehat{\gamma}(0) = (\eta, \theta)$ and $p \circ \widehat{\gamma} = \gamma$ ($\widehat{\gamma}$ is called the *lifting of the curve γ at* (η, θ)),

3. The first homotopy group $\pi_1(T)$ of the torus T is isomorphic to $\mathbb{Z}^2$. Its generators are the curves $\alpha(t) := (e^{2\pi it}, 1)$ and $\beta(t) := (1, e^{2\pi it})$, $t \in [0,1]$, called the *longitude* and the *meridian* of the torus.

4. If γ is a closed curve in T and $\gamma = n\alpha + m\beta$, $n, m \in \mathbb{Z}$ in $\pi_1(T)$, then the numbers n, m are characterized via p as follows. Let $\widehat{\gamma}$ be the lifting of γ at $(\eta_0, \theta_0) \in p^{-1}(\gamma(0))$. Since $\gamma(0) = \gamma(1)$ then $\widehat{\gamma}(1) - \widehat{\gamma}(0) \in \mathbb{Z}^2$. Then $\widehat{\gamma}(1) - \widehat{\gamma}(0) = (n, m)$. We say then that the curve γ *circles the torus T n-times along and m-times across.*

5. The first homotopy group $\pi_1(\mathbb{T})$ of the solid torus $\mathbb{T}$ is isomorphic to $\mathbb{Z}$ and its generator is the loop α. For every loop κ in $\mathbb{T}$ with the same origin as α we have $\kappa = \alpha^n$, where n is the index of the curve being projection of κ on $\mathbb{C}$ by pr_1 with respect to the point $0 \in \mathbb{C}$ ($n = \mathrm{Ind}_0\, pr_1 \circ \kappa$).

6. The first homotopy group $\pi_1(\partial P \setminus \mathbb{T})$ of the complement of the solid torus $\mathbb{T}$ is isomorphic also to $\mathbb{Z}$ and its generator is the loop β. Similarly as above for every loop κ in $\partial P \setminus \mathbb{T}$ with the same origin as β we have $\kappa = \beta^m$, where m is the index of the curve being projection of κ on $\mathbb{C}$ by pr_2 with respect to the point $0 \in \mathbb{C}$ ($m = \operatorname{Ind}_0 pr_2 \circ \kappa$).

Now we give some properties of the knot $T_{n,m}$.

Lemma 3.3. *The set $p^{-1}(T_{n,m})$ consists of the family of parallel lines*

$$L_k := \{(\eta, \theta) \in \mathbb{R}^2_{(\eta,\theta)} : \theta = \frac{m}{n}\eta + \frac{k}{n}\}, \quad k \in \mathbb{Z}.$$

Proof. If $(\eta, \theta) \in p^{-1}(T_{n,m})$, then there exists $t \in [0,1]$ such that $(e^{2\pi i\eta}, e^{2\pi i\theta}) = (e^{2\pi int}, e^{2\pi imt})$. From properties of the exponential function we get $\eta = nt + r$, $\theta = mt + s$, $r, s \in \mathbb{Z}$. Hence

$$\theta = \frac{m}{n}\eta + \frac{ns - mr}{n},$$

i.e. $(\eta, \theta) \in L_{ns-mr}$.

Vice versa, if $\theta = \frac{m}{n}\eta + \frac{k}{n}$ for a $k \in \mathbb{Z}$, then putting $t := \frac{\eta}{n} + \frac{ka}{n}$, where $a, b \in \mathbb{Z}$ and $am - bn = 1$ (such a, b always exist because $\mathrm{GCD}(n, m) = 1$), we get

$$e^{2\pi int} = e^{2\pi i(\eta + ka)} = e^{2\pi i\eta},$$

$$e^{2\pi imt} = e^{2\pi i(mt - kb)} = e^{2\pi i(\frac{m}{n}\eta + \frac{k(am-bn)}{n})} = e^{2\pi i(\frac{m}{n}\eta + \frac{k}{n})} = e^{2\pi i\theta}.$$

Then $(\eta, \theta) \in p^{-1}(T_{n,m})$. □

Lemma 3.4. *For every $P, Q \in T \setminus T_{n,m}$ there exists a curve connecting P, Q in $T \setminus T_{n,m}$.*

Proof. Let $p : \mathbb{R}^2_{(\eta,\theta)} \to T$ be the universal covering of the torus. Then by Lemma 3.3 $p^{-1}(T_{n,m})$ is a family of parallel lines L_k, $k \in \mathbb{Z}$, in the plane $\mathbb{R}^2_{(\eta,\theta)}$. Each of the strips lying between adjacent parallel lines is obviously a convex set. Thus, any two of its points can be connected by a segment. Then the image of this segment (via p) will obviously be a curve in T connecting the images of the ends of this segment. Therefore, it is enough to show that the image of each strip (open) is equal to $T \setminus T_{n,m}$. For simplicity we may consider the strip $\boldsymbol{P} := \{(\eta, \theta) : \eta \in \mathbb{R},\ \frac{m}{n}\eta < \theta < \frac{m}{n}\eta + \frac{1}{n}\}$. Let us take arbitrary point $Q = (e^{2\pi i\eta}, e^{2\pi i\theta}) \in T \setminus T_{n,m}$. We need to show that there is a point $(\eta', \theta') \in \boldsymbol{P}$ such that $(\eta' - \eta, \theta' - \theta) \in \mathbb{Z}^2$. Since $\mathrm{GCD}(n, m) = 1$, there exist $a, b \in \mathbb{N}$ such that

$$am - bn = 1. \tag{1}$$

Put $s := [n\theta - m\eta]$. Then the point $(\eta', \theta') := (\eta + as, \theta + bs)$ satisfies the conditions:

1. $p(\eta', \theta') = p(\eta, \theta) = Q$,
2. $(\eta', \theta') \in \boldsymbol{P}$.

The first condition is obvious and the second follows from the inequalities

$$\begin{aligned} 0 < n\theta - m\eta - [n\theta - m\eta] < 1,\\ 0 < n\theta - m\eta - s < 1,\\ 0 < n\theta - m\eta - s(am - bn) < 1,\\ m\eta + mas < n\theta + nbs < m\eta + mas + 1,\\ \frac{m}{n}(\eta + as) < \theta + bs < \frac{m}{n}(\eta + as) + \frac{1}{n}. \end{aligned}$$

The first inequality is obvious because $n\theta - m\eta \notin \mathbb{Z}$ (which follows from the assumption that $(\eta, \theta) \notin T_{n,m}$). □

Consider, in particular, two points Q, R on the torus not belonging to $T_{n,m}$ differing only in the argument $\frac{2\pi}{m}$ of the first coordinate. Therefore

$$\begin{aligned} Q &:= (e^{2\pi i\eta}, e^{2\pi i\theta}) \notin T_{n,m},\\ R &:= (e^{2\pi i(\eta - \frac{1}{m})}, e^{2\pi i\theta}) \notin T_{n,m}. \end{aligned}$$

They lie on the circle $\{(e^{2\pi it}, e^{2\pi i\theta}),\ t \in [0,1]\}$, and are separated by "one thread" of the knot $T_{n,m}$ (on this circle lie m points of $T_{n,m}$ arranged symmetrically, see Figure 7).

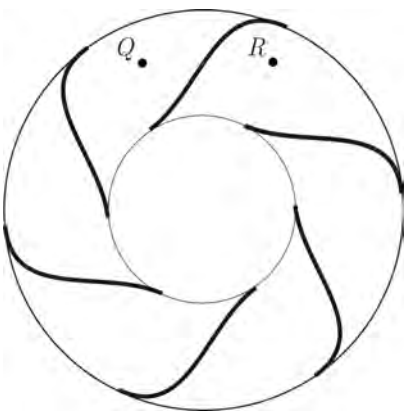

Fig. 7. The points Q and R.

Lemma 3.5. *For the above specified points $Q, R \notin T_{n,m}$ a curve connecting Q and R in $T \setminus T_{n,m}$ is the image of the segment $\overline{[(\eta, \theta), (\eta - \frac{1}{m} + a, \theta + b)]}$ via p, where $am - bn = 1$.*

Proof. Obviously $p(\eta, \theta) = Q$ and $p(\eta - \frac{1}{m} + a, \theta + b) = R$. Moreover the coefficient of the line containing the segment is equal to $\frac{m}{n}$ because

$$\frac{\theta + b - \theta}{\eta - \frac{1}{m} + a - \eta} = \frac{bm}{am - 1} = \frac{bm}{bn} = \frac{m}{n}.$$

Hence the segment is parallel to lines L_k. Therefore it lies in one of the strips. □

Lemma 3.6. *The set of points (η', θ') equivalent to (η, θ) in $\mathbb{R}^2$ (i.e. $p(\eta', \theta') = p(\eta, \theta)$) and lying in the same strip as (η, θ) is equal to $\{(\eta + kn, \theta + km) : k \in \mathbb{Z}\}$.*

Proof. Obviously the points $(\eta + kn, \theta + km)$, $k \in \mathbb{Z}$, are equivalent to (η, θ). Moreover they lie in the same strip as (η, θ) because the vectors $[km, kn]$ are parallel to L_k.

Take arbitrary point (η', θ') equivalent to (η, θ) w $\mathbb{R}^2$ and lying in the same strip as (η, θ). For simplicity we may assume

$$\frac{m}{n}\eta < \theta < \frac{m}{n}\eta + \frac{1}{n}. \tag{2}$$

Hence $(\eta', \theta') = (\eta + r, \theta + s)$ for some $r, s \in \mathbb{Z}$ and

$$\frac{m}{n}(\eta + r) < \theta + s < \frac{m}{n}(\eta + r) + \frac{1}{n}. \tag{3}$$

By (2) i (3) we get two inequalities

$$0 < n\theta - m\eta < 1,$$
$$0 < (n\theta - m\eta) + (ns - mr) < 1.$$

Since $ns - mr \in \mathbb{Z}$, therefore $ns - mr = 0$. Hence and by the assumption $\mathrm{GCD}(n, m) = 1$ we obtain $r = kn$ and $s = km$ for some $k \in \mathbb{Z}$. □

The last lemma implies a description of the first homotopy group of the complement of the knot $T_{n,m}$ in the torus T.

Proposition 3.7. *For every point $* \in T \setminus T_{n,m}$*

$$\pi_1(T \setminus T_{n,m}; *) \cong \mathbb{Z}. \tag{4}$$

The closed curve κ which is the image by p of the segment $\overline{(\eta, \theta), (\eta + n, \theta + m)}$, where $p(\eta, \theta) = *$, is a generator of $\pi_1(T \setminus T_{n,m}; *)$.

Proof. By definition $\kappa(t) := p((\eta + tn, \theta + tm))$, $t \in [0, 1]$, and for every $k \in \mathbb{Z}$, $\kappa^k(t) = p((\eta + tkn, \theta + tkm))$, $t \in [0, 1]$. Take any closed curve ι in $T \setminus T_{n,m}$ at $*$. Its lifting $\widehat{\iota}$ with initial point (η, θ) has the end at a point $(\eta + kn, \theta + km)$ for some $k \in \mathbb{Z}$ (by the previous lemma) and lies in a strip containing (η, θ). Since this strip is a simply connected set the curve $\widehat{\iota}$ is homotopic to the segment joining its ends i.e. to the segment $\overline{(\eta, \theta), (\eta + kn, \theta + km)}$. Hence after composition with p the curve ι is homotopic to κ^k. Then κ is a generator of $\pi_1(T \setminus T_{n,m}; *)$.

To show (4) it suffices to prove that no curve κ^k for $k \in \mathbb{Z} \setminus \{0\}$ is homotopic to the constant curve at $*$. In fact, otherwise for the lifting $\widehat{\kappa^k}$ of κ^k with initial point (η, θ) we would have $\widehat{\kappa^k}(1) = (\eta, \theta)$. On the other hand $\widehat{\kappa^k}(1) = (\eta + kn, \theta + km)$, which implies $k = 0$. □

We will now describe the knot group of $T_{n,m}$. By definition $\pi(T_{n,m}) = \pi_1(\partial P \setminus T_{n,m}; *) = \pi_1(\mathbb{S}^3 \setminus T_{n,m}; *)$, where $* \notin T_{n,m}$. For simplicity we take a point $* \in$

$T \setminus T_{n,m}$. Consider two loops γ, δ as in Figure 8 and 9

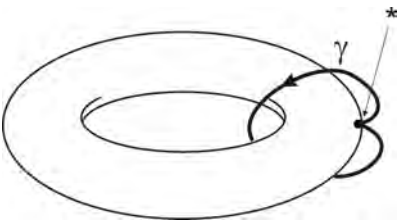

Fig.8. The loop γ.

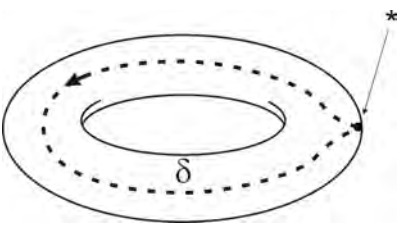

Fig. 9. The loop δ.

We first show

Lemma 3.8. *The loops γ and δ are generators of $\pi(T_{n,m})$.*

Proof. Take arbitrary loop κ in $\mathbb{S}^3 \setminus T_{n,m}$ with the initial and final point at $*$ (in short a loop κ based at $*$). Changing κ homotopically we may assume κ is a broken line. Hence κ has a finite number of common points $A_1, \ldots, A_l$ with T. So, we may represent κ as a finite sum of curves

$$\kappa = \kappa_1 \ldots \kappa_k,$$

where each curve κ_i lies either in $\mathbb{R}^3 \setminus \mathbb{T}$ (i.e. outside the solid torus with exception of ends - see Fig. 10)

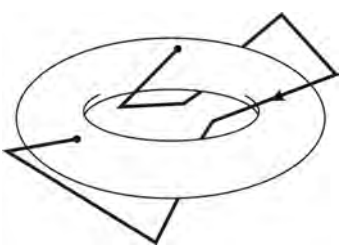

Fig. 10. The loop κ_i outside $\mathbb{R}^3 \setminus \mathbb{T}$.

or κ_i lies in $\operatorname{Int} \mathbb{T}$ (i.e. in the interior of the solid torus with exception of ends). By Lemma 3.4 each point A_i can be joined with the point $*$ by a curve in $T \setminus T_{n,m}$. Then changing homotopically κ (by moving each point A_i with the entire curve to the point $*$ along such a curve) we may assume that $A_1 = \ldots = A_l = *$. Hence κ is

homotopic to a sum of curves $\widetilde{\kappa}_1 \dots \widetilde{\kappa}_k$, where each curve $\widetilde{\kappa}_i$ is a loop in $\mathbb{R}^3 \setminus T_{n,m}$ based at $*$ which lies entirely either in $\mathbb{R}^3 \setminus \mathbb{T}$ or in $\operatorname{Int}\mathbb{T}$. So, $\widetilde{\kappa}_i$ is homotopic to a multiple of γ or a multiple of δ. Hence γ and δ are generators of the group $\pi_1(\overline{\mathbb{R}^3} \setminus T_{n,m}; *)$. □

Now we describe the relation between γ and δ in $\pi(T_{n,m})$. Let $* = p(\eta_0, \theta_0)$. Consider the loop κ_0 which is the image of the segment $\overline{(\eta_0, \theta_0), (\eta_0 + n, \theta_0 + m)}$ by p. It is a loop based at $*$ which lies in $T \setminus T_{n,m}$ and is "parallel" to $T_{n,m}$ in T and which circles the torus T n-times along and m-times across.

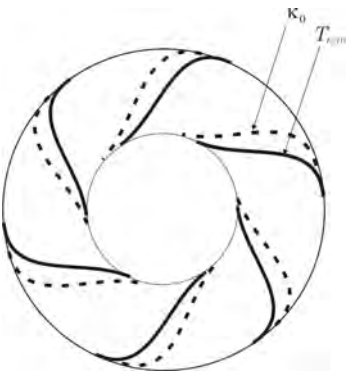

Fig. 12. The loop κ_0.

Let us change κ_0 (leaving the initial and final point fixed) in two ways:

1. "pulling out" the loop κ_0 from T. We get a loop lying in $\mathbb{R}^3 \setminus \mathbb{T}$ circling $\mathbb{T}$ m-times around. Hence κ_0 is homotopic to γ^m.

2. "pushing" the loop κ_0 into the interior of T. We get a loop lying in $\mathbb{T}$ circling $\mathbb{T}$ n-times along. Hence κ_0 is homotopic to δ^n.

In consequence

Lemma 3.9. *For the loops γ and δ in $\pi(T_{n,m})$ we have*

$$\gamma^m = \delta^n.$$

It is a unique non-trivial relation between γ and δ in $\pi(T_{n,m})$. We will use the Seifert-van Kampen theorem (see [CF]) to justify this fact precisely.

Theorem 3.10. *For relatively prime $n, m \in \mathbb{N}$*

$$\pi(T_{n,m}) = \mathcal{F}(\gamma, \delta) / \left(\delta^n \gamma^{-m}\right).$$

Proof. By the canonical homeomorphism $F : \partial P \to \mathbb{S}^3$ we will lead coniderations in $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. Recall $T_{n,m} \subset T^{st} \subset \mathbb{S}^3$. Let's consider two open sets U_1 and U_2 in $\mathbb{S}^3$. The first U_1 is such that its intersection with each half-plane containing the axis Oz is an open disc with a center at the point $(2, 0)$ and radius $3/2$ (notice that intersection T^{st} with such a half-plan is a circle with a center at the point $(2, 0)$

and radius 1 in which n points of $T_{n,m}$ lie) with removed segments of length $1/2$ along the radii of this disc with one end at a point $T_{n,m}$ (see Fig. 13)

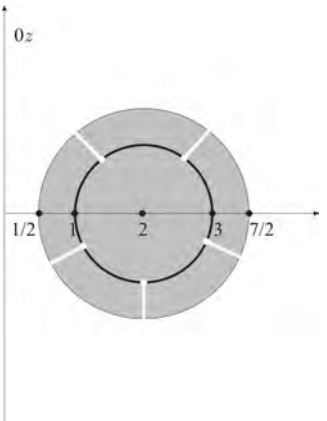

Fig. 13. The set U_1.

The second set U_2 is such that its intersection with each half-plane containing the Oz is the complement (together with the point ∞) of the closed disc with the center at the point $(2,0)$ and radius $1/2$ with removed segments of length $1/2$ along the radii of this disc with one end at a point $T_{n,m}$ (see Fig. 14).

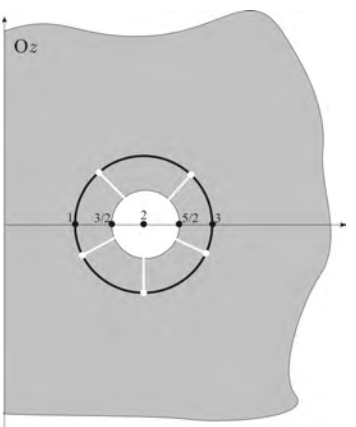

Fig. 14. The set U_2.

The sets U_1, U_2 are connected, arc connected, $U_1 \cup U_2 = \mathbb{S}^3 \setminus T_{n,m}$ and $\pi_1(U_1, *) = \mathcal{F}(\delta)$, $\pi_1(U_2, *) = \mathcal{F}(\gamma)$. The set $U_1 \cap U_2$ is arc connected and its first homotopy group is $\pi_1(T \setminus T_{n,m}; *)$ because each loop in $U_1 \cap U_2$ with beginning and end at $*$ is obviously homotopic to a loop lying in $T \setminus T_{n,m}$. But $\pi_1(T \setminus T_{n,m}; *) = \mathcal{F}(\kappa)$ (see Proposition 3.7), where the loop κ is the image of the segment $\overline{(\eta, \theta), (\eta + n, \theta + m)}$

by p and $p(\eta,\theta)=*$. The homomorphisms

$$\varphi_i : \pi_1(U_1\cap U_2,*)\to\pi_1(U_i,*),\quad i=1,2,$$

are defined on the generator κ by

$$\varphi_1(\kappa)=\delta^n,$$
$$\varphi_2(\kappa)=\gamma^m.$$

In fact, $pr_1\circ\kappa(t)=pr_1\circ p(\eta+tn,\theta+tm)=e^{2\pi i(\eta+tn)}=e^{2\pi i\eta}e^{2\pi itn}$ for $t\in[0,1]$, whence $\operatorname{Ind}_0 pr_1\circ\kappa=n$. Then κ, treated as a loop in U_1, is homotopic to δ^n. Similarly we show that κ treated as a loop in U_2 is homotopic to γ^m. Hence by the Seifert-van Kampen theorem

$$\pi(T_{n,m})=\pi_1(\partial P\setminus T_{n,m};*)=\mathcal{F}(\gamma,\delta)/\left(\delta^n\gamma^{-m}\right).$$

□

Consider the particular loop t based at $*$ (see Fig. 15) where the point Q differs from

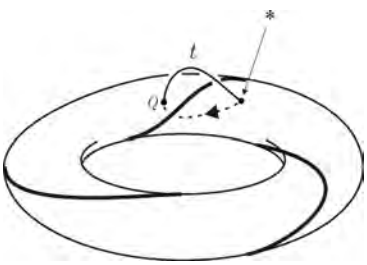

Fig. 15. The loop t.

point $*$ by the argument $\frac{2\pi}{m}$ of the first coordinate. Precisely, if $Q=(e^{2\pi i\eta},e^{2\pi i\theta})$, then $*=(e^{2\pi i(\eta-1/m)},e^{2\pi i\theta})$. Firstly we represent t by generators.

Lemma 3.11. *In* $\pi(T_{n,m})$

$$t=\delta^a\gamma^{-b},$$

where $a,b\in\mathbb{N}$ *and* $am-bn=1$.

Proof. Let $p(\eta,\theta)=Q$. By Lemma 3.5 the point Q can be joined to the point $*$ in $T\setminus T_{n,m}$ with a curve which is the image of the segment $\overline{[(\eta,\theta),(\eta-\frac{1}{m}+a,\theta+b)]}$ via p. By Property 4 of the universal cover p this curve circles the torus T $(a-\frac{1}{m})$-times along and b-times across. Since Q and $*$ differ by the argument $\frac{2\pi}{m}$ of the first coordinate we obtain as in the proof of Lemma 3.8

$$t=\delta^a\gamma^{-b}.$$

□

By Proposition 2.5 it follows that the loop t, and precisely its abstract class $[t]$, is a generator of abelianization $\pi(T_{n,m})':=\pi(T_{n,m})/[\pi(T_{n,m}),\pi(T_{n,m})]$. In particular the classes $[\gamma]$ and $[\delta]$ are generated by $[t]$. In fact

Lemma 3.12. *In* $\pi(T_{n,m})'$

$$[\gamma] = [t]^n,$$
$$[\delta] = [t]^m.$$

Proof. Since $\pi(T_{n,m})'$ is isomorphic to $H_1(\mathbb{S}^3 \setminus T_{n,m}, \mathbb{Z})$, we may lead considerations in the language of homology. Let $p(\eta, \theta) = *$. The loop (= cycle) t is homologic to any cycle t_i lying in the plane $\{e^{2\pi i\eta}\} \times \mathbb{C}$ which circles the one thread of $T_{n,m}$ and the loop (=cycle) γ is homologous to γ_0 which circles all the points of $T_{n,m}$ (see Fig. 16).

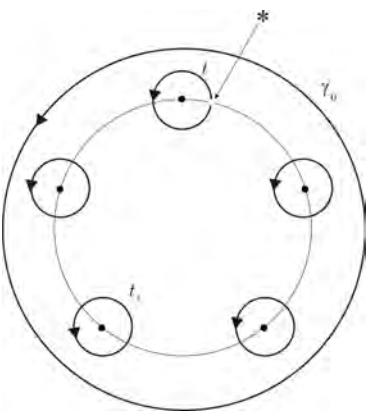

Fig. 16. The cycles t, t_i, γ_0

Since in the plane $\{e^{2\pi i\eta}\} \times \mathbb{C}$ the set $\left(\{e^{2\pi i\eta}\} \times \mathbb{C}\right) \cap T_{n,m}$ has n points the cycle γ_0 is homologous to the sum of cycles t_i, circling these points. Then $[\gamma_0] = n[t]$ in $H_1(\mathbb{S}^3 \setminus T_{n,m}, \mathbb{Z})$. Hence in $\pi(T_{n,m})'$ we have $[\gamma] = [t]^n$. We do the similar reasoning for the loop δ. We obtain in $\pi(T_{n,m})'$, $[\delta] = [t]^m$. □

We can now proceed to calculate the Alexander polynomial of the torus knots $T_{n,m}$.

Theorem 3.13. *For every relatively prime postive integers* n, m *we have*

$$A_{T_{n,m}}(t) = \frac{(t^{nm} - 1)(t - 1)}{(t^n - 1)(t^m - 1)}.$$

Proof. Because $\pi(T_{n,m}) = \mathcal{F}(x, y)/\left(x^n y^{-m}\right)$ then using formal derivatives we obtain

$$\frac{\partial\left(x^n y^{-m}\right)}{\partial x} = 1 + x + \ldots + x^{n-1} = \frac{x^n - 1}{x - 1},$$
$$\frac{\partial\left(x^n y^{-m}\right)}{\partial y} = -x^n y^{-1} - x^n y^{-2} - \ldots - x^n y^{-m} = -x^n y^{-m} \frac{y^m - 1}{y - 1},$$

whence

$$M_{\pi(T_{n,m})} = \left[\frac{x^n - 1}{x - 1}, -x^n y^{-m} \frac{y^m - 1}{y - 1}\right].$$

By Lemma 3.11 $t = x^a y^{-b}$ is a generator of $\pi(T_{n,m})'$ and from Lemma 3.12 $x = t^m$ and $y = t^n$. Then

$$M'_{\pi(T_{n,m})} = \left[\frac{t^{mn}-1}{t^m-1}, -\frac{t^{nm}-1}{t^n-1}\right].$$

Hence

$$A_{T_{n,m}}(t) = \mathrm{GCD}\left(\frac{t^{mn}-1}{t^m-1}, -\frac{t^{mn}-1}{t^n-1}\right) = \frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)}.$$

The last equality follows from the assumption $\mathrm{GCD}\,(n,m) = 1$ and simple facts about roots of the unity.

This ends the proof. □

In particular for the trivial knot $T_{1,1}$ we obtain $A_{T_{1,1}}(t) \equiv 1$. Hence we get

Corollary 3.14. *If T is a trivial knot then $A_T(t) \equiv 1$.*

Hence we get a topological classification of torus knots of the first order.

Theorem 3.15. *The torus knot of the first order $T_{n,m}$ is trivial if and only if $n = 1$ or $m = 1$. Two torus knots of the first order $T_{n,m}$ and $T_{k,l}$, $n, m, k, l \geq 2$, are equivalent if and only if $(n,m) = (k,l)$ or $(n,m) = (l,k)$.*

Proof. The first part of the theorem follows from Remark 3.2 and the fact that for $n, m \geq 2$ the Aleksander polynomial of $T_{n,m}$ is not constant ($\deg A_{T_{n,m}} > 0$).

Assume now that torus knots $T_{n,m}$ i $T_{k,l}$, $n, m, k, l \geq 2$, are equivalent. Then their groups are isomorphic. Hence $A_{T_{n,m}} = A_{T_{k,l}}$, that is

$$\frac{(t^{mn}-1)(t-1)}{(t^m-1)(t^n-1)} = \frac{(t^{kl}-1)(t-1)}{(t^k-1)(t^l-1)}. \tag{5}$$

This equality implies

$$mn = kl \tag{6}$$

because otherwise, for instance $mn > kl$, some primitive root of unity of degree mn would be a root of left hand side of (5) and not of right hand side, which is impossible. If $mn = kl$ then again from (5) it follows in similar way that

$$m = k \text{ or } m = l. \tag{7}$$

From equalities (6) and (7) we obtain $(n,m) = (k,l)$ or $(n,m) = (l,k)$.

If $(n,m) = (k,l)$ or $(n,m) = (l,k)$, then the identity mapping in the first case and the permutation of coordinates $(x,y) \mapsto (y,x)$ in the second one are homeomorphisms which give the equivalence of knots.

This ends the proof. □

Remark 3.16. *Of course, we can also consider torus knots $T_{n,m}$ for any integers $n, m \in \mathbb{Z} \setminus \{0\}$ which satisfy $\mathrm{GCD}\,(n,m) = 1$. In these cases all the above reasoning are analogous with obvious changes. Limiting our considerations to positive n, m results from the fact that we obtain such knots from curves singularities.*

4. Torus knots of higher orders

Recall that by definition $T := \{(x,y) \in \mathbb{C}^2 : |x| = 1,\ |y| = 1\} \subset \partial P$. Let $n_1, m_1 \in \mathbb{N}$ and $\mathrm{GCD}(n_1, m_1) = 1$. Consider the torus knot $T_{n_1,m_1} \subset \partial P$. Then

$$T_{n_1,m_1} = \{(e^{2\pi i n_1 t}, e^{2\pi i m_1 t}),\ \ t \in [0,1]\} \subset T \subset \partial P.$$

Take r_1, $0 < r_1 < 1$. Instead of T_{n_1,m_1} we consider the equivalent to it the torus knot in ∂P

$$\{(e^{2\pi i n_1 t}, r_1 e^{2\pi i m_1 t}),\ \ t \in [0,1]\} \subset \partial P.$$

We will also denote it by T_{n_1,m_1}. We define closed *tubular neighbourhood of* T_{n_1,m_1} contained in ∂P by

$$\mathrm{Tube}(T_{n_1,m_1}) = \bigcup_{(x,y)\in T_{n_1,m_1}} (\{x\} \times \overline{K(y,r_2)}),$$

where r_2 is so small that the closed discs with centers in the points $y_1, \ldots, y_{m_1}$ and radius r_2, where

$$\pi_1^{-1}(x) \cap T_{n_1,m_1} = x \times \{y_1, \ldots, y_{m_1}\},$$

are contained in the disc $K(0,1)$ and are pairwise disconnected (see Fig.17).

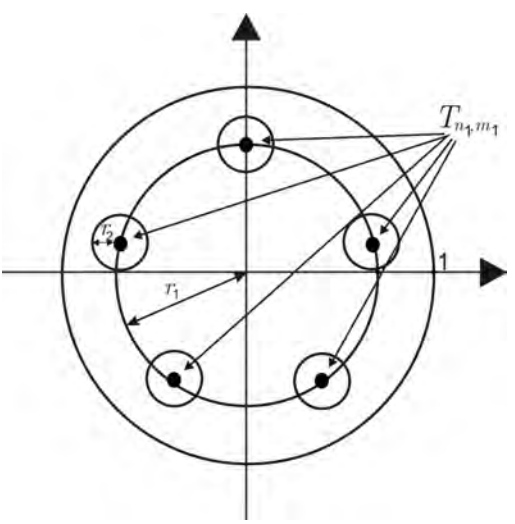

Fig. 17. A tubular neighbourhood of T_{n_1,m_1}.

Then $\mathrm{Tube}(T_{n_1,m_1})$ is given parametrically by

$$\mathrm{Tube}(T_{n_1,m_1}) = \{(e^{2\pi i n_1 t}, r_1 e^{2\pi i m_1 t} + r e^{2\pi i s}),\ \ t,s \in [0,1],\ r \in [0,r_2]\}.$$

Obviously $\mathrm{Tube}(T_{n_1,m_1})$ is contained in ∂P and its boundary $\partial(\mathrm{Tube}(T_{n_1,m_1}))$ is homeomorphic to the torus. We choose the following homeomorphism

$$\Phi_1 : T \to \partial(\mathrm{Tube}(T_{n_1,m_1})),$$
$$\Phi_1(e^{2\pi it}, e^{2\pi is}) = (e^{2\pi in_1 t}, r_1 e^{2\pi im_1 t} + r_2 e^{2\pi is}),\ t, s \in [0,1].$$

Let T_{n_2,m_2} be an arbitrary torus knot of the first order lying in $T \subset \partial P$. Then $n_2, m_2 \in \mathbb{N}$ and $\mathrm{GCD}(n_2, m_2) = 1$. So $\Phi_1(T_{n_2,m_2})$ is a knot in ∂P, and thus (through the homeomorphism F) a knot in $\mathbb{S}^3$. These types of knots are called the *torus knots of the second order* and denote by $T_{(n_1,m_1)(n_2,m_2)}$ (both in ∂P and in $\mathbb{S}^3$). The type of this knot in ∂P does not depend on the choice of the radii r_1, r_2 as long as they satisfy the above assumptions (because there is a homeomorphism transforming the unit disc into oneself, being an identity on the boundary, carrying points $T_{(n_1,m_1)(n_2,m_2)}$ to the points of the same knot with different radii r_1', r_2'). Because the knot T_{n_2,m_2} in T is given by the formula

$$T_{n_2,m_2} = \{(e^{2\pi in_2 t}, e^{2\pi im_2 t}),\quad t \in [0,1]\} \subset T,$$

then $T_{(n_1,m_1)(n_2,m_2)}$ is described by the formula (see Fig. 18)

$$T_{(n_1,m_1)(n_2,m_2)} = \{(e^{2\pi in_1 n_2 t}, r_1 e^{2\pi im_1 n_2 t} + r_2 e^{2\pi im_2 t}),\quad t \in [0,1]\}.$$

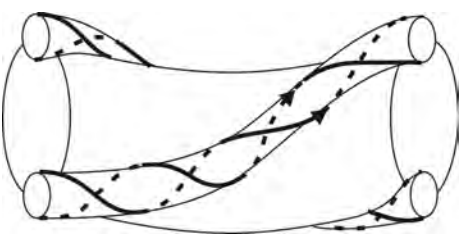

Fig. 18. The torus knot of the second order.

Higher-order torus knots are defined inductively. For a given torus knot of the k-th order $T_{(n_1,m_1)\cdots(n_k,m_k)} \subset \partial P$, $k \geq 1$, given by the formula

$$[0,1] \ni t \mapsto (e^{2\pi in_1 \ldots n_k t}, r_1 e^{2\pi im_1 n_2 \ldots n_k t} + r_2 e^{2\pi im_2 n_3 \ldots n_k t} + \ldots + r_k e^{2\pi im_k t}) \in \partial P$$

we consider its closed tubular neighbourhood $\mathrm{Tube}(T_{(n_1,m_1)\cdots(n_k,m_k)})$ with sufficiently small radius r_{k+1} (such that this neighbourhood is contained in ∂P_1 and that the closed discs of this neighbourhood in each plane $\{x\} \times \mathbb{C}$ are pairwise disjoined). The boundary $\partial(\mathrm{Tube}(T_{(n_1,m_1)\cdots(n_k,m_k)}))$ is homeomorphic to the torus T. We fix the following homeomorphism

$$\Phi_k : T \to \partial(\mathrm{Tube}(T_{(n_1,m_1)\cdots(n_k,m_k)})),$$
$$\Phi_k(e^{2\pi it}, e^{2\pi is}) = (e^{2\pi in_1 \ldots n_k t}, r_1 e^{2\pi im_1 n_2 \ldots n_k t} + r_2 e^{2\pi im_2 n_3 \ldots n_k t} +$$
$$\ldots + r_k e^{2\pi im_k t} + r_{k+1} e^{2\pi is}),\qquad t, s \in [0,1])\}.$$

Let $T_{n_{k+1},m_{k+1}}$ be any torus knot of the first order lying in $T \subset \partial P$. Then $\Phi_k(T_{n_{k+1},m_{k+1}})$ is a knot in ∂P and thus a knot in $\mathbb{S}^3$. These types of knots are called the *torus knots of the* $(k+1)$*-th order*. It is given by the formula

$$t \mapsto (e^{2\pi i n_1 \dots n_{k+1} t}, r_1 e^{2\pi i m_1 n_2 \dots n_{k+1} t} + r_2 e^{2\pi i m_2 n_3 \dots n_{k+1} t} + \dots + r_k e^{2\pi i m_k n_{k+1} t} + r_{k+1} e^{2\pi i m_{k+1} t}) \quad \text{for } t \in [0,1]. \tag{8}$$

According to Remark 3.2 the torus knot of the first order $T_{n,m}$ is trivial if and only if $n = 1$ or $m = 1$. Due to the lack of symmetry between variables x and y in the definition of higher-order torus knots, this type of theorem only takes place for the first index.

Proposition 4.1. *Let $T_{(n_1,m_1)\cdots(n_k,m_k)}$ be a torus knot of the k-th order and $n_i = 1$ for some $i \in \{1,\dots,k\}$. Then*

$$T_{(n_1,m_1)\cdots(n_k,m_k)} \sim T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(n_{i+1},m_{i+1})\cdots(n_k,m_k)}.$$

Proof. First we show that $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ is equivalent to $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$ in ∂P. For any $x = e^{2\pi i t}$ in the plane $\{x\} \times \mathbb{C}$ we have $n_1 \cdot \ldots \cdot n_{i-1}$ points of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$ lying in the unit disc. Around each of these points a circle with a sufficiently small radius r_i is given (such that closed discs with these radii are disjoined and are contained in the interior of unit disc). On each of these circles is given one point of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ and these points depend continuously on x. Assuming that r_i are small enough, we can include these discs in discs with a greater radii $\widetilde{r}_i > r_i$ and the same centers such that their closures are still disjoined and contained in the open unit disc (see Fig. 19).

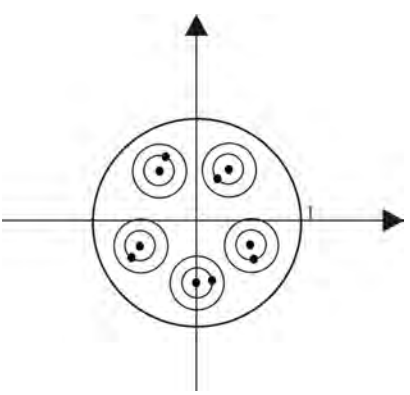

Fig. 19.

It is easy to show that there exists a homeomorphism h_x of the unit disc on itself carrying out the points of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ on corresponding them points of the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$ and being the identity on boundaries of the unit disc and discs of the radii $\widetilde{r}_i$. Moreover, we may choose h_x such that they

depend continously on x. Then the mapping

$$F : \partial P \to \partial P,$$
$$F|_{\partial P_1}(x, y) := (x, h_x(y)),$$
$$F|_{\partial P_2}(x, y) := (x, y)$$

is a homeomorphism of ∂P transforming $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ on the knot $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$. Further constructions of torus knots of successive orders applied to the equivalent ones $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})(1,m_i)}$ and $T_{(n_1,m_1)\cdots(n_{i-1},m_{i-1})}$ lead to equivalent knots. □

Remark 4.2. *There is no similar theorem when $m_i = 1$ for some $i \in \{1, \ldots, k\}$. See Remark 4.8.*

We will now describe the knot group of $T_{(n_1,m_1)\cdots(n_k,m_k)}$. We described the knot groups of torus knots of the first order in Theorem 3.10. For every $n_1, m_1 \in \mathbb{N}$, $\mathrm{GCD}(n_1, m_1) = 1$, we have

$$\pi(T_{n_1,m_1}) = \mathcal{F}(x, y)/\left(x^{n_1} y^{-m_1}\right).$$

We compute now the knot groups of torus knot of the second order $T_{(n_1,m_1)(n_2,m_2)}$. By definition

$$\pi(T_{(n_1,m_1)(n_2,m_2)}) = \pi_1(\partial P \setminus T_{(n_1,m_1)(n_2,m_2)}; *) = \pi_1(\mathbb{R}^3 \setminus \Phi_1(T_{(n_1,m_1)(n_2,m_2)}); *),$$

where the point $* \notin T_{(n_1,m_1)(n_2,m_2)}$. Take the point $*$ lying on the boundary of $\mathrm{Tube}(T_{n_1,m_1})$. As generators we fix the following three loops based at $*$:

1. the loop γ as in the case of torus knot of the first order $T_{n,m}$; call it here γ_0 (see Fig. 20),

2. the loop δ as in the case of torus knot of the first order $T_{n,m}$; call it here δ_1 (see Fig. 20),

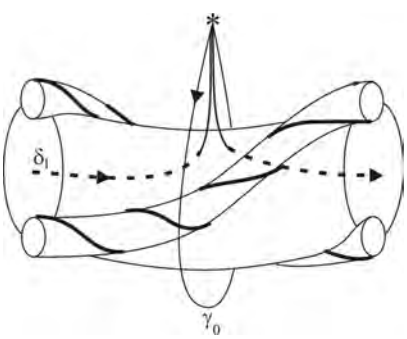

Fig. 20. The loops γ_0, δ_1.

(for a better geometrical representation of these loops, we have drawn a point $*$ outside the torus),

3. the loop being the "axis" of the tubular neighbourhood Tube(T_{n_1,m_1}); call it here δ_2 (see Fig. 21). It is equivalent to T_{n_1,m_1} in ∂P.

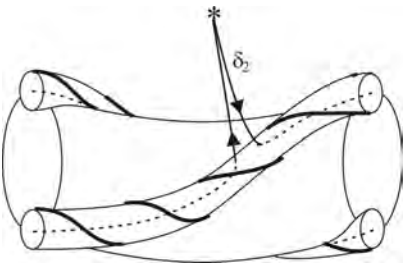

Fig. 21. The loop δ_2.

We show now

Lemma 4.3. *The loops γ_0, δ_1 and δ_2 are generators of $\pi(T_{(n_1,m_1)(n_2,m_2)})$.*

Proof. The proof is similar to the case of the first-order torus knots $T_{n,m}$. By the homeomorphism $F : \partial P \to \mathbb{S}^3$ we move considerations to $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. Take any loop κ in $\mathbb{R}^3 \setminus T_{(n_1,m_1)(n_2,m_2)}$ based at $*$ (recall we chose the point $*$ in $\partial(\text{Tube}(T_{n_1,m_1})) \setminus T_{(n_1,m_1)(n_2,m_2)}$. Changing κ by a homotopy we may assume κ is a broken line. Hence κ has a finite number of common points with $\partial(\text{Tube}(T_{n_1,m_1}))$. So, we may represent κ as a finite sum of curves

$$\kappa = \kappa_1 \dots \kappa_k,$$

where each curve κ_i lies either in $\mathbb{R}^3 \setminus \text{Tube}(T_{n_1,m_1})$ (except the ends of the curve) or in the interior of Tube(T_{n_1,m_1}) (except the ends of the curve). By Lemma 3.4 each common points of κ with $\partial(\text{Tube}(T_{n_1,m_1})$ can be joined by a curve with the chosen point $*$ in $\partial(\text{Tube}(T_{n_1,m_1})) \setminus T_{(n_1,m_1)(n_2,m_2)}$. Then changing homotopically κ (by moving each common point with the entire curve to the point $*$ along such a curve) we obtain that κ is homotopic to a sum of curves $\widetilde{\kappa}_1 \dots \widetilde{\kappa}_k$, where each curve $\widetilde{\kappa}_i$ is a loop in $\mathbb{R}^3 \setminus T_{(n_1,m_1)(n_2,m_2)}$ based at $*$ and lies entirely either in $\mathbb{R}^3 \setminus \text{Tube}(T_{n_1,m_1})$ or in Int(Tube(T_{n_1,m_1})) (except the ends of the curve). Those running in $\mathbb{R}^3 \setminus \text{Tube}(T_{n_1,m_1})$ are obviously generated by γ_0 i δ_1, and those running in Tube(T_{n_1,m_1}) are a multiple of δ_2. Then γ_0, δ_1 and δ_2 are generators of the group $\pi(T_{(n_1,m_1)(n_2,m_2)})$. □

We describe relations between γ_0, δ_1 and δ_2 in $\pi(T_{(n_1,m_1)(n_2,m_2)})$. Of course, the relationship between γ_0 and δ_1 is the same as in case of torus knot $T_{(n_1,m_1)}$

$$\gamma_0^{m_1} = \delta_1^{n_1}. \tag{R1}$$

To determine the relationship between δ_2 and the pair γ_0, δ_1 we consider an auxiliary loop γ_1 (Fig. 21); it corresponds to the loop t in the case of $T_{n,m}$ from

Lemma 3.11.

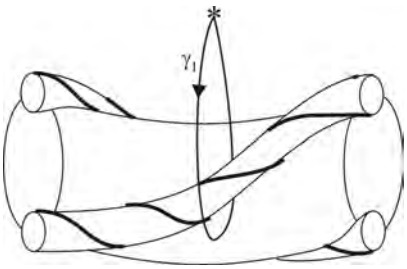

Fig. 21. The loop γ_1.

By this lemma

$$\gamma_1 = \delta_1^a \gamma_0^{-b}, \tag{9}$$

where $a, b \in \mathbb{N}$ and $am_1 - bn_1 = 1$.

Consider the loop κ_0 (Fig. 22) based at $*$, lying in $\partial(\mathrm{Tube}(T_{n_1,m_1}))$ "parallel" to $T_{(n_1,m_1)(n_2,m_2)}$, so circulating $\partial(\mathrm{Tube}(T_{n_1,m_1}))$ n_2-times along and m_2-times across. It means that the homeomorphism Φ_1^{-1} transforms the curve κ_0 into a curve which circles T n_2-times along and m_2-times across. Hence the projection κ_0 on the unit circle (via the projection $pr_1 : \mathbb{C}^2 \to \mathbb{C}$ on the first axis) goes around this circle n_1n_2-times in a positive direction.

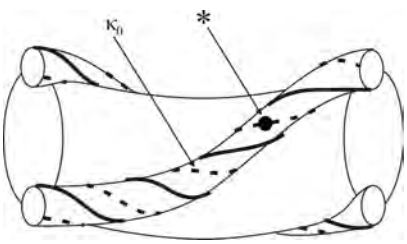

Fig. 22. The loop κ_0.

If we change homotopically κ_0 so that it will lie inside $\mathrm{Tube}(T_{n_1,m_1})$ (with exception of $*$), then obviously

$$\kappa_0 \ \sim \ \delta_2^{n_2} \tag{10}$$

(because δ_2 is an "axis" of $\mathrm{Tube}(T_{n_1,m_1})$, and κ_0 circles n_2-times along this tubular neighbourhood).

If we change homotopically κ_0 so that it will lie outside $\mathrm{Tube}(T_{n_1,m_1})$ (with exception of $*$), then

$$\kappa_0 \ \sim \ \gamma_1^{m_2 - m_1 n_2} \delta_1^{n_1 n_2}. \tag{11}$$

To justify this, let's first determine the integer s such that the curve $\gamma_1^s \kappa_0$ is homotopic to a multiple of δ_1 and precisely homotopic to $\delta_1^{n_1 n_2}$ (because the projection of κ_0 on the first axis circles the unit circle n_1n_2-times). For one turn κ_0

around Tube(T_{n_1,m_1}) (there are exactly m_2 of them) therefore corresponds $\frac{n_1n_2}{m_2}$ rotation of the projection on the first axis. So the point P in Figure 23 will do $\alpha := \frac{n_1n_2}{m_2} \cdot \frac{m_1}{n_1} = \frac{m_1n_2}{m_2}$ rotation.

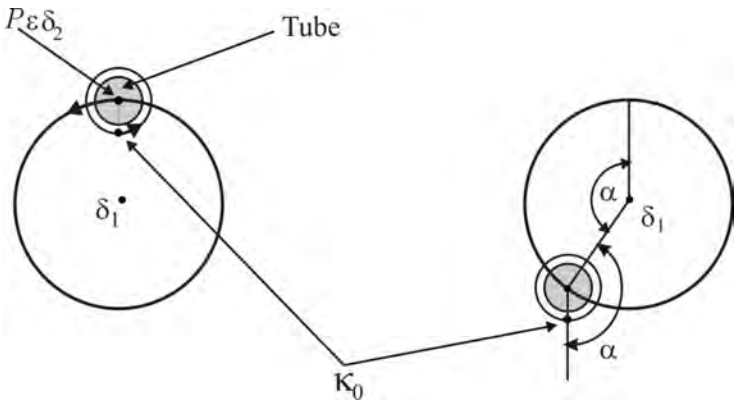

Fig. 23.

Therefore, to obtain a curve being a multiple of δ_1, (it is the axis of T_{n_1,m_1}, so it is represented in the Figure 23 by the center of the circle) $-1+\alpha$ turn should be made.

Because κ_0 turns around Tube(T_{n_1,m_1}), m_2-times then $s = m_2(-1+\alpha) = -m_2 + m_1n_2$. Hence $\gamma_1^{-m_2+m_1n_2}\kappa_0 \sim \delta_1^{n_1n_2}$. This gives (11).

From (10) and (11) we get the relation

$$\delta_2^{n_2} \sim \gamma_1^{m_2-m_1n_2}\delta_1^{n_1n_2}. \tag{12}$$

In turn, by (9) we get the relation between $\gamma_0, \delta_1, \delta_2$

$$\delta_2^{n_2} \sim \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}. \tag{R2}$$

(R1) and (R2) are the only relations between $\gamma, \delta_1, \delta_2$. To prove this, it is sufficient to use the Seifert-van Kampen theorem (see the proof of Theorem 3.10). Then we get

Theorem 4.4. *For any torus knot of the second order $T_{(n_1,m_1)(n_2,m_2)}$ we have*

$$\pi(T_{(n_1,m_1)(n_2,m_2)}) \cong \mathcal{F}(\gamma_0,\delta_1,\delta_2)/\left(\delta_1^{n_1}\gamma_0^{-m_1}, \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}\delta_2^{-n_2}\right).$$

We can now calculate Alexander's polynomial of torus knots of the second order. For any relatively prime positive integers m, n we define the polynomial

$$W_{n,m}(t) := \frac{(t^{nm}-1)(t-1)}{(t^n-1)(t^m-1)}.$$

These are indeed polynomials by properties of the roots of unity and the assumption that GCD$(m,n) = 1$. Then the Alexander polynomial of the torus knot of the first order $T_{n,m}$ is equal to $W_{n,m}(t)$.

Theorem 4.5. *The Alexander polynomial of the torus knot of the second order* $T^2 := T_{(n_1,m_1)(n_2,m_2)}$ *is given by the formula*

$$A_{T^2}(t) = W_{n_1,m_1}(t^{n_2})W_{n_2,m_2-m_1n_2+m_1n_1n_2}(t).$$

Proof. From Theorem 4.4 we have

$$\pi(T^2) \cong \mathcal{F}(\gamma_0,\delta_1,\delta_2)/\left(\delta_1^{n_1}\gamma_0^{-m_1}, \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}\delta_2^{-n_2}\right).$$

Denoting the first relation by R_1 and the second by R_2 we get through formal differentiation

$$\frac{\partial R_1}{\partial\gamma_0} = -\delta_1^{n_1}\gamma_0^{-m_1}\frac{\gamma_0^{m_1}-1}{\gamma_0-1}, \quad \frac{\partial R_1}{\partial\delta_1} = \frac{\delta_1^{n_1}-1}{\delta_1-1}, \quad \frac{\partial R_1}{\partial\delta_2} = 0.$$

and

$$\begin{aligned}
\frac{\partial R_2}{\partial\gamma_0} &= -\delta_1^a\gamma_0^{-b}\frac{\left(\left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}-1\right)}{\delta_1^a\gamma_0^{-b}-1}\frac{\gamma_0^b-1}{\gamma_0-1},\\
\frac{\partial R_2}{\partial\delta_1} &= \frac{\left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}-1}{\delta_1^a\gamma_0^{-b}-1}\frac{\delta_1^a-1}{\delta_1-1} + \left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\frac{\delta_1^{n_1n_2}-1}{\delta_1-1},\\
\frac{\partial R_2}{\partial\delta_2} &= -\left(\delta_1^a\gamma_0^{-b}\right)^{m_2-m_1n_2}\delta_1^{n_1n_2}\delta_2^{-n_2}\frac{\delta_2^{n_2}-1}{\delta_2-1}.
\end{aligned}$$

Taking into account the equality $\gamma_1 = \delta_1^a\gamma_0^{-b}$, where $am_1 - bn_1 = 1$, and that in $\pi(T_{(n_1,m_1)(n_2,m_2)})'$ we have $\gamma_0 = \gamma_1^{n_1}$, $\delta_1 = \gamma_1^{m_1}$, we obtain equalities in $\pi(T_{(n_1,m_1)(n_2,m_2)})'$

$$\frac{\partial R_1}{\partial\gamma_0} = -\frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{n_1}-1}, \quad \frac{\partial R_1}{\partial\delta_1} = \frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{m_1}-1}, \quad \frac{\partial R_1}{\partial\delta_2} = 0.$$

and

$$\begin{aligned}
\frac{\partial R_2}{\partial\gamma_0} &= -\gamma_1\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{\gamma_1^{bn_1}-1}{\gamma_1^{n_1}-1} = -\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{\gamma_1^{am_1}-\gamma_1}{\gamma_1^{n_1}-1},\\
\frac{\partial R_2}{\partial\delta_1} &= \frac{\gamma_1^{m_2-m_1n_2}-1}{\gamma_1-1}\frac{\gamma_1^{am_1}-1}{\gamma_1^{m_1}-1} + \gamma_1^{m_2-m_1n_2}\frac{\gamma_1^{m_1n_1n_2}-1}{\gamma_1^{m_1}-1},\\
\frac{\partial R_2}{\partial\delta_2} &= -\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\delta_2-1}.
\end{aligned}$$

Then minors of the second degree of the Aleksander matrix of the group $\pi(T^2)'$ are

$$M_1 = \begin{vmatrix} \frac{\partial R_1}{\partial \gamma_0} & \frac{\partial R_1}{\partial \delta_1} \\ \frac{\partial R_2}{\partial \gamma_0} & \frac{\partial R_2}{\partial \delta_1} \end{vmatrix} = \frac{\partial R_1}{\partial \gamma_0}\frac{\partial R_2}{\partial \delta_1} - \frac{\partial R_1}{\partial \delta_1}\frac{\partial R_2}{\partial \gamma_0}$$

$$= -\frac{\gamma_1^{n_1 m_1}-1}{\gamma_1^{n_1}-1}\left(\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{(\gamma_1^{am_1}-1)}{\gamma_1^{m_1}-1}+\gamma_1^{m_2-m_1n_2}\frac{\gamma_1^{m_1n_1n_2}-1}{\gamma_1^{m_1}-1}\right)$$

$$-\frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{m_1}-1}\left(-\frac{\left(\gamma_1^{m_2-m_1n_2}-1\right)}{\gamma_1-1}\frac{(\gamma_1^{am_1}-\gamma_1)}{\gamma_1^{n_1}-1}\right)$$

$$= -\frac{\gamma_1^{m_2-m_1n_2}\,(\gamma_1^{n_1m_1}-1)\,(\gamma_1^{m_1n_1n_2}-1)}{(\gamma_1^{n_1}-1)\,(\gamma_1^{m_1}-1)} - \frac{(\gamma_1^{m_2-m_1n_2}-1)\,(\gamma_1^{n_1m_1}-1)}{(\gamma_1^{n_1}-1)\,(\gamma_1^{m_1}-1)}$$

$$= -\frac{(\gamma_1^{n_1m_1}-1)\left(\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1\right)}{(\gamma_1^{n_1}-1)\,(\gamma_1^{m_1}-1)} = -W_{n_1,m_1}(\gamma_1)\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\gamma_1-1},$$

$$M_2 = \begin{vmatrix} \frac{\partial R_1}{\partial \gamma_0} & \frac{\partial R_1}{\partial \delta_2} \\ \frac{\partial R_2}{\partial \gamma_0} & \frac{\partial R_2}{\partial \delta_2} \end{vmatrix} = \frac{\partial R_1}{\partial \gamma_0}\frac{\partial R_2}{\partial \delta_2} = \frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{n_1}-1}\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\delta_2-1},$$

$$M_3 = \begin{vmatrix} \frac{\partial R_1}{\partial \delta_1} & \frac{\partial R_1}{\partial \delta_2} \\ \frac{\partial R_2}{\partial \delta_1} & \frac{\partial R_2}{\partial \delta_2} \end{vmatrix} = \frac{\partial R_1}{\partial \delta_1}\frac{\partial R_2}{\partial \delta_2} = \frac{\gamma_1^{n_1m_1}-1}{\gamma_1^{m_1}-1}\frac{\gamma_1^{m_2-m_1n_2+m_1n_1n_2}-1}{\delta_2-1}.$$

Because the loop γ_2 (see Figure 24) is a generator of the group $\pi(T^2)'$,

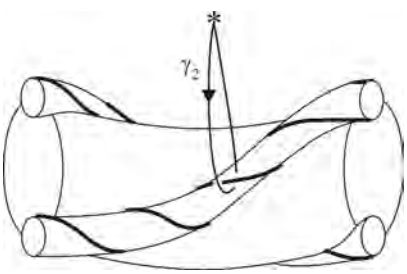

Fig. 24. The loop γ_2.

then in $\pi(T^2)'$ we have $\gamma_1 = \gamma_2^{n_2}$ and $\delta_1 = \gamma_2^{m_1n_2}$. Hence from equality (12) we get $\delta_2 = \gamma_2^{m_2-m_1n_2+m_1n_1n_2}$. Then

$$M_1' = -W_{n_1,m_1}(\gamma_2^{n_2})\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{n_2}-1},$$

$$M_2' = \frac{\gamma_2^{n_1m_1n_2}-1}{\gamma_2^{n_1n_2}-1}\,\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1},$$

$$M_3' = \frac{\gamma_2^{n_1m_1n_2}-1}{\gamma_2^{n_1n_2}-1}\,\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1}.$$

Since $\mathrm{GCD}(n_2, m_2 - m_1n_2 + m_1n_1n_2) = 1$, we easily show

$$\begin{aligned}
&\mathrm{GCD}(M_1', M_2', M_3') = \mathrm{GCD}(M_1', \mathrm{GCD}(M_2', M_3'))\\
&= \mathrm{GCD}\left(M_1', \frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1}\,\mathrm{GCD}\left(\frac{\gamma_1^{n_1m_1n_2}-1}{\gamma_1^{n_1n_2}-1}, \frac{\gamma_1^{n_1m_1n_2}-1}{\gamma_1^{m_1n_2}-1}\right)\right)\\
&= \mathrm{GCD}\left(M_1', \frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1} W_{n_1,m_1}(\gamma_2^{n_2})\right)\\
&= W_{n_1,m_1}(\gamma_2^{n_2})\,\mathrm{GCD}\left(\frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{n_2}-1}, \frac{\gamma_2^{(m_2-m_1n_2+m_1n_1n_2)n_2}-1}{\gamma_2^{m_2-m_1n_2+m_1n_1n_2}-1}\right)\\
&= W_{n_1,m_1}(\gamma_2^{n_2}) W_{n_2,m_2-m_1n_2+m_1n_1n_2}(\gamma_2).
\end{aligned}$$

Putting $t = \gamma_2$ we get the assertion of the theorem. □

We will now discribe the general case of the torus knots of the g-order $T^g := T_{(n_1,m_1)\cdots(n_g,m_g)} \subset \partial P$. We will calculate $\pi(T^g) = \pi_1(\partial P \setminus T^g, *)$, where $* \notin T^g$. We choose the point $*$ on $\partial(\mathrm{Tube}(T^{g-1})) \setminus T^g$. As in the case of the second order torus knots, it can be shown that the generators of $\pi(T^g)$ are γ_0 and the "axes" $\delta_1, \ldots, \delta_g$ of consecutive tubular neighbourhoods (see Fig. 25).

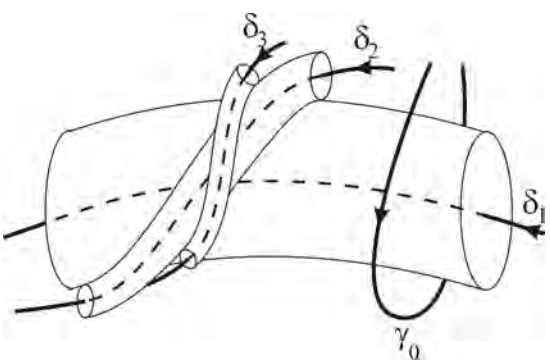

Fig. 25. Generators of $\pi(T^g)$.

Of course $\delta_1 \sim T^0 := T_{1,1}$, $\delta_2 \sim T^1, \ldots, \delta_g \sim T^{g-1}$ in ∂P. The same reasoning as in the case of the second-order toru knots (using the Seifert-van Kampen theorem) we will obtain that there are the following relations between these generators

$$\begin{aligned}
&R_1 : \delta_1^{n_1} = \gamma_0^{m_1},\\
&R_2 : \delta_2^{n_2} = \gamma_1^{m_2-m_1n_2}\delta_1^{n_1n_2},\\
&\ldots\ldots\ldots\ldots\ldots\\
&R_g : \delta_g^{n_g} = \gamma_{g-1}^{m_g-m_{g-1}n_g}\delta_{g-1}^{n_{g-1}n_g},
\end{aligned} \tag{13}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_{g-1}$ (see Fig. 26) are loops circling one thread of the torus knots $T^0, T^1, \ldots, T^{g-1}$, respectively.

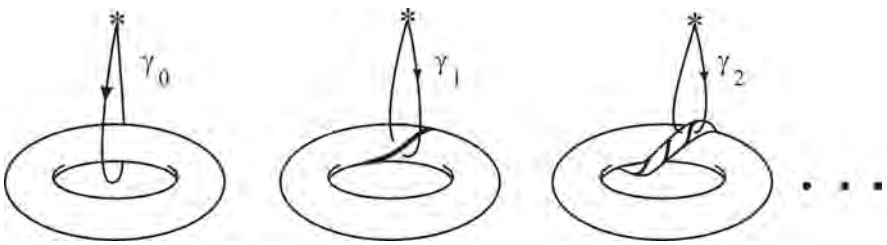

Fig. 26. The loops $\gamma_0, \ldots, \gamma_g - 1$

The loops satisfy the relations

$$
\begin{aligned}
&\gamma_1 = \delta_1^{a_1}\gamma_0^{-b_1}, \text{ where } a_1, b_1 \in \mathbb{N},\ a_1 m_1 - b_1 n_1 = 1,\\
&\gamma_2 = \delta_2^{a_2}\gamma_1^{-b_2+a_2 m_1}\delta_1^{-a_2 n_1}, \text{ where } a_2, b_2 \in \mathbb{N},\ \ a_2 m_2 - b_2 n_2 = 1,\\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\gamma_{g-1} = \delta_{g-1}^{a_{g-1}}\gamma_{g-2}^{-b_{g-1}+a_{g-1}m_{g-2}}\delta_{g-2}^{-a_{g-1}n_{g-2}}, \text{ where } a_{g-1}, b_{g-1} \in \mathbb{N},\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a_{g-1}m_{g-1} - b_{g-1}n_{g-1} = 1.
\end{aligned}
\tag{14}
$$

In fact, we will show this only for γ_2, because the reasoning in the general case is analogous. We have to express γ_2 by $\gamma_1, \delta_1, \delta_2$. Fix point $*$ on $\partial(\mathrm{Tube}(T^1)) \setminus T^2$. Let's denote by Q the point on $\partial(\mathrm{Tube}(T^1)) \setminus T^2$ that differs from the point $*$ by $1/m$ rotation of the projection on the first axis (in coordinates given by the canonical homeomorphism $\Phi_1 : T \to \partial(\mathrm{Tube}(T^1)))$ (see Fig. 27).

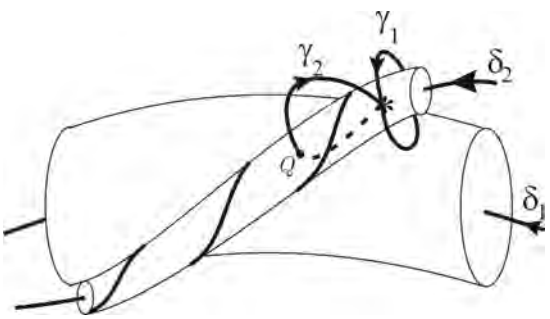

Fig. 27.

By Lemma 3.4 the point Q can be connected to the point $*$ by a curve lying in $\partial(\mathrm{Tube}(T^1))$ "parallel" to T^2. In coordinates given by the canonical homeomorphism $\Phi_1 : T \to \partial(\mathrm{Tube}(T^1)))$ this curve circles the torus $\partial(\mathrm{Tube}(T^1))$ $a_2 - 1/m_2$-times along and b_2-times across. Therefore, by moving the point Q along this curve, together with the entire curve γ_2, we get a curve homotopic to curve γ_2 in $\partial P \setminus T^2$ with the initial and final point in $*$, whose the first part lies inside $\mathrm{Tube}(T^1)$ (except the point $*$) and the second one out of $\mathrm{Tube}(T^1)$. Denoting these curves by κ_1 and

κ_2, we have $\gamma_2 = \kappa_1\kappa_2$. We will now express κ_1 and κ_2 by $\gamma_1, \delta_1, \delta_2$. Because the curve connecting Q with $*$ makes $a_2 - 1/m_2$ and the first part of γ_2 (from the point $*$ to Q) makes $1/m_2$ rotation along $\partial(\mathrm{Tube}(T^1))$, and δ_2 is the axis of $\mathrm{Tube}(T^1)$, then

$$\kappa_1 = \delta_2^{a_2}.$$

Determining κ_2 is much more difficult. The curve κ_2 makes $-b_2$ rotations across $\partial(\mathrm{Tube}(T^1))$. We will determine $s \in \mathbb{Z}$ such that $\gamma_1^s\kappa_2$ is a multiple of δ_1, more precisely equal to $\delta_1^{-a_2n_1}$ (because the projection of κ_2 on the first axis makes $-a_2n_1$ rotations; in fact, $-a_2$ rotations in coordinates of Φ_1 but each such rotation corresponds to n_1 rotations of the projection on the first axis in $\mathbb{C}^2$). Let's analyze one rotation around $\partial(\mathrm{Tube}(T^1))$ towards negative orientation (see Fig. 28).

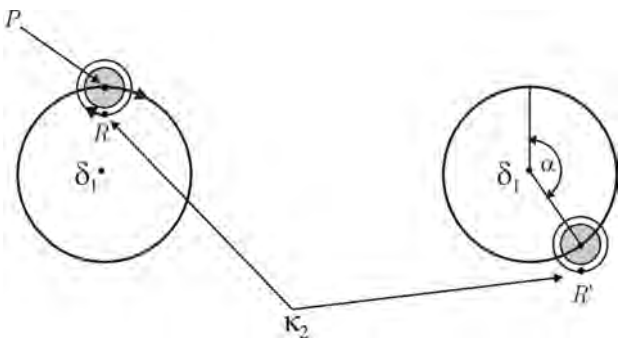

Fig. 28.

After one rotation in the negative direction around $\partial(\mathrm{Tube}(T^1))$ a point R goes to a point R' and the projection of the path made by the point R on the first axis will make of course $n_1\frac{a_2}{b_2}$ rotation (because the rotational speed of R is $\frac{a_2}{b_2}$) towards negative orientation. Then the point P (its rotational speed is $\frac{m_1}{n_1}$) will make rotation $\alpha = \frac{m_1}{n_1}n_1\frac{a_2}{b_2} = \frac{m_1a_2}{b_2}$. Then to get a curve which is a multiple of δ_1 it is not enough to make 1 rotation (for every single rotation of κ_2), but you should also add an α rotation in the negative direction. Because κ_2 makes b_2 rotations so $s = b_2(1-\alpha) = b_2 - m_1a_2$. Hence $\gamma_1^{b_2-m_1a_2}\kappa_2 = \delta_1^{-a_2n_1}$, which gives $\kappa_2 = \gamma_1^{-b_2+m_1a_2}\delta_1^{-a_2n_1}$. Consequently

$$\gamma_2 = \kappa_1\kappa_2 = \delta_2^{a_2}\gamma_1^{-b_2+m_1a_2}\delta_1^{-a_2n_1}.$$

Then we obtain

Theorem 4.6. *For any sequence of pairs of natural numbers* $((n_1,m_1),\dots,(n_g,m_g))$ *such that* $\mathrm{GCD}\,(n_i,m_i) = 1$, $i = 1,\dots,g$, *we have*

$$\pi(T_{(n_1,m_1)\cdots(n_g,m_g)}) = \mathcal{F}(\gamma_0,\delta_1,\dots,\delta_g)/(R_1,\dots,R_g),$$

where $R_1,\dots,R_g$ *are relations given in (13) and (14).*

We can now give the Aleksander polynomial of the knot $T^g := T_{(n_1,m_1)\cdots(n_g,m_g)}$.

Theorem 4.7.

$$A_{T^g}(t) = W_{n_1,\lambda_1}(t^{n_2\cdots n_g})W_{n_2,\lambda_2}(t^{n_3\cdots n_g})\dots W_{n_{g-1},\lambda_{g-1}}(t^{n_g})W_{n_g,\lambda_g}(t), \tag{15}$$

where the sequence $\lambda_1, \cdots, \lambda_g$ is defined recursively

$$\begin{aligned} \lambda_1 &= m_1, \\ \lambda_k &= m_k - m_{k-1}n_k + \lambda_{k-1}n_{k-1}n_k, \quad k \geq 2. \end{aligned} \tag{16}$$

Proof. We proved it for $g = 1$ and $g = 2$. Proof of the general case can be found in [Le]. □

Remark 4.8. *In particular for the knot $T := T_{(2,1)(2,1)}$ we have $A_T(t) = W_{2,3}(t)$, so it's not trivial knot.*

We will now show that, under additional assumptions, the Alexander polynomial of the torus knots uniquely characterizes it. We will prove it under assumptions

$$n_k > 1, \quad k = 1, \dots, g \tag{17}$$

$$m_k - m_{k-1}n_k > 0, \quad k = 2, \dots, g, \tag{18}$$

This condition is always satisfied for torus knots associated with curves singularities. First, we will prove a lemma.

Lemma 4.9. *If inequalities (17) and (18) holds, then*

$$\lambda_g > \lambda_i n_i \dots n_g, \quad i = 1, \dots, g-1, \tag{19}$$

$$\lambda_g n_g > \lambda_i n_i \dots n_g, \quad i = 1, \dots, g-1. \tag{20}$$

Proof. Because the second inequality follows from the first one, it is enough to prove the first one. From inequalities (17) and (18) we get

$$\begin{aligned} \lambda_g &= m_g - m_{g-1}n_g + \lambda_{g-1}n_{g-1}n_g > \lambda_{g-1}n_{g-1}n_g \\ &= (m_{g-1} - m_{g-2}n_{g-1} + \lambda_{g-2}n_{g-2}n_{g-1})\, n_{g-1}n_g \\ &> \lambda_{g-2}n_{g-2}n_{g-1}^2 n_g \geq \lambda_{g-2}n_{g-2}n_{g-1}n_g \\ &\geq \dots \geq \lambda_i n_i \dots n_g. \end{aligned}$$

□

Theorem 4.10. *Let $T := T_{(n_1,m_1)\cdots(n_g,m_g)}$ and $T' := T_{\left(n'_1,m'_1\right)\cdots\left(n'_h,m'_h\right)}$ be two torus knots such that*

$$\begin{aligned} n_i > 1, \quad i = 1, \dots, g, \qquad n'_i > 1, \quad i = 1, \dots, h, \\ m_i - m_{i-1}n_i > 0, \quad i = 2, \dots, g, \\ m'_i - m'_{i-1}n'_i > 0, \quad i = 2, \dots, h. \end{aligned} \tag{21}$$

Assume the Aleksander polynomials $A_T(t)$ and $A_{T'}(t)$ are equal. Then

$$\begin{aligned} g &= h, \\ n_i &= n'_i, \quad i = 1, \dots, g, \\ m_i &= m'_i, \quad i = 1, \dots, g. \end{aligned}$$

Proof. By assumption $A_T(t) = A_{T'}(t)$. The polynomial $A_T(t)$ is given by formulas (15) and (16). Analogously

$$\begin{aligned} A_{T'}(t) &= W_{n'_1,\lambda'_1}(t^{n'_2\cdots n'_h})W_{n'_2,\lambda'_2}(t^{n'_3\cdots n'_h})\dots W_{n'_{h-1},\lambda'_{h-1}}(t^{n'_h})W_{n'_h,\lambda'_h}(t),\\ \lambda'_1 &= m'_1,\\ \lambda'_k &= m'_k - m'_{k-1}n'_k + \lambda'_{k-1}n'_{k-1}n'_k, \quad k \geq 2. \end{aligned}$$

We will show first that

$$(n_g, \lambda_g) = (n'_h, \lambda'_h). \tag{22}$$

From the form of factors of $A_T(t)$ and $A_{T'}(t)$, namely

$$W_{n_i,\lambda_i}(t^{n_{i+1}\cdots n_g}) = \frac{\left(t^{\lambda_i n_i\cdots n_g}-1\right)\left(t^{n_{i+1}\cdots n_g}-1\right)}{\left(t^{\lambda_i n_{i+1}\cdots n_g}-1\right)\left(t^{n_i\cdots n_g}-1\right)},$$

$$W_{n_g,\lambda_g}(t) = \frac{\left(t^{\lambda_g n_g}-1\right)(t-1)}{\left(t^{\lambda_g}-1\right)\left(t^{n_g}-1\right)}$$

and inequality (20) and analogous for T' it follows that

$$\lambda_g n_g = \lambda'_h n'_h. \tag{23}$$

Indeed, otherwise e.g. if $\lambda_g n_g > \lambda'_h n'_h$, then from the assumption $n_g > 1,\ n'_h > 1$ a primitive root of unity of degree $\lambda_g n_g$ would be a root of the polynomial $A_T(t)$ and would not be a root of $A_{T'}(t)$, which is impossible. From the equality (23) follows in a similar manner to the above that

$$\lambda_g = \lambda'_h. \tag{24}$$

From (23) and (24) we get (22). Hence

$$W_{n_g,\lambda_g}(t) = W_{n'_h,\lambda'_h}(t).$$

Then dividing $A_T(t)$ and $A_{T'}(t)$ by this polynomial and substituting $u = t^{n_g}$ we get the equality of polynomials

$$\begin{aligned} &W_{n_1,\lambda_1}(u^{n_2\cdots n_{g-1}})W_{n_2,\lambda_2}(u^{n_3\cdots n_{g-1}})\dots W_{n_{g-1},\lambda_{g-1}}(u)\\ &\qquad = W_{n'_1,\lambda'_1}(u^{n'_2\cdots n'_{h-1}})W_{n'_2,\lambda'_2}(u^{n'_3\cdots n'_{h-1}})\dots W_{n'_{h-1},\lambda'_{h-1}}(u). \end{aligned}$$

The polynomials on both sides of the equality are the Alexander polynomials of torus knots of $(g-1)$-order $T_{(n_1,m_1)\dots(n_{g-1},m_{g-1})}$ and $T_{(n'_1,m'_1)\dots(n'_{h-1},m'_{h-1})}$. Repeating the above reasoning, we will receive successively

$$\begin{gathered} (n_{g-1}, \lambda_{g-1}) = (n'_{h-1}, \lambda'_{h-1})\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ (n_1, \lambda_1) = (n'_1, \lambda'_1) \end{gathered}$$

Hence $g = h$ and $n_i = n'_i$ for $i = 1, \dots, g$. Since $\lambda_1 = m_1$ and $\lambda'_1 = m'_1$, we have $m_1 = m'_1$. Further using the formulas for λ_i we easily get tha $m_i = m'_i$ for $i = 2, \dots, g$, too. This ends the proof. □

5. The knot of an irreducible curve

At this section we will recall the known basic properties of analytic curves in the complex plane $\mathbb{C}^2$. Details can be found in many textbooks on complex curves [KP], [W], [BK], [L].

For a given set $V \subset \mathbb{C}^n$ by $\boldsymbol{V}$ or $\widehat{V}$ we denote its germ at $0 \in \mathbb{C}^n$. A *local analytic curve* (for short a *curve*) is any germ $\boldsymbol{V}$ at $0 \in \mathbb{C}^2$ of the zero set of a holomorphic function $f \in \mathbb{C}\{x, y\}$ satisfying the conditions: $f \neq \text{const}$, $f(0,0) = 0$. Then $\operatorname{ord} f > 0$. The curve described by a holomorphic function $f \in \mathbb{C}\{x, y\}$ we denote by $V(f)$. When f is defined in a certain neighbourhood U of point 0, we denote the set of zeros of f in U by $V_U(f)$. Because $\mathbb{C}\{x, y\}$ is the unique factorization domain, we will always assume that the function f describing a curve $\boldsymbol{V}$ is reduced, i.e. there are no multiple factors in the factorization of f in $\mathbb{C}\{x, y\}$ into irreducible factors. Each curve $\boldsymbol{V} = V(f)$ has the unique decomposition into irreducible components

$$\boldsymbol{V} = \boldsymbol{V}_1 \cup \ldots \cup \boldsymbol{V}_k,$$

called *branches* of $\boldsymbol{V}$. The branches uniquely correspond to irreducible factors of f in $\mathbb{C}\{x, y\}$, i.e. if $f = f_1 \ldots f_l$ in $\mathbb{C}\{x, y\}$ and f_i are irreducible and not associated then $k = l$ and after renumbering $\boldsymbol{V}_i = V(f_i)$ for $i = 1, \ldots, k$. Because we are interested in the properties of analytic curves, invariant with respect to biholomorphisms of neighbourhoods of the zero in $\mathbb{C}^2$, we can always assume that the function f describing an analytic curve V satisfies the condition

$$\operatorname{ord} f = \operatorname{ord} f(0, y) \tag{25}$$

(we get this condition by linear change of variables in $\mathbb{C}^2$). Moreover, by the Weierstrass theorem we can additionally assume that f is a distinguished polynomial, i.e. $f \in \mathbb{C}\{x\}[y]$ and has the form

$$f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x), \quad n > 0, \quad \operatorname{ord} a_i \geq i, \quad i = 1, \ldots, n. \tag{26}$$

Then each branch $\boldsymbol{V}_i$ of the curve $V(f)$ has the *Puiseux parameterization*, i.e. there is a holomorphic, one-to-one mapping $\Phi_i(t) = (t^{n_i}, \varphi_i(t))$, $\operatorname{ord} \varphi_i \geq n_i$, defined in a neighbourhood of $0 \in \mathbb{C}$ such that $\boldsymbol{V}_i = \widehat{\operatorname{Im} \Phi_i}$. Moreover, if f is irreducible in $\mathbb{C}\{x\}[y]$, then we may assume that in (26) $\operatorname{ord} a_i > i$, $i = 1, \ldots, n$. Then for a Puiseux parameterization $\Phi(t) = (t^n, \varphi(t))$ of the unique branch of f the inequality $\operatorname{ord} \varphi > n$ holds.

The basic theorem on which the study of the topological structure of $V_U(f)$ in U is based is the theorem on the cone structure of isolated singularity. Before we give this theorem we will define the concept of a cone with a given base. For any $A \subset \mathbb{C}^n \setminus \{\mathbf{0}\}$ the *cone with base* A is the union of segments connecting point 0 with points A (see Fig. 29). We denote it $\operatorname{cone}(A)$. Therefore

$$\operatorname{cone}(A) := \{\boldsymbol{z} \in \mathbb{C}^n : \boldsymbol{z} = t\boldsymbol{a},\ t \in [0, 1],\ \boldsymbol{a} \in A\}$$

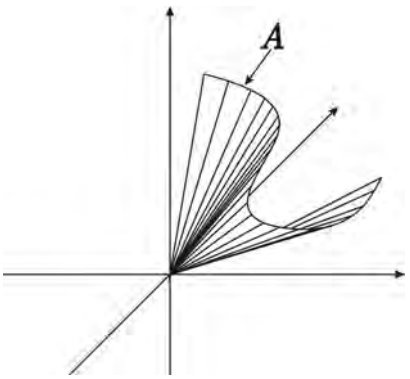

Fig. 29. The cone with a base A.

Note that for any closed polycylinder $P(\varepsilon,\eta) := \{(x,y) \in \mathbb{C}^2 : |x| \le \varepsilon,\ |y| \le \eta\} \subset \mathbb{C}^2$ with center at zero and radii $\varepsilon, \eta > 0$ and its boundary

$$\partial P(\varepsilon,\eta) = \{(x,y) \in \mathbb{C}^2 : |x| = \varepsilon,\ |y| \le \eta\} \cup \{(x,y) \in \mathbb{C}^2 : |x| \le \varepsilon,\ |y| = \eta\}$$

we have

$$\operatorname{cone}(\partial P(\varepsilon,\eta)) = P(\varepsilon,\eta).$$

Theorem 5.1 (on the cone structure of irreducible curve singularity)**.** *Let* $\boldsymbol{V}$ *be an irreducible curve and* $V := V_U(f)$ *its representative. Suppose* f *is a distinguished polynomial, i.e.* $f \in \mathbb{C}\{x\}[y]$ *has the form (26) and in the Puiseux parameterization* $\Phi(t) = (t^n, \varphi(t))$ *of the unique branch of* f *there is* $\operatorname{ord}\varphi > n$*. Then there exists* $\widetilde{\varepsilon} > 0$ *such that* $P(\widetilde{\varepsilon},\widetilde{\varepsilon}) \subset U$ *and for every* ε, $0 < \varepsilon < \widetilde{\varepsilon}$, *there exists a homeomorphism of the pairs (see Fig. 30)*

$$(P(\varepsilon,\varepsilon), V \cap P(\varepsilon,\varepsilon)) \overset{\text{top}}{\approx} (P(\varepsilon,\varepsilon), \operatorname{cone}(V \cap \partial P(\varepsilon,\varepsilon))$$

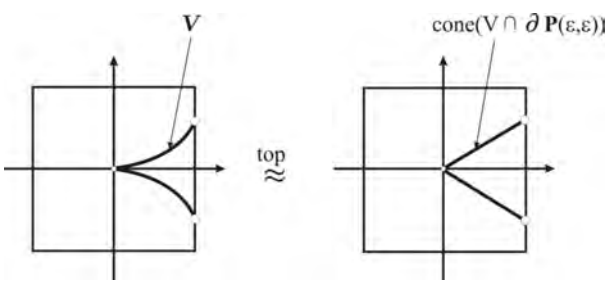

Fig. 30. The cone structure of a local curve.

and for any $\varepsilon, \varepsilon'$, $0 < \varepsilon < \varepsilon' < \widetilde{\varepsilon}$ *there exists a homeomorphism of the pairs*

$$(\partial P(\varepsilon,\varepsilon), V \cap \partial P(\varepsilon,\varepsilon)) \overset{\text{top}}{\approx} (\partial P(\varepsilon',\varepsilon'), V \cap \partial P(\varepsilon',\varepsilon'))\,.$$

A proof can be found in [M], [P], [W]. This theorem says that the immersion of V in P is determined, up to a homeomorphism of P, by the trace of V in the boundary of this polycylinder.

Remark 5.2. *The theorem is usually proven for closed balls. Then assumptions about the form of the function f are superfluous.*

Remark 5.3. *The theorem holds for any isolated singularity (of arbitrary dimension).*

Let $\boldsymbol{V} = V(f)$ be an irreducible curve, where $f \in \mathbb{C}\{x\}[y]$ is a distinguished poynomial of degree n, $n := \operatorname{ord} f = \operatorname{ord} f(0,y)$. Then $\boldsymbol{V}$ has a Puiseux parametrization $\Phi(t) = (t^n, \varphi(t))$, $t \in K$ – a neighbourhood of the origin in $\mathbb{C}$, $\operatorname{ord}\varphi > n$, in a neighbourhood U of zero in $\mathbb{C}^2$, i.e. f is defined in U and

$$V_U(f) = \{\Phi(t) : t \in K\} \ . \tag{27}$$

Denote $V := V_U(f)$. By the theorem on the cone structure there exists $\widetilde{\varepsilon} > 0$ such that $P(\widetilde{\varepsilon}, \widetilde{\varepsilon}) \subset U$ and for every ε, $0 < \varepsilon < \widetilde{\varepsilon}$ the traces of V on boundaries $\partial P(\varepsilon,\varepsilon)$ of policylinders $P(\varepsilon,\varepsilon)$ are topologically equivalent i.e. for every two $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon$ there exists a homeomorphism $H : \partial P(\varepsilon_1,\varepsilon_1) \to \partial P(\varepsilon_2,\varepsilon_2)$ such that $H(V \cap \partial P(\varepsilon_1,\varepsilon_1)) = V \cap \partial P(\varepsilon_2,\varepsilon_2)$. Moreover the form of Φ implies that by diminishing $\widetilde{\varepsilon}$ we may assume that $|\varphi(t)| < |t|$ dla $t \in K$.

Hence and again from the form of Puiseux parameterization $\Phi(t) = (t^n, \varphi(t))$ it follows that for ε, $0 < \varepsilon < \widetilde{\varepsilon}$, we have $V \cap \partial P(\varepsilon,\varepsilon) = \Phi(S(\varepsilon^{1/n}))$, where $S(r)$ is the circle in $\mathbb{C}$ with the center at 0 and radius $r > 0$. Thus, the trace of V in the boundary of each of these polycylinders is homeomorphic to S^1. On the other hand, the boundary $\partial P(\varepsilon,\eta)$ is homeomorphic to a three-dimensional sphere $S^3 = \mathbb{R}^3 \cup \{0\}$ and so $V \cap \partial P(\varepsilon,\varepsilon)$ is a knot. It follows from the above that this knot does not depend on the choice of the radius ε. We call it the knot of the curve $\boldsymbol{V}$ and denote it by $K_{\boldsymbol{V}}$. Then, by definition, the knot group $\pi(K_{\boldsymbol{V}})$ is equal to $\pi_1(\partial P(\varepsilon,\varepsilon) \setminus V, *)$, $* \in \partial P(\varepsilon,\varepsilon) \setminus V$. By the theorem on the cone structure we may calculate this group differently.

Lemma 5.4. $\pi(K_{\boldsymbol{V}}) \cong \pi_1(P(\varepsilon,\varepsilon) \setminus V)$.

Proof. Take the point $* \in \partial P(\varepsilon,\varepsilon) \setminus V$ as the base point for both groups $\pi(K_{\boldsymbol{V}})$ and $\pi_1(P(\varepsilon,\varepsilon) \setminus V)$. We have to show that

$$\pi_1(\partial P(\varepsilon,\varepsilon) \setminus V, *) \cong \pi_1(P(\varepsilon,\varepsilon) \setminus V, *).$$

This isomorphism follows from the theorem on cone structure of V in $P(\varepsilon,\varepsilon)$ because each loop in $P(\varepsilon,\varepsilon) \setminus V$ with beginning and end at $*$ is homotopic to a loop lying in $\partial P(\varepsilon,\varepsilon) \setminus V$. □

Let $(\beta_0, \beta_1, \ldots, \beta_h)$ be the characteristic of the curve $\boldsymbol{V}$ and $((m_1,n_1), \ldots, (m_h,n_h))$ be the sequence of characteristic pairs of $\boldsymbol{V}$. Recall that if $\varphi(t) = a_{p_1}t^{p_1} + a_{p_2}t^{p_2} + \ldots$, $a_{p_i} \neq 0$, $i \geq 1$, then

$$\beta_0 = n,$$
$$\beta_i = \min\{p_k : \operatorname{GCD}(\beta_0, \beta_1, \ldots, \beta_{i-1}, p_k) < \operatorname{GCD}(\beta_0, \beta_1, \ldots, \beta_{i-1})\},\ i = 1, \ldots, h$$

and

$$\frac{\beta_1}{\beta_0} = \frac{m_1}{n_1}, \ \ \mathrm{GCD}(m_1, n_1) = 1,$$
$$\frac{\beta_2}{\beta_0} = \frac{m_2}{n_1 n_2}, \ \ \mathrm{GCD}(m_2, n_2) = 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{\beta_h}{\beta_0} = \frac{m_h}{n_1 \cdots n_h}, \ \ \mathrm{GCD}(m_h, n_h) = 1.$$

Since

$$\frac{\beta_1}{\beta_0} < \frac{\beta_2}{\beta_0} < \ldots < \frac{\beta_h}{\beta_0},$$

then

$$m_i < m_{i-1} n_i \text{ dla } i = 2, \ldots, h. \tag{28}$$

Theorem 5.5. *Under the above assumptions on* $\boldsymbol{V}$ *the knot* $K_{\boldsymbol{V}}$ *is the torus knot of the h-order of the type* $(m_1, n_1), \ldots, (m_h, n_h)$, *i.e.*

$$K_{\boldsymbol{V}} \sim T_{(m_1,n_1),\ldots,(m_h,n_h)}.$$

Proof. Fix the above assumptions and notations. Denote by pr_1 the projection of $\mathbb{C}^2$ onto the first axis: $pr_1(x, y) = x$. We will define an auxiliary characteristic sequence $((n'_1, m'_1), \ldots, (n'_{\beta_h}, m'_{\beta_h}))$ of the curve V. Its construction is analogous to the construction of the sequence $((m_1, n_1), \ldots, (m_h, n_h))$ with the difference that we allow equality $n'_i = 1$. If $y(t) = a_{p_1} t^{p_1} + a_{p_2} t^{p_2} + \ldots$, $a_{p_i} \neq 0$, we put

$$\frac{p_1}{n} = \frac{m'_1}{n'_1}, \ \ \mathrm{GCD}(m'_1, n'_1) = 1,$$
$$\frac{p_2}{n} = \frac{m'_2}{n'_1 n'_2}, \ \ \mathrm{GCD}(m'_2, n'_2) = 1,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{p_\beta}{n} = \frac{m'_{p_\beta}}{n'_1 \cdots n'_{p_\beta}}, \ \ \mathrm{GCD}(m'_{p_\beta}, n'_{p_\beta}) = 1,$$

where $p_\beta = \beta_h$. Note that the sequence of characteristic pairs $((m_1, n_1), \ldots, (m_h, n_h))$ is a subsequence of $((m'_1, n'_1), \ldots, (m'_{p_\beta}, n'_{p_\beta}))$ and $(m_h, n_h) = (m'_{p_\beta}, n'_{p_\beta})$. More precisely, it suffices to omit from the sequence $((m'_1, n'_1), \ldots, (m'_{p_\beta}, n'_{p_\beta}))$ the pairs for which $n'_i = 1$. We will show that the knot of $\boldsymbol{V}$ is eqivalent to $T_{(n'_1,m'_1)\ldots(n'_{p_\beta},m'_{p_\beta})}$. Then by Proposition 4.1 $T_{(n'_1,m'_1)\ldots(n'_{p_\beta},m'_{p_\beta})} = T_{(m_1,n_1)\ldots(m_h,n_h)}$, which will give the assertion.

We will show the equality $K_{\boldsymbol{V}} = T_{(n'_1,m'_1)\ldots(n'_{p_\beta},m'_{p_\beta})}$ by approximation of the knot $K_{\boldsymbol{V}}$ by knots received by "truncation" of the parameterization Φ. More specifically, we will prove that for any $i = 1, \ldots, p$ the image of the mapping

$$\Phi_i(t) := (t^n, a_{p_1} t^{p_1} + \ldots + a_{p_i} t^{p_i}), \quad t \in S(\varepsilon^{1/n}),$$

is a knot of the type $T_{(n'_1,m'_1)\dots(n'_i,m'_i)}$. In particular for $i = p_\beta$ the image Φ_{p_β} of the circle $S(\varepsilon^{1/n})$ has the type $T_{(n'_1,m'_1)\dots(n'_{p_\beta},m'_{p_\beta})}$. Then we will notice that the images by Φ_{p_β} and Φ of the circle $S(\varepsilon^{1/n})$ have the same type in $\partial P(\varepsilon,\varepsilon)$, whence we obtain $T_{(n'_1,m'_1)\dots(n'_{p_\beta},m'_{p_\beta})} \sim K_{\boldsymbol{V}}$, which gives the assertion.

Let's consider first the case $i = 1$, i.e.

$$\Phi_1(t) = (t^n, a_{p_1}t^{p_1}),\ t \in S(\varepsilon^{1/n}).$$

Decreasing ε we may assume $|a_{p_1}t^{p_1}| < \varepsilon$ for $t \in S(\varepsilon^{1/n})$. The image of Φ_1 is obviously equal to the image of the mapping

$$\Phi_1^{\mathrm{red}}(t) := (t^{n'_1}, a_{p_1}t^{m'_1}),\ t \in S(\varepsilon^{1/n'_1}),$$

and this is the first order torus knot of type (m'_1, n'_1). This knot lies in the torus $\{(x,y) : |x| = \varepsilon,\ |y| = |a_{p_1}\varepsilon^{m'_1/n'_1}|\}$. Let's denote this knot by T_1.

Consider now the case $i = 2$, i.e.

$$\Phi_2(t) = (t^n, a_{p_1}t^{p_1} + a_{p_2}t^{p_2}),\ t \in S(\varepsilon^{1/n}).$$

Decreasing ε we may assume that $|a_{p_1}t^{p_1} + a_{p_2}t^{p_2}| < \varepsilon$ for $t \in S(\varepsilon^{1/n})$. Notice the image of mapping Φ_2 lies in the boundary of tubular neighbourhood of T_1 with the radius $\left|a_{p_2}\varepsilon^{p_2/n}\right|$. In fact, for every $t \in S(\varepsilon^{1/n})$ we have $\Phi_2(t) = \Phi_1(t) + (0, a_{p_2}t^{p_2})$ and if ε is sufficiently small, then discs with radius $|a_{p_2}|\,\varepsilon^{p_2/n}$ and centers in points of the set $\pi_1^{-1}(x) \cap T_1$, $|x| = \varepsilon$, are contained in the disc $K(0,\varepsilon)$ and they are pairwise disjoined (the latter follows from the fact that the distance of any two different points of T_1 in $\pi_1^{-1}(x)$ is greater or equal to $|a_{p_1}(1-\rho)t^{p_1}| = |a_{p_1}(1-\rho)|\,\varepsilon^{p_1/n}$, where ρ is a primitive root of unity of degree p_1 and the inequality $p_1 < p_2$). Moreover, the image of Φ_2 is of course equal to the image of the mapping

$$\Phi_2^{\mathrm{red}}(t) := (t^{n'_1n'_2}, a_{p_1}t^{m'_1n'_2} + a_{p_2}t^{m'_2}),\ t \in S(\varepsilon^{1/n'_1n'_2}),$$

and this is a torus knot of the second order of the type $(n'_1, m'_1)\,(n'_2, m'_2)$. Let's denote this knot by T_2. By repeating this reasoning (decreasing ε each time, if necessary) we will finally get that the image of the circle $S(\varepsilon^{1/n})$ by Φ_{p_β} is a torus knot in $P(\varepsilon,\varepsilon)$ which has the type $(n'_1, m'_1)\dots(n'_{p_\beta}, m'_{p_\beta})$. Moreover it lies in the boundary of $\partial P(\varepsilon,\varepsilon)$. Let's denote this knot by T_{p_β}.

It remains to compare the knot T_{p_β} with the knot $K_{\boldsymbol{V}}$, i.e. the images of the circle $S(\varepsilon^{1/n})$ by Φ_{p_β} and Φ in $\partial P(\varepsilon,\varepsilon)$. Since $p_\beta = \beta_h$ we have to compare the images of Φ_{β_h} and Φ. We have

$$\Phi_{\beta_h}(t) = (t^n, a_{p_1}t^{p_1} + \dots + a_{\beta_h}t^{\beta_h}),\quad t \in S(\varepsilon^{1/n}),$$
$$\Phi(t) = (t^n, a_{p_1}t^{p_1} + \dots + a_{\beta_h}t^{\beta_h} + \dots),\quad t \in S(\varepsilon^{1/n}).$$

For a fixed x, $|x| = \varepsilon$, and every t such that $t^n = x$ we have

$$pr_1^{-1}(x) \cap T_{\beta_h} = \{(t^n, a_{p_1}(\rho t)^{p_1} + \dots + a_{\beta_h}(\rho t)^{\beta_h}) : \rho \in U(n)\},$$
$$pr_1^{-1}(x) \cap K_{\boldsymbol{V}} = \{(t^n, a_{p_1}(\rho t)^{p_1} + \dots + a_{\beta_h}(\rho t)^{\beta_h} + \dots) : \rho \in U(n)\},$$

where $U(n)$ is the set of roots of unity of degree n. Because

$$\mathrm{GCD}(n, p_1, \ldots, \beta_h) = 1,$$

then for sufficiently small ε each of this set has n elements. We show this for the set $pr_1^{-1}(x) \cap T_{\beta_h}$, because the reasoning for the set $pr_1^{-1}(x) \cap K_{\boldsymbol{V}}$ is analogous. If for some $\rho, \widetilde{\rho} \in U(n)$, $\rho \neq \widetilde{\rho}$, and $t \in S(\varepsilon^{1/n})$ it holds

$$a_{p_1} (\rho t)^{p_1} + \ldots + a_{\beta_h} (\rho t)^{\beta_h} = a_{p_1} (\widetilde{\rho} t)^{p_1} + \ldots + a_{\beta_h} (\widetilde{\rho} t)^{\beta_h},$$

then

$$a_{p_1} t^{p_1} (\rho^{p_1} - \widetilde{\rho}^{p_1}) + \ldots + a_{\beta_h} t^{\beta_h} \left(\rho^{\beta_h} - \widetilde{\rho}^{\beta_h}\right) = 0.$$

Then, if this equality hold for an infinite number of $t \to 0$, then

$$\rho^{p_1} - \widetilde{\rho}^{p_1} = 0, \ldots, \rho^{\beta_h} - \widetilde{\rho}^{\beta_h} = 0.$$

Hence

$$\left(\frac{\rho}{\widetilde{\rho}}\right)^{p_1} = 1, \ldots, \left(\frac{\rho}{\widetilde{\rho}}\right)^{\beta_h} = 1.$$

From properties of the roots of unity we conclude $\mathrm{GCD}(n, p_1, \ldots, \beta_h) > 1$, which is contrary to the assumption.

Denote these points as follows

$$pr_1^{-1}(x) \cap T_{\beta_h} = \{P_\rho : \rho \in U(n)\},$$
$$pr_1^{-1}(x) \cap K_{\boldsymbol{V}} = \{\widetilde{P}_\rho : \rho \in U(n)\}.$$

The distance of each two different points P_ρ and $P_{\rho'}$ for $\rho, \rho' \in U(n)$ and ε small enough satisfies the inequality

$$\|P_\rho - P_{\rho'}\| \geq C\varepsilon^{\beta_h/n} \tag{29}$$

for some constant $C > 0$. In fact, since $\rho \neq \rho'$, the condition $\mathrm{GCD}(n, p_1, \ldots, \beta_h) = 1$ implies the existence of p_i such that $\rho^{p_i} - \widetilde{\rho}^{p_i} \neq 0$. Let p_{i_0} be the least such p_i. The for sufficiently small t we have

$$\|P_\rho - P_{\rho'}\| = |a_{p_{i_0}} t^{p_{i_0}} (\rho^{p_{i_0}} - \widetilde{\rho}^{p_{i_0}}) + \ldots + a_{\beta_h} t^{\beta_h} \left(\rho^{\beta_h} - \widetilde{\rho}^{\beta_h}\right)|$$
$$\geq |a_{p_{i_0}} t^{p_{i_0}} (\rho^{p_{i_0}} - \widetilde{\rho}^{p_{i_0}})| - |a_{p_{i_0+1}} t^{p_{i_0+1}} (\rho^{p_{i_0+1}} - \widetilde{\rho}^{p_{i_0+1}}) + \ldots + a_{\beta_h} t^{\beta_h} \left(\rho^{\beta_h} - \widetilde{\rho}^{\beta_h}\right)|$$
$$\geq \widetilde{C}_1 |t^{p_{i_0}}| - \widetilde{C}_2 |t^{p_{i_0+1}}|$$

for some positive constants $\widetilde{C}_1, \widetilde{C}_2$. Since $p_{i_0} < p_{i_0+1}$, for sufficiently small t

$$\widetilde{C}_1 |t^{p_{i_0}}| - \widetilde{C}_2 |t^{p_{i_0+1}}| \geq \frac{\widetilde{C}_1}{2} |t^{p_{i_0}}|.$$

In turn $p_{i_0} \leq \beta_h$, whence

$$\frac{\widetilde{C}_1}{2} |t^{p_{i_0}}| \geq \frac{\widetilde{C}_1}{2} |t^{\beta_h}| = \frac{\widetilde{C}_1}{2} \varepsilon^{\beta_h/n},$$

which gives (29). On the other hand, the distance of points P_ρ and $\widetilde{P}_\rho$ for the same $\rho \in U(n)$ and ε small enough satisfies the inequality

$$\|P_\rho - \widetilde{P}_\rho\| = |a_{\beta_h+1} t^{\beta_h+1} \rho^{\beta_h+1} + \ldots| \leq C' \left|t^{\beta_h+1}\right| = C' \varepsilon^{(\beta_h+1)/n}.$$

Since $\beta_h < \beta_h + 1$, decreasing ε we may assume that points $\widetilde{P}_\rho$ belong to discs with centers at P_ρ and these dics are contained in $K(0,\varepsilon)$ and are pairwise disjoined (see Fig. 31).

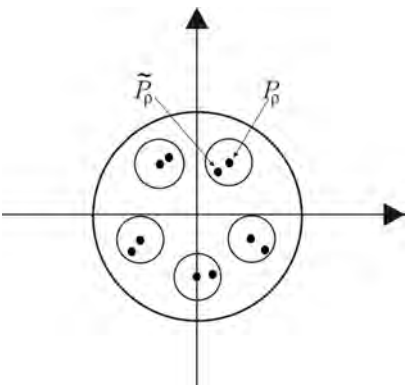

Fig. 31. Points P_ρ and $\widetilde{P}_\rho$.

Of course, points P_ρ and $\widetilde{P}_\rho$ depend continuously on the point x. It is not difficult to prove for each x the existence of homeomorphism h_x of the disc $K(0,\varepsilon)$ on itself, continuously depending on x, transforming points P_ρ into points $\widetilde{P}_\rho$ and being the identity on the boundary.

By extending these homeomorphisms by identity to the rest of the boundary $\partial P(\varepsilon,\varepsilon)$ we get that the knot T_{β_h} is equivalent to the knot $K_{\boldsymbol{V}}$.

Hence $K_{\boldsymbol{V}}$ is a torus knot of the type $((m_1,n_1),\dots,(m_h,n_h))$. □

6. Topological equivalence of irreducible curves

We can now prove the basic characterization of topological types of irreducible curves. First we will give the necessary definitions. Two curves $\boldsymbol{V} = V(f)$ and $\widetilde{\boldsymbol{V}} = V(\widetilde{f})$ are *topologically equivalent*, if there exist neighbourhoods $U, \widetilde{U}$ of the origin in $\mathbb{C}^2$ such that pairs $(U, V_U(f))$ i $(\widetilde{U}, V_{\widetilde{U}}(\widetilde{f}))$ are homeomorphic, i.e.

$$(U, V_U(f)) \overset{\text{top}}{\approx} (\widetilde{U}, V_{\widetilde{U}}(\widetilde{f})).$$

It means, there exists a homeomorphism $H : U \to \widetilde{U}$ leaving the point 0 fixed, which sends $V_U(f)$ on $V_{\widetilde{U}}(\widetilde{f})$. Of course, this relation is a relation of equivalence in the set of curves. Abstract classes of this relation are called *topological types of curves.*

Theorem 6.1. *Two irreducible curves have the same topological type if and only if they have the same characteristics.*

Proof. 1. $\Leftarrow$. Let $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ be two irreducible curves with the same characteristic $(\beta_0,\beta_1,\dots,\beta_h)$. Using a linear change of variables in $\mathbb{C}^2$, we may assume that $\boldsymbol{V} = \widehat{V_U(f)}$ and $\widetilde{\boldsymbol{V}} = \widehat{V_{\widetilde{U}}(\widetilde{f})}$, $f, \widetilde{f} \in \mathbb{C}\{x\}[y]$ are distinguished polynomials, ord $f =$

$\operatorname{ord} f(0,y) = \operatorname{ord} \widetilde{f} = \operatorname{ord} \widetilde{f}(0,y)$. Denote $V := V_U(f)$, $\widetilde{V} := V_{\widetilde{U}}(\widetilde{f})$. By assumption on common characteristics, it follows that f and $\widetilde{f}$ have the same sequence of characteristic pairs

$$((m_1,n_1),\dots,(m_h,n_h)).$$

By Theorem 5.5 for their knots $K_{\boldsymbol{V}}$ and $K_{\widetilde{\boldsymbol{V}}}$ we have

$$K_{\boldsymbol{V}} \sim T_{(m_1,n_1),\dots,(m_h,n_h)},$$
$$K_{\widetilde{\boldsymbol{V}}} \sim T_{(m_1,n_1),\dots,(m_h,n_h)}.$$

Hence $K_{\boldsymbol{V}} \sim K_{\widetilde{\boldsymbol{V}}}$ as knots in $\partial P(\varepsilon,\varepsilon)$ for sufficiently small ε. Then the pairs $(\partial P(\varepsilon,\varepsilon), K_{\boldsymbol{V}})$ and $(\partial P(\varepsilon,\varepsilon), K_{\widetilde{\boldsymbol{V}}})$ are homeomorphic. Hence, by extending this homeomorphism (along the segments connecting the points of $\partial P(\varepsilon,\varepsilon)$ with 0) we will get a homeomorphism of the pair $(P(\varepsilon,\varepsilon), \operatorname{cone}(K_{\boldsymbol{V}}))$ with the pair $(P(\varepsilon,\varepsilon), \operatorname{cone}(K_{\widetilde{\boldsymbol{V}}}))$. By the theorem on the conic structure these pairs are homeomorphic to $(P(\varepsilon,\varepsilon), V \cap P(\varepsilon,\varepsilon))$ and $(P(\varepsilon,\varepsilon), \widetilde{V} \cap P(\varepsilon,\varepsilon))$, respectively. Then

$$(P(\varepsilon,\varepsilon), V \cap P(\varepsilon,\varepsilon)) \overset{\text{top}}{\approx} (P(\varepsilon,\varepsilon), \widetilde{V} \cap P(\varepsilon,\varepsilon)).$$

Hence, after restriction this homomorphism to the interior of $P(\varepsilon,\varepsilon)$ we get

$$(\operatorname{Int} P(\varepsilon,\varepsilon), V \cap \operatorname{Int} P(\varepsilon,\varepsilon)) \overset{\text{top}}{\approx} (\operatorname{Int} P(\varepsilon,\varepsilon), \widetilde{V} \cap \operatorname{Int} P(\varepsilon,\varepsilon)),$$

i.e. $\boldsymbol{V}$ i $\widetilde{\boldsymbol{V}}$ are topologically equivalent.

2. $\Rightarrow$. Suppose that irreducibles curves $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ are topologically equivalent. Using a linear change of variables in $\mathbb{C}^2$, we may assume that $\boldsymbol{V} = \widehat{V_U(f)}$, $\widetilde{\boldsymbol{V}} = \widehat{V_{\widetilde{U}}(\widetilde{f})}$ where $f, \widetilde{f} \in \mathbb{C}\{x\}[y]$ are distinguished polynomials and

$$\Phi(t) = (t^n, \varphi(t)), \ \operatorname{ord}\varphi > n, \ t \in K,$$
$$\widetilde{\Phi}(t) = (t^m, \widetilde{\varphi}(t)), \ \operatorname{ord}\widetilde{\varphi} > m, \ t \in \widetilde{K},$$

are their parametrizations in neighbourhoods of U and $\widetilde{U}$, respectively. Let

$$((m_1,n_1),\dots,(m_g,n_g)),$$
$$((\widetilde{m}_1,\widetilde{n}_1),\dots,(\widetilde{m}_h,\widetilde{n}_h))$$

be characteristic pairs of $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ "read off" from the parameterizations Φ and $\widetilde{\Phi}$. By Theorem 5.5

$$K_{\boldsymbol{V}} \sim T_{(m_1,n_1),\dots,(m_g,n_g)},$$
$$K_{\widetilde{\boldsymbol{V}}} \sim T_{(\widetilde{m}_1,\widetilde{n}_1),\dots,(\widetilde{m}_h,\widetilde{n}_h)}.$$

We will now show that the knot groups $\pi(K_{\boldsymbol{V}})$ and $\pi(K_{\widetilde{\boldsymbol{V}}})$ are isomorphic. By assumption $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ are topologically equivalent. Then decreasing U and $\widetilde{U}$, if necessary, there exists a homeomorphism $F : U \to \widetilde{U}$ which maps $V_U(f)$ on $V_{\widetilde{U}}(\widetilde{f})$. Take $\varepsilon > 0$ such that $P(\varepsilon,\varepsilon) \subset U$. Assume $\varepsilon < \widetilde{\varepsilon}$, where $\widetilde{\varepsilon}$ is the radius of a

policylinder, appearing in the theorem on the conic structure, "good" for both $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$. Take r_1, r_2, r_3, r_4 such that

$$0 < r_2 < r_1 < \varepsilon, \quad 0 < r_4 < r_3 < \varepsilon$$

and

$$F(P(r_4, r_4)) \subset P(r_2, r_2) \subset F(P(r_3, r_3)) \subset P(r_1, r_1) \subset \widetilde{U}.$$

Hence

$$F(P(r_4, r_4)) \setminus \widetilde{V} \subset P(r_2, r_2) \setminus \widetilde{V} \subset F(P(r_3, r_3)) \setminus \widetilde{V} \subset P(r_1, r_1) \setminus \widetilde{V}.$$

These inclusions induce a sequence of homomorphisms of the first homotopy groups of these sets.

$$\pi_1(F(P(r_4, r_4))\setminus\widetilde{V}) \xrightarrow{f_1} \pi_1(P(r_2, r_2)\setminus\widetilde{V}) \xrightarrow{f_2} \pi_1(F(P(r_3, r_3))\setminus\widetilde{V}) \xrightarrow{f_3} \pi_1(P(r_1, r_1)\setminus\widetilde{V}).$$

Of course, the superposition $f_3 \circ f_2$, induced by the embedding $P(r_2, r_2) \setminus \widetilde{V} \hookrightarrow P(r_1, r_1) \setminus \widetilde{V}$, is an isomorphism by Lemma 5.4. Since F is a homeomorphism of U on $\widetilde{U}$ mapping V on $\widetilde{V}$, then

$$\pi_1(F(P(r_4, r_4)) \setminus \widetilde{V}) \cong \pi_1(P(r_4, r_4) \setminus V),$$
$$\pi_1(F(P(r_3, r_3)) \setminus \widetilde{V}) \cong \pi_1(P(r_3, r_3) \setminus V).$$

Hence we get the sequence of homomorphisms

$$\pi_1(P(r_4, r_4) \setminus V) \xrightarrow{\widetilde{f}_1} \pi_1(P(r_2, r_2) \setminus \widetilde{V}) \xrightarrow{\widetilde{f}_2} \pi_1(P(r_3, r_3) \setminus V) \xrightarrow{\widetilde{f}_3} \pi_1(P(r_1, r_1) \setminus \widetilde{V})$$

in which the superposition (induced by emeddings) $\widetilde{f}_3 \circ \widetilde{f}_2$ and $\widetilde{f}_2 \circ \widetilde{f}_1$ are isomorphisms. From here we can easily check that the homomorphism $\widetilde{f}_2$ is also an isomorphism. By Lemma 5.4

$$\pi_1(P(r_3, r_3) \setminus V) \cong \pi(\boldsymbol{V}),$$
$$\pi_1(P(r_2, r_2) \setminus \widetilde{V}) \cong \pi(\widetilde{\boldsymbol{V}}),$$

whence $\pi(\boldsymbol{V}) \cong \pi(\widetilde{\boldsymbol{V}})$. Hence their Aleksander polynomials $A(K_{\boldsymbol{V}})$ and $A(K_{\widetilde{\boldsymbol{V}}})$ are equal. But by the Theorem 5.5 $K_{\boldsymbol{V}} = T_{(m_1,n_1),\ldots,(m_g,n_g)}$ and $K_{\widetilde{V}} = T_{(\widetilde{m}_1,\widetilde{n}_1),\ldots,(\widetilde{m}_h,\widetilde{n}_h)}$, which implies

$$A(T_{(m_1,n_1),\ldots,(m_g,n_g)}) = A(T_{(\widetilde{m}_1,\widetilde{n}_1),\ldots,(\widetilde{m}_h,\widetilde{n}_h)}).$$

Because for characteristic pairs $((m_1, n_1), \ldots, (m_g, n_g))$, $((\widetilde{m}_1, \widetilde{n}_1), \ldots, (\widetilde{m}_h, \widetilde{n}_h))$ of curves $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$, inequalities (28) are satisfied, then by Theorem 4.10 we get $g = h$ and $(m_1, n_1), \ldots, (m_g, n_g) = (\widetilde{m}_1, \widetilde{n}_1), \ldots, (\widetilde{m}_h, \widetilde{n}_h)$. Then the characterics of $\boldsymbol{V}$ and $\boldsymbol{V}'$ are identical. □

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Faculty of Mathematics and Computer Science, University of Łódź, 90-238 Łódź, ul. Banacha 22

E-mail address: tadeusz.krasinski@wmii.uni.lodz.pl

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ON THE DUAL HESSE ARRANGEMENT

MAGDALENA LAMPA-BACZYŃSKA AND DANIEL WÓJCIK

Abstract. In the present note we investigate to which extent the configuration of 9 lines intersecting in triples in 12 points is determined by these incidences. We show that up to a projective automorphism there is exactly one such configuration in characteristic zero and one in characteristic 3. We pin down the geometric difference between these two realizations.

1. Introduction

In projective geometry, a point-line configuration consists of a finite set of points, and a finite arrangement of lines, such that each point is incident to the same number of lines and each line is incident to the same number of points. Their systematic study has been initiated by Theodor Reye in 1876 but they are a much more classical subject of study.

To a configuration there is assigned a symbol (p_γ, ℓ_π), where p is the number of points, ℓ is the number of lines, γ is the number of lines through each point and π is the number of points on each line. For example the famous Pappus configuration is denoted by $(9_3, 9_3)$, see Figure 1. One should bear in mind that the same symbol might be assigned to many non-isomorphic configurations. In the present note we are interested in the configuration $(12_3, 9_4)$. This is a remarkable configuration because there are no incidences among the 9 lines other than the 12 configuration points. One incarnation of this configuration, namely the dual Hesse configuration plays an important role in testing various properties in the theory of arrangements and recently also in commutative algebra (see e.g. the work of Dumnicki, Szemberg

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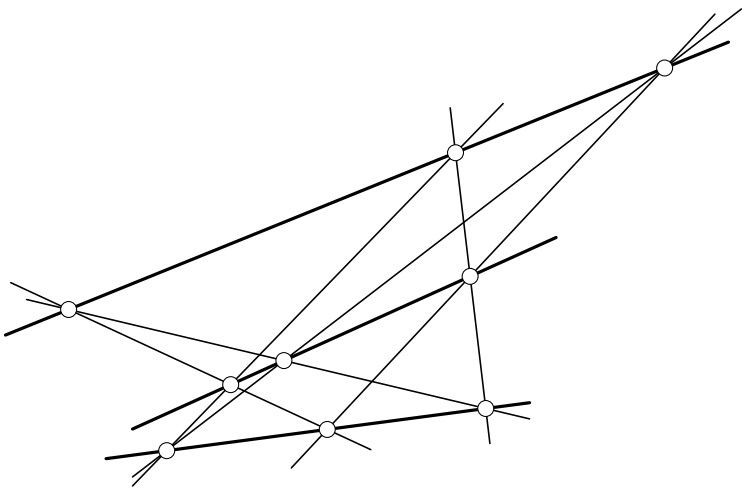

FIGURE 1. Pappus configuration

and Tutaj-Gasińska [3]) and algebraic geometry, more precisely in the theory of unexpected hypersurfaces (see e.g. the work of Bauer, Malara, Szemberg and Szpond [2]). All this has motivated our research. We want to find out to what extent the configuration is determined by the combinatorics involved.

2. Dual Hesse configuration

Ludwig Otto Hesse published in 1844 in Crelle's Journal an article [5] which contains, in particular, a description of a remarkable point-line configuration. The configuration consists of 9 points and 12 lines arranged so that there are 4 lines through every point and 3 points on every line, it is a $(9_4, 12_3)$ configuration. The incidences are indicated in Figure 2, which we borrowed from Wikipedia.

It is not possible to draw this configuration in the real plane without bending the lines. This follows from the celebrated Sylvester-Gallai Theorem. In fact, it seems that Hesse's discovery has prompted Sylvester to ask in [8] if there are non-trivial (i.e. not a pencil) point-line configurations in the *real* projective plane such that there are no intersection points among configuration lines where only 2 configuration lines meet.

Hesse construction works over complex numbers. Taking 9 inflection points of an elliptic curve C embedded into $\mathbb{P}^2$ as a smooth cubic curve (which is equivalent to taking 3–torsion points on C) one gets the Hesse configuration joining all pairs of these points. Because of the arithmetic properties of an elliptic curve, a line

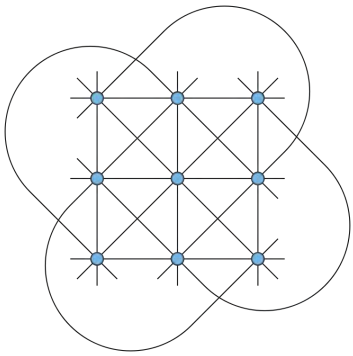

FIGURE 2. Hesse configuration

intersecting it in two 3–torsion points must go through a third 3–torsion point. Thus there are 12 such lines altogether. Another, very illuminating description comes from the *Hesse pencil*, which in appropriate coordinates can be written as

$$s\,xyz \,+\, t\,(x^3+y^3+z^3) \,=\, 0. \tag{1}$$

There are exactly 4 singular members in this pencil, each of which splits into 3 lines, see the work of Artebani and Dolgachev [1] for a beautiful account on various aspects of this pencil.

Passing to the dual we obtain what is known as the *dual Hesse configuration*. Over complex numbers it can be given by linear factors of the polynomial

$$(x^3-y^3)\,(y^3-z^3)\,(z^3-x^3) \,=\, 0. \tag{2}$$

In this description it is immediately recognized as a member of an infinite family of Fermat arrangements, see [9] for an extensive survey on this kind of arrangements. The dual Hesse configuration is of course of type $(9_3, 12_4)$. It is one of very few examples of point-line configurations where all intersection points between configuration lines belong to exactly 3 of these lines. The other examples are:

- trivial configuration: 3 lines meeting in a single point;
- the finite projective plane $\mathbb{P}^2(\mathbb{F}_3)$ defined in characteristic 3 with the whole pencil of lines through a single point removed.

The purpose of this note is to investigate to which extent incidences of the dual Hesse configuration determine it up to a projective automorphism. Our main results are the following.

Theorem 1. *Any* $(12_3, 9_4)$ *point-line configuration in the projective plane defined over a field* $\mathbb{F}$ *of characteristic zero is projectively equivalent to that given by linear factors of the Fermat equation (2).*
In particular, the field $\mathbb{F}$ *must contain roots of unity of order* 3.

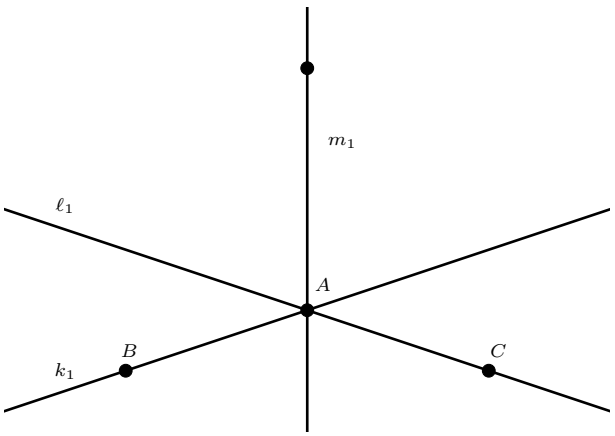

FIGURE 3. First step

As a by-product of our proof we obtain also the following complementary result.

Theorem 2. *Any* $(12_3, 9_4)$ *point-line configuration in the projective plane defined over a field* $\mathbb{F}$ *of positive characteristic is projectively equivalent to that obtained from* $\mathbb{P}^2(\mathbb{F}_3)$ *by removing the whole pencil of lines passing through a point.*
In particular, $\mathbb{F}$ *must be of characteristic* 3*.*

We prove these statements in the subsequent section.

3. Coordinatization of the dual Hesse configuration

In this section we use incidences of the $(12_3, 9_4)$ point-line configuration to recover step by step its coordinates in a conveniently chosen system of coordinates. This approach is motivated by Sturmfels [7]. It has been applied successfully recently in the study of parameter spaces of Böröczky arrangements, see [6] and [4].

Let $\mathbb{F}$ be an arbitrary field with sufficiently many elements so that $\mathbb{P}^2(\mathbb{F})$ has at least 12 points. Since all points in the configuration have the same properties, we pick one, call it A and we assume that $A = (1 : 1 : 1)$. Then we name the lines passing through A as k_1, ℓ_1 and m_1. Since any configuration line contains 4 configuration points, we pick $B = (1 : 0 : 0)$ on k_1. On ℓ_1 we pick a point C which is not connected by a configuration line to the point B. It is possible, because there are only 3 lines passing through B and they intersect ℓ_1 in A and two other points. We take C to be the remaining point and we set its coordinates to $(0 : 1 : 0)$. Thus the lines k_1, ℓ_1 have equations $y - z = 0$ and $x - z = 0$ respectively. The choices made so far are depicted in Figure 3.

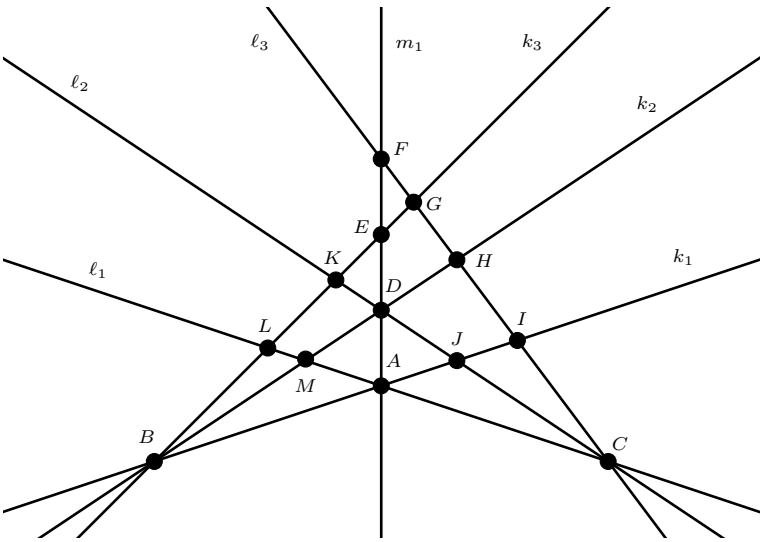

FIGURE 4. Case 1

Let k_2, k_3 be the configuration lines passing through B distinct from k_1, and similarly let ℓ_2, ℓ_3 be the configuration lines passing through C distinct from ℓ_1. Then we have two possibilities on how these lines meet the line m_1.

Case 1. We assume that k_2, k_3, ℓ_2, ℓ_3 intersect m_1 in 3 distinct points. Renumbering the lines if necessary, we may assume that these are k_2 and ℓ_2 which meet m_1 in the same point, which we call D. This situation is indicated in Figure 4.

Then A, B, C, D and intersection points between lines

$$E = k_3 \cap m_1,\ F = \ell_3 \cap m_1,\ G = k_3 \cap \ell_3,\ H = k_2 \cap \ell_3, I = k_1 \cap \ell_3,$$

$$J = k_1 \cap \ell_2,\ K = k_3 \cap \ell_2,\ L = k_3 \cap \ell_1,\ M = k_2 \cap \ell_1$$

must be all mutually distinct (otherwise some 2 distinct lines would intersect in 2 distinct points). But then there would be already 13 points in the configuration. A contradiction.

Case 2. Thus we are left with the case in which the lines k_2, ℓ_2 and k_3, ℓ_3 intersect m_1 pairwise in the same point. Let $E = k_2 \cap \ell_2$ and $F = k_1 \cap \ell_2$. In this situation let D be the point on m_1 not connected neither to A, nor to B by a configuration line.

We have now again two possibilities depending on whether the points B, C, D are collinear or not.

Subcase 2.1. The points B, C, D are collinear.

In this situation we can assume after change of coordinates if necessary that $D = (1 : 1 : 0)$. Let $E = (a : a : 1)$ be a point on the line m_1 distinct from A and

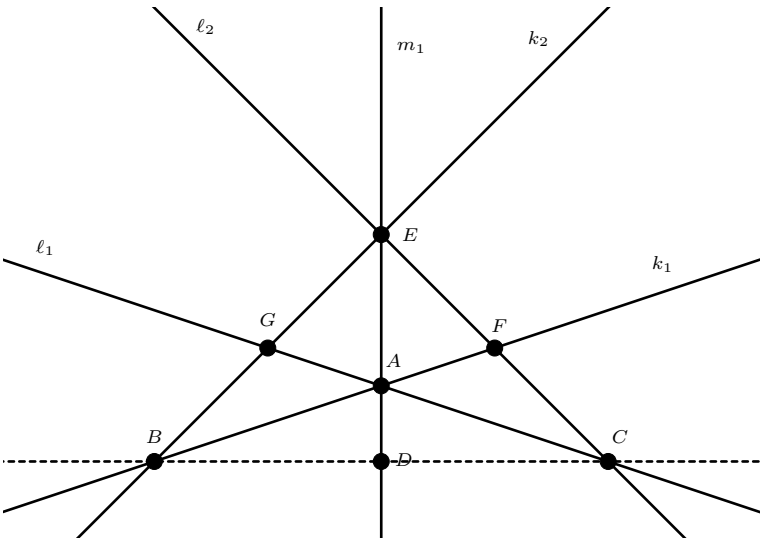

FIGURE 5. Case 2.1 first step of the construction

D, so that $a \neq 1$ is an element of $\mathbb{F}$. Then we have

$$k_2 : \ y - az = 0 \ \text{ and } \ \ell_2 : \ x - az = 0,$$

for the lines BE and CE, respectively. Then we compute

$$F = k_1 \cap \ell_2 = (a : 1 : 1), \ \text{ and } \ G = k_2 \cap \ell_1 = (1 : a : 1).$$

The situation so far is depicted in Figure 5.

Next we choose another point $H = (b : b : 1)$ on the line m_1. In order to keep points mutually distinct, we require now $b \neq 1$ and $b \neq a$. Then we have

$$k_3 : \ y - bz = 0 \ \text{ and } \ \ell_3 : \ x - bz = 0$$

for the lines joining B and H, and C and H respectively. This determines all other configuration points:

$$I = k_1 \cap \ell_3 = (b : 1 : 1), \ J = k_3 \cap \ell_1 = (1 : b : 1),$$

$$K = k_3 \cap \ell_2 = (a : b : 1), \ L = k_2 \cap \ell_3 = (b : a : 1).$$

The configuration is indicated in Figure 6.

We need still to determine the lines m_2 and m_3. Let m_2 be the line joining D and I. Then we have

$$m_2 : \ -x + y + (b-1)z = 0.$$

Since $K = m_2 \cap \ell_2$, we obtain $a = -1$.

Then it must be $m_2 \cap \ell_1$ equal either G or J. In the first case we get $b = 3$, in the second $b = 1$, which is a contradiction with the assumption $b \neq a$. Hence m_2

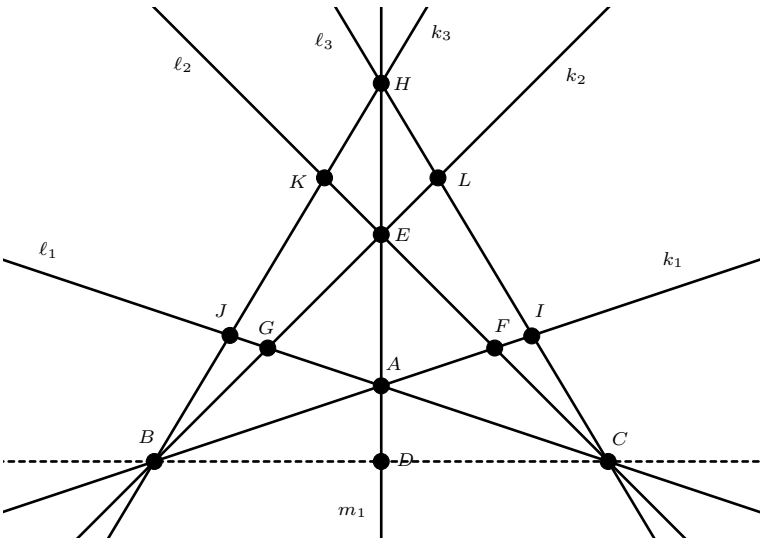

FIGURE 6. Case 2.1 second step of the construction

is the line through D, I, K and G. Consequently m_3 is the line through D and J. Thus we have

$$m_3 : \ x - y + (b-1)z = 0.$$

Since L is also a point on m_3 we obtain $2b = a + 1 = 0$. It is not possible that $\mathbb{F}$ is of characteristic 2, because otherwise it would be $b = 1$, which is excluded by our assumptions. Thus we can divide by 2 and we have now $b = 3$ and $b = 0$. This is possible only in characteristic 3. Assuming this characteristic, we complete the construction by verifying that the condition $F \in m_3$ is satisfied.

Subcase 2.2. The points B, C, D are not collinear.

In this situation we can assume after change of coordinates if necessary that $D = (0 : 0 : 1)$. We proceed as in the Subcase 2.1 and obtain the same coordinates of points and lines passing through B and C. We collect and present them below for convenience.

$$A = (1 : 1 : 1),\ B = (1 : 0 : 0),\ C = (0 : 1 : 0),\ D = (0 : 0 : 1),$$

$$E = (a : a : 1),\ F = (a : 1 : 1),\ G = (1 : a : 1),\ H = (b : b : 1),$$

$$I = (b : 1 : 1),\ J = (1 : b : 1),\ K = (a : b : 1),\ L = (b : a : 1),$$

with $a, b \in \mathbb{F}$ such that $a \neq 1 \neq b \neq a$.

$$k_1 : \ y - z = 0,\ k_2 : \ y - az = 0,\ k_3 : \ y - bz = 0,$$

$$\ell_1 : \ x - z = 0,\ \ell_2 : \ x - az = 0,\ \ell_3 : \ x - bz.$$

The incidences above are depicted in Figure 7.

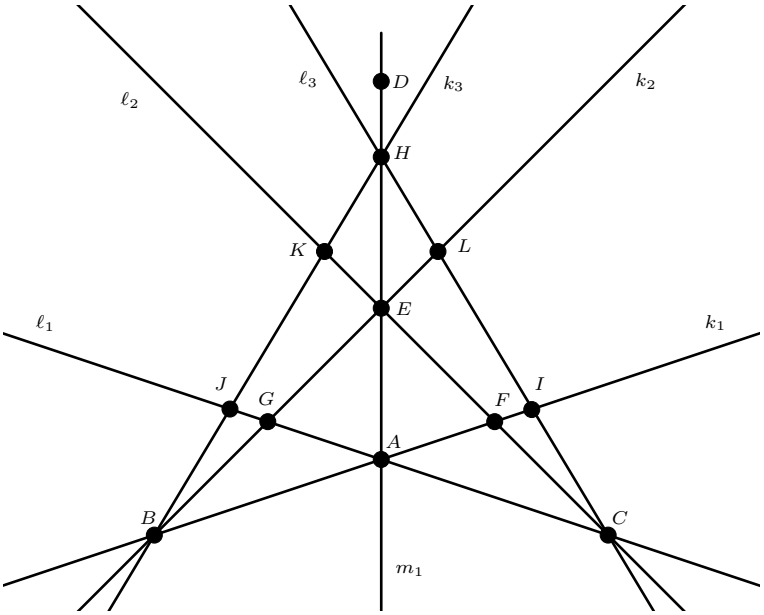

FIGURE 7. Case 2.2 the construction

The only difference compared to Subcase 2.1 occurs in lines m_2 and m_3 passing through D. We focus now on these lines. Let m_2 be the line determined by D and I. Then

$$m_2 : \ x - by = 0.$$

Checking the conditions for points F, G, K and L to lie on m_2, we get an immediate contradiction for F and L, so that it must be G and K on m_2. The incidence conditions are then

$$\begin{cases} 1 - ab &= 0 \\ a - b^2 &= 0 \end{cases}$$

and we see that it must be $b^3 = 1$. Since b cannot be equal 1, it must be a primitive root of 1 of order 3. Then $a = b^2$ is the other primitive order 3 root of 1.

It remains to check that the line m_3 determined by D and J with equation

$$m_3 : \ bx - y = 0$$

contains points F and L. As this is elementary, we are done with the proof of Theorems 1 and 2.

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(Magdalena Lampa-Baczyńska) Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: lampa.baczynska@wp.pl

(Daniel Wójcik) Department of Mathematics, Pedagogical University of Cracow, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: daniel.wojcik@krakow.up.pl

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FINITELY GENERATED SUBRINGS OF R[X]

ANDRZEJ NOWICKI

Abstract. In this article all rings and algebras are commutative with identity, and we denote by $R[x]$ the ring of polynomials over a ring R in one variable x. We describe rings R such that all subalgebras of $R[x]$ are finitely generated over R.

Introduction

Let K be a field and let L be a subfield of $K(x_1, \ldots, x_n)$ containing K. In 1954, Zariski in [15], proved that if $n \leqslant 2$, then the ring $L \cap K[x_1, \ldots, x_n]$ is finitely generated over K. This is a result concerning the fourteenth problem of Hilbert. Today we know ([8], [9], [7]) that a similar statement for $n \geqslant 3$ is not true. Many results on this subject one can find, for example, in [4], [5], [10], [13], and also in the author articles ([11], [12]) published by University of Lodz in Materials of the Conferences of Complex Analytic and Algebraic Geometry.

We are interested in the case $n = 1$. It is well known that every K-subalgebra A of $K[x_1]$ is finitely generated over K. In this case we do not assume that A has a form $L \cap K[x_1]$. We recall it (with a proof) as Theorem 2.1. An elementary proof one can find, for example, in [6]. The assumption that K is a field is here very important. What happens in the case when K is a commutative ring and K is not a field? In this article we will give a full answer to this question.

Throughout this article all rings and algebras are commutative with identity, and we denote by $R[x]$ the ring of polynomials over a ring R in one variable x. We say that a ring R is an *sfg-ring*, if every R-subalgebra of $R[x]$ is finitely generated over R. We already know that if R is a field then R is an sfg-ring. We will show

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that the rings $\mathbb{Z}$ and $\mathbb{Z}_4$ are not sfg-rings. But, for instance, the rings $\mathbb{Z}_6$ and $\mathbb{Z}_{105}$ are sfg-rings.

The main result of this article states that R is an sfg-ring if and only if R is a finite product of fields. For a proof of this fact we prove, in Section 3, many various lemmas. A crucial role plays the Artin-Tate Lemma (Lemma 1.3). If R is an sfg-ring then we successively prove that R is Noetherian, reduced, that every prime ideal of R is maximal, and by this way we obtain that R is a finite product of fields. Moreover, in the last section, we present a proof that every finite product of fields is an sfg-ring.

1. Preliminary lemmas and notations

We start with the following well known lemma (see for example [2] Proposition 6.5).

Lemma 1.1. *If R is a Noetherian ring and M is a finitely generated R-module, then M is a Noetherian module.*

Let A be an algebra over a ring R. If S is a subset of A, then we denote by $R[S]$ the smallest R-subalgebra of A containing R and S. Several times we will use the following obvious lemma.

Lemma 1.2. *Let $A = R[S]$. If the algebra A is finitely generated over R, then there exists a finite subset S_0 of S such that $A = R[S_0]$.*

The next lemma comes from [14] (Lemma 2.4.3). This is a particular case of the Artin and Tate result published in [1]. Since this lemma plays an important role in our article, we present also its simple proof.

Lemma 1.3 (Artin, Tate, 1951)**.** *Let R be a Noetherian ring, B a finitely generated R-algebra, and A an R-subalgebra of B. If B is integral over A, then the algebra A is finitely generated over R.*

Proof. Let $B = R[b_1, \ldots, b_s]$, where $b_1, \ldots, b_s$ are some elements of B. Since each b_i is integral over A, we have equalities of the form

$$b_i^{n_i} + a_{i1} b^{n_i - 1} + \cdots + a_{in_i} = 0, \quad \text{for } i = 1, \ldots, s,$$

where all coefficients a_{ij} belong to A, and $n_1, \ldots, n_s$ are positive integers. Let $\{a_1, \ldots, a_m\}$ be the set of all the coefficients a_{ij}, and put

$$A' = R[a_1, \ldots, a_m].$$

It is clear that A' is a Noetherian ring and B is an A'-module generated by all elements of the form $b_1^{j_1} b_2^{j_2} \cdots b_s^{j_s}$, where $0 \leqslant j_1 < n_1, \ldots, 0 \leqslant j_s < n_s$. Thus, B is a finitely generated A'-module and so, by Lemma 1.1, B is a Noetherian A'-module. This means that every submodule of B is finitely generated. In particular,

A is a finitely generated A'-module. Assume that $a_{m+1}, a_{m+2}, \ldots, a_n \in A$ are its generators. Then

$$A = A'a_{m+1} + \cdots + A'a_n = R[a_1, \ldots, a_n]\,,$$

and we see that the algebra A is finitely generated over R. □

Let us fix some notations. For a given subset I of a ring R, we denote by $I[x]$ the set of all polynomials from $R[x]$ with the coefficients belonging to I. If I is an ideal of R, then $I[x]$ is an ideal of $R[x]$, and then the rings $R[x]/I[x]$ and $(R/I)[x]$ are isomorphic.

Let $f : S \to T$ be a homomorphism of rings. We denote by $\overline{f}$ the mapping from $S[x]$ to $T[x]$ defined by the formula

$$\overline{f}\left(\sum_j s_j x^j\right) = \sum_j \varphi(s_j)x^j$$

for all $\sum_j s_j x^j \in S[x]$. This mapping is a homomorphism of rings and $\operatorname{Ker}\overline{f} = (\operatorname{Ker} f)\,[x]$. We will say that $\overline{f}$ is the *homomorphism associated with* f. If f a surjection, then $\overline{f}$ is also a surjection. It is clear that if S and T are R-algebras, and $f : S \to T$ is a homomorphism of R-algebras, then $\overline{f} : S[x] \to T[x]$ is also a homomorphism of R-algebras.

In next sections we will use the following two lemmas.

Lemma 1.4. *Let I be an ideal of a ring R, and let $A = R\,[ax;\ a \in I]$. If the ideal I is not finitely generated, then the algebra A is not finitely generated over R.*

Proof. Assume that I is not finitely generated and suppose that A is finitely generated over R. Then, by Lemma 1.2, there exists a finite subset $\{a_1, \ldots, a_n\}$ of I such that $A = R[a_1x, \ldots, a_nx]$. Then of course $(a_1, \ldots, a_n) \neq I$ so, there exists $b \in I \setminus (a_1, \ldots, a_n)$. Since $bx \in A = R[a_1x, \ldots, a_nx]$, we have $bx = F(a_1x, \ldots, a_nx)$, where F is a polynomial belonging to $R[t_1, \ldots, t_n]$. Let

$$F = r_0 + r_1t_1 + r_2t_2 + \cdots + r_nt_n + G$$

where $r_0, r_1, \ldots, r_n \in R$ and $G \in R[t_1, \ldots, t_n]$ is a polynomial in which the degrees of all nonzero monomials are greater than 1. Then, in the ring $R[x]$ we have

$$bx = F\,(a_1x, \ldots, a_nx) = r_0 + r_1a_1x + \cdots + r_na_nx + hx^2,$$

where h is some element of $R[x]$. This implies that $b = r_1a_1 + \cdots + r_na_n \in (a_1, \ldots, a_n)$, but it is a contradiction, because $b \notin (a_1, \ldots, a_n)$. □

Lemma 1.5. *Let $A = R\left[bx, bx^2, \ldots, bx^n\right]$, where $n \geqslant 1$, $0 \neq b \in R$ and $b^2 = 0$. Then every element u of A is of the form $u = r_0 + r_1bx + r_2bx^2 + \cdots + r_nbx^n$ for some $r_0, r_1, \ldots, r_n \in R$.*

Proof. Let $u \in A$. Then $u = F(bx, bx^2, \ldots, bx^n)$ for some n, where F is a polynomial in n variables belonging to the polynomial ring $R[t_1, \ldots, t_n]$. Let

$$F(t_1, \ldots, t_n) = r_0 + r_1 t_1 + r_2 t_2 + \cdots + r_n t_n + G(t_1, \ldots, t_n),$$

where $r_0, \ldots, r_n \in R$ and $G \in R[t_1, \ldots, t_n]$ is a polynomial such that the degrees of all nonzero monomials of F are greater than 1. Then $G(bx, \ldots, bx^n) = b^2 H(x)$, gdzie $H(x) \in R[x]$. But $b^2 = 0$, so $u = r_0 + r_1 bx + r_2 bx^2 + \cdots + r_n bx^n$. □

2. Subalgebras of K[x]

Let us start with the following consequence of Lemma 1.3.

Theorem 2.1. *If $K[x]$ is the polynomial ring in one variable over a field K, then every K-subalgebra of $K[x]$ is finitely generated over K.*

Proof. Let $A \subset K[x]$ be a K-subalgebra. If $A = K$ then of course A is finitely generated over K. Assume that $A \neq K$ and let $f \in A \setminus K$. Multiplying f by the inverse of its initial coefficient, we may assume that f is monic. Let $f = x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$, where $n \geqslant 1$ and $a_1, \ldots, a_n \in K$. It follows from the equality

$$x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + (a_n - f) = 0,$$

that the variable x is integral over A. This implies that the ring $K[x]$ is integral over A and, by Lemma 1.3, the algebra A is finitely generated over K. □

For the polynomial rings in two or bigger number of variables, a similar assertion is not true.

Example 2.2. *Let $K[x, y]$ be the polynomial ring in two variables over a field K, and*

$$A = K\left[xy, xy^2, xy^3, \ldots\right].$$

The algebra A is not finitely generated over K.

Proof. For every positive integer n, consider the ideal I_n of A, generated by the monomials $xy, xy^2, \ldots, xy^n$. Observe that $xy^{n+1} \notin I_n$. Indeed, suppose $xy^{n+1} = F_1 xy + F_2 xy^2 + \cdots + F_n xy^n$, where $F_1, \ldots, F_n \in A$. Every element of A is of the form $a + Gxy$ with $a \in K$ and $G \in K[x, y]$. In particular $F_j = a_j + G_j xy$, where $a_j \in K$, $G_j \in K[x, y]$ for all $j = 1, \ldots, n$. Thus, in $K[x, y]$ we have

$$y^{n+1} = a_1 y + a_2 y^2 + \cdots + a_n y^n + \left(G_1 y^2 + G_2 y^3 + \cdots + G_n y^n\right) x.$$

Let $\varphi : K[x, y] \to K[y]$ be the homomorphism of K-algebras defined by $x \mapsto 0$ and $y \mapsto y$. Then in the ring $K[y]$, we have the false equality $y^{n+1} = \varphi\left(y^{n+1}\right) = a_1 y + a_2 y^2 + \cdots + a_n y^n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. The ring A is not Noetherian. In particular, the algebra A is not finitely generated over K. □

In Theorem 2.1 we assumed that K is a field. This assumption is here very important. For instance, if K is the ring of integers $\mathbb{Z}$, then a similar assertion is not true.

Example 2.3. *Let* $A = \mathbb{Z}\left[2x, 2x^2, 2x^3, \dots\right]$. *Then* A *is a subalgebra of* $\mathbb{Z}[x]$ *and* A *is not finitely generated over* $\mathbb{Z}$.

Proof. For every positive integer n, consider the ideal I_n of A, generated by the monomials $2x, 2x^2, \dots, 2x^n$. Observe that $2x^{n+1} \notin I_n$. Indeed, suppose $2x^{n+1} = 2xF_1 + 2x^2F_2 + \cdots + 2x^nF_n$, where $F_1, \dots, F_n \in A$. Every element of A is of the form $a + 2xG$ with $a \in \mathbb{Z}$ and $G \in \mathbb{Z}[x]$. In particular, $F_j = a_j + 2xG_j$, where $a_j \in \mathbb{Z}$, $G_j \in \mathbb{Z}[x]$ for all $j = 1, \dots, n$. Thus, in $\mathbb{Z}[x]$ we have the equality

$$x^{n+1} = a_1x + a_2x^2 + \cdots + a_nx^n + 2\left(G_1x^2 + G_2x^3 + \cdots + G_nx^{n+1}\right).$$

For an integer u, denote by $\overline{u}$ the element u modulo 2. Then, in the ring $\mathbb{Z}_2[x]$ we have the false equality $x^{n+1} = \overline{a_1}x + \overline{a_2}x^2 + \cdots + \overline{a_n}x^n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. The ring A is not Noetherian. In particular, the algebra A is not finitely generated over $\mathbb{Z}$. □

3. Properties of sfg-rings

Let us recall that a ring R is said to be an *sfg-ring*, if every R-subalgebra of $R[x]$ is finitely generated over R. We already know (by Theorem 2.1) that if R is a field then R is an sfg-ring. Moreover we know (by Example 2.3) that $\mathbb{Z}$ is not an sfg-ring. In this section we will prove that every sfg-ring is a finite product of fields. For a proof of this fact we need the following 9 successive lemmas. In all the lemmas we assume that R is an sfg-ring.

Lemma 3.1. *R is Noetherian.*

Proof. Suppose R is not Noetherian. Then there exists an ideal I of R which is not finitely generated. Consider the R-algebra $A = R\,[ax;\ a \in I]$. It follows from Lemma 1.4 that this algebra is not finitely generated over R. But this contradicts our assumption that R is an sfg-ring. □

Now we know, by this lemma, that if R is an sfg-ring, then every R-subalgebra of $R[x]$ is a Noetherian ring.

Lemma 3.2. *If I is an ideal of R, then R/I is also an sfg-ring.*

Proof. Put $\overline{R} := R/I$. Let $\varphi : R \to \overline{R}$, $r \mapsto r+I$ be the natural ring homomorphism, and let $\overline{\varphi} : R[x] \to \overline{R}[x]$ be the homomorphism associated with φ. Let B be an $\overline{R}$-subalgebra of $\overline{R}[x]$. We need to show that B is finitely generated over $\overline{R}$. For this aim consider the R-algebra $A := \overline{\varphi}^{-1}(B)$. It is an R-subalgebra of $R[x]$. Since R is an sfg-ring, the algebra A is finitely generated over R. Let $W \subset A$ be a finite set of generators of A. Then it is easy to check that $\overline{\varphi}(W)$ is a finite set of generators of B over $\overline{R}$. □

Lemma 3.3. *Every non-invertible element of R is a zero divisor.*

Proof. Suppose there exists a non-invertible element $b \in R$ such that b is not a zero divisor of R. Then $b \neq 0$ and b is not a zero divisor of $R[x]$. Consider the R-subalgebra $A = R\left[bx, bx^2, bx^3, \dots\right]$. For every positive integer n, let I_n be the ideal of A, generated by the monomials $bx, bx^2, \dots, bx^n$. Observe that $bx^{n+1} \notin I_n$. Indeed, suppose $bx^{n+1} = bxF_1 + bx^2F_2 + \cdots + bx^nF_n$, where $F_1, \dots, F_n \in A$. Every element of A is of the form $a + bxG$ with $a \in R$ and $G \in R[x]$. In particular, $F_j = a_j + bxG_j$, where $a_j \in \mathbb{R}$, $G_j \in R[x]$ for all $j = 1, \dots, n$. Since the element b is not a zero divisor of $R[x]$, we have in $R[x]$ the following equality

$$x^{n+1} = a_1x + a_2x^2 + \cdots + a_nx^n + b\left(G_1x^2 + G_2x^3 + \cdots + G_nx^n\right).$$

Consider the factor ring $R/(b)$. Let $\varphi : R \to R/(b)$, $r \mapsto r + (b)$, be the natural homomorphism and $\overline{\varphi} : R[x] \to R/(b)[x]$ be the homomorphism associated with φ. Using $\overline{\varphi}$, from the above equality we obtain that $x^{n+1} = \varphi(a_1)x + \varphi(a_2)x^2 + \cdots + \varphi(a_n)x^n$. This is a false equality in the polynomial ring $\mathbb{R}/(b)[x]$. Therefore, $bx^{n+1} \notin I_n$. Hence, the infinite sequence $I_1 \subset I_2 \subset I_3 \subset \cdots$ is strictly increasing. This means that the ring A is not Noetherian. In particular, by Lemma 3.1, the algebra A is not finitely generated over R. But this contradicts our assumption that R is an sfg-ring. □

It follows from the above lemma that every ring without zero divisors, which is not a field, is not an sfg-ring. Thus, we see again, for instance, that $\mathbb{Z}$ is not an sfg-ring.

Lemma 3.4. *R is a reduced ring, that is, R is without nonzero nilpotent elements.*

Proof. Suppose that there exists $c \in R$ such that $c \neq 0$ and $c^m = 0$ for some $m \geqslant 2$. Assume that m is minimal and put $b := c^{m-1}$. Then $0 \neq b \in R$ and $b^2 = 0$. Consider the R-algebra $A = R[bx, bx^2, bx^3, \dots]$. It is an R-subalgebra of $R[x]$. Since R is an sfg-ring, this algebra is finitely generated over R. Hence, by Lemma 1.2, $A = R\left[bx, bx^2, \dots, bx^n\right]$ for some fixed n. But $bx^{n+1} \in A$ so, by Lemma 1.5,

$$bx^{n+1} = r_0 + r_1bx + r_2bx^2 + \cdots + r_nbx^n,$$

where $r_0, r_1, \dots, r_n \in R$. It is an equality in the polynomial ring $R[x]$. This implies that $b = 0$ and we have a contradiction. Therefore, the algebra A is not finitely generated over R, and this contradicts our assumption that R is an sfg-ring. □

Lemma 3.5. *$(b) = (b^2)$ for all $b \in R$.*

Proof. It is clear when R is a field. Assume that R is not a field. Let $b \in R$ and suppose $(b^2) \neq (b)$. Then $b \notin (b^2)$. Consider the ideal $I := (b^2)$ and the factor ring $\overline{R} := R/I$. Let $\overline{b} = b + I$. Then $0 \neq \overline{b} \in \overline{R}$ and $\overline{b}^2 = 0$, so the ring $\overline{R}$ has a nonzero nilpotent. Hence, by Lemma 3.4, $\overline{R}$ is not an sfg-ring. However, by Lemma 3.2, this is an sfg-ring. Thus, we have a contradiction. □

Lemma 3.6. *The Jacobson radical $J(R)$ is equal to zero.*

Proof. Put $J := J(R)$. It follows from Lemma 3.1 that J is a finitely generated R-module. If $b \in J$ then, by Lemma 3.5, $b = ub^2$ for some $u \in R$, and so, $b \in J^2$. Thus, we have the equality $J^2 = J$. Now, by Nakayama's Lemma, $J = 0$. □

Lemma 3.7. *If R is local, then R is a field.*

Proof. Assume that R is local and M is the unique maximal ideal of R. Then M is the Jacobson radical of R. It follows from Lemma 3.6 that $M = 0$. Thus R is a field. □

Lemma 3.8. *Every prime ideal of R is maximal.*

Proof. Let P be a prime ideal of R and suppose P is not maximal. Then there exists a maximal ideal M such that $P \subset M$ and $M \neq P$. Let $b \in M \setminus P$. It follows from Lemma 3.5 that $b = ub^2$ for some $u \in R$. Then

$$b(1 - ub) = 0 \in P.$$

But $b \notin P$, so $1 - ub \in P \subset M$. Hence, $b \in M$ and $1 - ub \in M$. This implies that $1 \in M$, that is, $M = R$. However $M \neq R$, so we have a contradiction. □

Lemma 3.9. *R is Artinian.*

Proof. We already know by Lemma 3.1 that R is Noetherian. Moreover we know, by Lemma 3.8 that the Krull dimension of R is equal to 0. Using a basic fact of commutative algebra (see for example [2] or [3] 99) we deduce that R is Artinian. □

Now we are ready to prove the mentioned proposition which is the main result of this section.

Proposition 3.10. *Every sfg-ring is a finite product of fields.*

Proof. Let R be an sfg-ring. We already know (by Lemma 3.9) that R is Artinian. It is known (see for example [2] or [3]) that every Artinian ring is a finite product of some local Artinian rings. Hence,

$$R = R_1 \times R_2 \times \cdots \times R_s,$$

where $R_1, \ldots, R_s$ are local Artinian rings. Since all projections $\pi_j : R \to R_j$ (for $j = 1, \ldots, s$) are surjections of rings, it follows from Lemma 3.2 that all the rings $R_1, \ldots, R_s$ are sfg-rings. Moreover, they are local so, by Lemma 3.7, they are fields. □

According to the above proposition we know that if R is an sfg-ring, then R is a finite product of fields. In the next sections we will prove that the opposite implication is also true.

4. Initial coefficients

Let us assume that R is a ring which is not a field, and A is an R-subalgebra of the R-algebra $R[x]$. Let us denote by $\mathcal{W}_A$ the set of all nonzero initial coefficients of polynomials of positive degree belonging to A. Note three lemmas concerning this set.

Lemma 4.1. *Let $a \in \mathcal{W}_A$. Then the polynomial ax is integral over A.*

Proof. There exists a polynomial $f(x) = ax^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 \in A$, with $n \geqslant 1$ and $r_0, \ldots, r_{n-1} \in R$. Let $g(x) = a^{n-1}f(x)$. Then

$$g(x) = (ax)^n + r_{n-1}(ax)^{n-1} + ar_{n-2}(ax)^{n-2} + \cdots + r_1a^{n-2}(ax) + r_0a^{n-1}$$

is also a polynomial belonging to A. Consider the polynomial

$$H(t) = t^n + r_{n-1}t^{n-1} + ar_{n-2}t^{n-2} + \cdots + r_1a^{n-2}t + r_0a^{n-1} - g(x)\,.$$

It is a monic polynomial in the variable t and all its coefficients are in A. Since $H(ax) = g(x) - g(x) = 0$, the element ax is integral over A. □

Lemma 4.2. *If R is Noetherian and $\mathcal{W}_A$ contains an invertible element, then the algebra A is finitely generated over R.*

Proof. Let $a \in \mathcal{W}_A$ be invertible in R. Then, by Lemma 4.1, the variable x is integral over A and this means that the ring $R[x]$ is integral over A. Hence, by Lemma 1.3, the algebra A is finitely generated over R. □

Lemma 4.3. *Let $a, r \in R$. If $a \in \mathcal{W}_A$ and $ra \neq 0$, then $ra \in \mathcal{W}_A$.*

Proof. Assume that $f = ax^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \in A$ with $n \geqslant 1$. Then rf is a polynomial belonging to A and the initial coefficient equals $ra \neq 0$. Hence, $ra \in \mathcal{W}_A$. □

Consider for example the ring $\mathbb{Z}_6$. Using the above lemmas we will show that $\mathbb{Z}_6$ is an sfg-ring. Let $R = \mathbb{Z}_6$, and let $A \subset R[x]$ be an R-subalgebra. We need to show that A is finitely generated over R. It is clear if $\mathcal{W}_A = \emptyset$, because in this case $A = R$. If $\mathcal{W}_A$ contains an invertible element of R (in our case 1 or 5) then, by Lemma 4.2, it is also clear.

Let us assume that $\mathcal{W}_A \subset \{2, 3, 4\}$. Since $2 \cdot 2 = 4$ and $2 \cdot 4 = 2$ in $\mathbb{Z}_6$, we have $4 \in \mathcal{W}_A \iff 2 \in \mathcal{W}_A$. If $3 \in \mathcal{W}_A$ and $4 \in \mathcal{W}_A$ then, by Lemma 4.1, the polynomials $4x$ and $3x$ are integral over A, and then $R[x]$ is integral over A, because $x = 4x - 3x$, and in this case, by Lemma 1.3, the algebra A is finitely generated over R.

Assume that $\mathcal{W}_A = \{2, 4\}$, and let $f(x) = 4x^n + r_{n-1}x^{n-1} + \cdots + r_1x + r_0 \in A$ where $n \geqslant 1$ and $r_0, \ldots, r_{n-1} \in \mathbb{Z}_6$. Since $r_0 = r_0 \cdot 1 \in A$, we may assume that $r_0 = 0$. The polynomial $3f(x)$ also belongs to A. Hence, $3r_{n-1}x^{n-1} + \cdots + 3r_1x \in A$.

Suppose that for some $j \in \{1, \ldots, n-1\}$ we have $3r_j \neq 0$. Let us take the maximal j. Then $3r_j \in \mathcal{W}_A = \{2, 4\}$, so $r_j = 0, 2$ or 4 and in every case we have a contradiction, because $3r_j \neq 0$. Therefore, all the elements $3r_1, \ldots, 3r_{n-1}$ are zeros. This means that $r_i = 4b_i$ with $b_i \in \mathbb{Z}_6$, for all $i = 1, \ldots, n-1$. Observe that 4 is an idempotent in $\mathbb{Z}_6$. We have $4 = 4^m$ for every positive integer m. Hence,

$$\begin{aligned} f(x) &= 4x^n + 4b_{n-1}x^{n-1} + 4b_{n-2}x^{n-2} + \cdots + 4b_1x \\ &= (4x)^n + b_{n-1}(4x)^{n-1} + \cdots + b_1(4x)^1 \end{aligned}$$

and hence, A is a $\mathbb{Z}_6$-subalgebra of the $\mathbb{Z}_6$-algebra $\mathbb{Z}_6[4x]$. In this case $4 \in \mathcal{W}_A$ so, by Lemma 4.1, the monomial $4x$ is integral over A and so, the ring $\mathbb{Z}_6[4x]$ is integral over A. Therefore, by Lemma 1.3, the algebra A is finitely generated over $R = \mathbb{Z}_6$.

Now let us assume that $\mathcal{W}_A = \{3\}$. In this case we use a similar way, as in the previous case. We show that A is a subalgebra of $\mathbb{Z}_6$-algebra $\mathbb{Z}_6[3x]$ and, using again Lemma 1.3, we see that A is finitely generated over $\mathbb{Z}_6$. Therefore we proved that $\mathbb{Z}_6$ is an sfg-ring.

5. Finite products of fields

In this section we prove that every finite product of fields is an sfg-ring. Throughout this section

$$R = K_1 \times K_2 \times \cdots \times K_n,$$

where $K_1, \ldots, K_n$ are fields. It is clear that the ring R is Noetherian, and even Artinian. Let A be an R-subalgebra of $R[x]$. We will show that A is finitely generated over R. We know, by Theorem 2.1, that it is true for $n = 1$. Now we assume that $n \geqslant 2$.

Let us fix the following notations:

$$\begin{aligned} &N = \{1, 2, \ldots, n\}\,; \\ &e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, \ldots, 0), \quad \ldots, \quad e_n = (0, 0, \ldots, 1)\,; \\ &I = \{i \in N;\ e_i \in \mathcal{W}_A\}\,; \\ &J = N \setminus I\,; \\ &\varepsilon = \sum_{i \in I} e_i\,. \end{aligned}$$

Observe that if $I = \emptyset$, then $A = R$ and nothing to prove. We know, by Lemma 4.1, that if $i \in I$, then $e_i x$ is an integral element over A. If $I = N$, then the variable x is integral over A, because $x = (1, 1, \ldots, 1)x = \sum_{i=1}^{n} e_i x$, and in this case, by Lemma 1.3, the algebra A is finitely generated over R. Hence, we will assume that $I \neq \emptyset$ and $I \neq N$. Without loss of generality we may assume that

$$I = \{1, 2, \ldots, s\}, \quad J = \{s+1, \ldots, n\}, \quad \text{where } 1 \leqslant s < n\,,$$

and $\varepsilon = e_1 + \cdots + e_s$. Note two simple lemmas. The first one is obvious.

Lemma 5.1. *Let u be an element of R such that $ue_j = 0$ for all $j \in J$. Then $u = \varepsilon u$.*

Lemma 5.2. *Let $u \in R$. If $u \in \mathcal{W}_A$, then $u = \varepsilon u$.*

Proof. Let $u = (u_1, \dots, u_n)$ and assume that $u \in \mathcal{W}_A$. Suppose there exists $j \in J$ such that $ue_j \neq 0$. Then u_j is a nonzero element of the field K_j, and $vu = e_j$, where $v = (0, \dots, 0, u_j^{-1}, 0, \dots, 0)$. Hence, $e_j = v \cdot ue_j$ and so, by Lemma 4.3, the element e_j belongs to $\mathcal{W}_A$. But it is a contradiction, because $j \in J = N \setminus I$. Therefore, $ue_j = 0$ for all $j \in J$ and so, by Lemma 5.1, we have $u = \varepsilon u$. □

Now consider the R-subalgebra B of $R[x]$, defined by

$$B = R\,[e_1 x, e_2 x, \dots, e_s x]\ .$$

We will prove that $A \subset B$, that is, that B is a subalgebra of A.

Let f be an arbitrary element of A. If $\deg f = 0$, then obviously $f \in B$. Assume that $\deg f \geqslant 1$ and $u \in R$ is the initial coefficient of f. Since $R \subset A$, we may assume that the constant term of f is equal to zero, Then we have

$$f = ux^n + d_1 x^{n_1} + d_2 x^{n_2} + \cdots + d_p x^{n_p},$$

where $d_1, \dots, d_p$ are nonzero elements of R, and $n > n_1 > n_2 > \cdots > n_p \geqslant 1$. It follows from Lemma 5.2 that $u = \varepsilon u$.

Let $j \in J$. Then $ue_j = u\,(\varepsilon e_j) = u0 = 0$ and then

$$e_j f = e_j d_1 x^{n_1} + e_j d_2 x^{n_2} + \cdots + e_j d_p x^{n_p} \in A\,.$$

Suppose $e_j d_q \neq 0$ for some $q \in \{1, \dots, p\}$. Let us take the minimal q. Then $0 \neq e_j d_q \in \mathcal{W}_A$. Put $d_q = (c_1, \dots, c_n)$ with $c_i \in K_i$ for all $i = 1, \dots, n$. Since $e_j d_q \neq 0$, we have $c_j \neq 0$ and so, $v d_q = e_j$, where $v = \big(0, \dots, 0, c_j^{-1}, 0, \dots, 0\big)$. This implies that $e_j = v(e_j d_q) \in \mathcal{W}_A$. But $e_j \notin \mathcal{W}_A$, because $j \in J = N \setminus I$. Hence, we have a contradiction.

Therefore, all the elements $e_j d_1, \dots, e_j d_p$ are zeros, and such situation is for all $j \in J$. This means, by Lemma 5.1, that $d_1 = \varepsilon d_1,\ \dots,\ d_p = \varepsilon d_p$. Observe that the element ε is an idempotent of R, so $\varepsilon = \varepsilon^m$ for $m \geqslant 1$. Hence,

$$\begin{aligned} f &= ux^n + d_1 x^{n_1} + d_2 x^{n_2} + \cdots + d_p x^{n_p} \\ &= u\varepsilon x^n + d_1 \varepsilon x^{n_1} + d_2 \varepsilon x^{n_2} + \cdots + d_p \varepsilon x^{n_p} \\ &= u\varepsilon^n x^n + d_1 \varepsilon^{n_1} x^{n_1} + d_2 \varepsilon^{n_2} x^{n_2} + \cdots + d_p \varepsilon^{n_p} x^{n_p} \\ &= u(\varepsilon x)^n + d_1 (\varepsilon x)^{n_1} + d_2 (\varepsilon x)^{n_2} + \cdots + d_p (\varepsilon x)^{n_p}\,, \end{aligned}$$

and hence, the polynomial f belongs to the ring $R[\varepsilon x]$. But

$$R[\varepsilon x] \subset R[e_1 x, e_2 x, \dots, e_s x] = B\,,$$

so $f \in B$. Thus, we proved that A is an R-subalgebra of B. Let us recall that all the monomials $e_1 x, \dots, e_s x$ are integral over A. Hence, the ring B is integral over A. It follows from Lemma 1.3 that A is finitely generated over R. Therefore, we proved the following proposition.

Proposition 5.3. *Every finite product of fields is an sfg-ring.*

Immediately from this proposition and Proposition 3.10 we obtain the following main result of this article.

Theorem 5.4. *A ring R is an sfg-ring if and only if R is a finite product of fields.*

Now, by this theorem and the Chinese Remainder Theorem, we have

Colorary 5.5. *The ring $\mathbb{Z}_m$ is an sfg-ring if and only if m is square-free.*

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Nicolaus Copernicus University, Faculty of Mathematics and Computer Sciences, ul. Chopina 12/18, 87-100 Toru, Poland

E-mail address: anow@mat.uni.torun.pl

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EXTREMAL PROPERTIES OF LINE ARRANGEMENTS IN THE COMPLEX PROJECTIVE PLANE

PIOTR POKORA

Abstract. In the present note we study some extreme properties of point-line configurations in the complex projective plane from a viewpoint of algebraic geometry. Using Hirzebruch-type inequalites we provide some new results on r-rich lines, symplicial arrangements of lines, and the so-called free line arrangmenets.

1. Introduction

In the present note we study some classical questions in the theory of point-line configurations in the complex projective plane using tools from algebraic geometry. This path is rather classical, and it dates back to the famous Hirzebruch's inequality [9] which can be viewed as a main tool in the subject. Let us recall that if $\mathcal{L} = \{\ell_1, ..., \ell_d\} \subset \mathbb{P}^2_{\mathbb{C}}$ is an arrangement of $d \geq 6$ lines such that there is no point where all the lines meet and there is no point where $d-1$ lines meet simultaneously, then

$$t_2 + t_3 \geq d + \sum_{r \geq 5} (r-4)t_r,$$

where t_r denotes the number of r-fold points, i.e., points where exactly r-lines from the arrangement meet. Hirzebruch's inequality can be found in many papers devoted to combinatorics, and one of the most important problems is to prove Hirzebruch's inequality using only combinatorial methods [3, p. 315; Problem 7]. This problem is motivated mostly due to Hirzebruch's approach, namely he used the theory of Hirzebruch-Kummer covers of the complex projective plane branched along line arrangmenets. Moreover, Hirzebruch's inequality is (only) a very strong by-product of the whole story since the main aim was to construct new examples of

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complex compact ball-quotient surfaces, i.e., minimal complex compact algebraic surfaces of general type such that the universal cover is the complex unit ball. The very first observation which comes from Hirzebruch's inequality is that every complex line arrangement has always double or triple intersection points. The real counterpart of Hirzebruch's inequality is the classical Melchior's result [10] which tells us that for a real line arrangement $\mathcal{A}$ (defined over the real numbers) which is not a pencil of lines one always has

$$t_2 \geq 3 + \sum_{r\geq 4}(r-3)t_r,$$

and the equality holds if and only if $\mathcal{A}$ is a simplicial line arrangement. Melchior's inequality provides an alternative proof of the dual orchard problem – every real line arrangement which is not a pencil has at least one double intersection point.

It is worth emphasizing that Hirzebruch's inequality is proved using, in the final step, the Bogmolov-Miyaoka-Yau inequality [11] which tells us that for a smooth complex projective surface with Kodaira dimension ≥ 0 one always has

$$K_X^2 \leq 3e(X),$$

where K_X is the canonical divisor and $e(X)$ denotes the topological Euler characteristic. It was very desirable to have meaningful generalizations of the Bogomolov-Miyaoka-Yau inequality to the case of pairs (X, D), where X is a normal complex projective surface and D is a boundary divisor, and now we have several choices – the most powerful is the orbifold Euler characteristic. It turns out that using it we can show the following result which is due to Bojanowski [2].

Theorem 1.1 (Bojanowski)**.** *Let $\mathcal{L} = \{\ell_1, ..., \ell_d\}$ be a finite set of lines in the complex projective plane. Assume that $t_r = 0$ for $r \geq \frac{2d}{3}$, then*

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{r\geq 5}\left(\frac{r^2}{4} - r\right)t_r.$$

The main aim of the present note is to apply Bojanowski's result in the context of certain questions, extremal in their nature, for point-line configurations. The note is inspired mostly by F. de Zeeuw's paper [6], and we are going to follow his path in the context of r-rich lines.

2. On r-rich lines

Let $\mathcal{P} = \{P_1, ..., P_n\}$ be a finite set of mutually distinct points in the complex projective plane (some of our results should be also formulated over the reals where obtained bounds are usually much better). Then we denote by ℓ_r the number of r-rich lines, i.e., those lines in the plane containing exactly r-points from the configuration $\mathcal{P}$. We are going to use the dual version of Bojanowski's inequality.

Theorem 2.1 (Bojanowski)**.** *Let $\mathcal{P} = \{P_1, ..., P_n\}$ be a finite set of mutually distinct points in the complex projective plane. Assume that there is no subset of $\frac{2n}{3}$ points which are collinear, then*

$$\ell_2 + \frac{3}{4}\ell_3 \geq n + \sum_{r\geq 5}\left(\frac{r^2}{4} - r\right)\ell_r.$$

Using Bojanowski's inequality, we can derive very strong bounds on r-rich lines, namely

a) $f_1 := \sum_{r\geq 2} r\ell_r \geq \frac{n(n+3)}{3}$;
b) $f_2 := \sum_{r\geq 2} r^2\ell_r \geq \frac{4n^2}{3}$.

The first result is (strong) Beck's theorem on point configurations in the complex projective plane which was proved by de Zeeuw [6].

Theorem 2.2. *Let $\mathcal{P} = \{P_1, ..., P_n\}$ be a finite set of mutually distinct points in the complex projective plane. Assume that there is no subset of $\frac{2n}{3}$ points which is collinear, then*

$$\sum_{r\geq 2}\ell_r \geq \frac{n^2 + 6n}{12}.$$

Now we are ready to give our proof of Beck's theorem.

Proof. Using (dual) Hirzebruch's inequality we see that

$$4f_0 - f_1 \geq n + \ell_2,$$

where $f_0 := \sum_{r\geq 2}\ell_r$. Then

$$4f_0 - f_1 + f_1 \geq n + \ell_2 + f_1 \geq n + \frac{n^2 + 3n}{3} \geq \frac{n^2 + 6n}{3},$$

so we arrive at

$$f_0 \geq \frac{n^2 + 6n}{12}.$$

□

Looking at Hirzebruch's inequality, we see that for point configurations (except the case when all the points are collinear or all but one point are collinear) one has

$$\ell_2 + \ell_3 \geq n.$$

Taking into account that Bojanowski's inequality is more acurate, we can formulate the following conjecture as it was suggested by de Zeeuw [7, Conjecture 4.5].

Conjecture 2.3. *For point configurations in $\mathbb{C}^2$ which do not have large pencils as subconfigurations (i.e., not too many points are collinear) one has*

$$\ell_2 + \ell_3 \geq c \cdot n^2$$

for a positive constant c.

If we restrict our attention to a real point configuration, one can show that if $\mathcal{P} \subset \mathbb{P}^2_{\mathbb{R}}$ is a finite set of n points such that at most $\alpha \cdot n$ are collinear, where $\alpha = \frac{6+\sqrt{3}}{9}$, then

$$(\triangle) \qquad \ell_2 + \ell_3 \geq \frac{1}{18}n^2.$$

This bound follows from an improvement on Beck's theorem on two extremes proved by de Zeeuw [6, Corollary 2.3].

Theorem 2.4 (Beck's theorem on two extremes)**.** *Let $\mathcal{P}$ be a finite set of n points in $\mathbb{P}^2_{\mathbb{R}}$, then one of the following is true:*

- *There is a line which contains more than $\alpha \cdot n$ points of $\mathcal{P}$, where $\alpha = \frac{6+\sqrt{3}}{9}$.*
- *There are at least $\frac{n^2}{9}$ lines spanned by $\mathcal{P}$.*

Now we are ready to show $(\triangle)$.

Proof. If $\mathcal{P}$ is a finite set of points, then we have

$$\ell_2 \geq 3 + \sum_{r \geq 4}(r-3)t_r.$$

Adding $\ell_2 + 2\ell_3$ on both sides we obtain

$$2\ell_2 + 2\ell_3 \geq 3 + \ell_2 + 2\ell_3 + \sum_{r \geq 4}(r-3)t_r \geq 3 + \sum_{r \geq 2}\ell_r.$$

If at most $\alpha \cdot n$ points from $\mathcal{P}$ are collinear with $\alpha = \frac{6+\sqrt{3}}{9}$, then

$$2\ell_2 + 2\ell_3 \geq 3 + \sum_{r \geq 2}\ell_r \geq \frac{n^2}{9},$$

which completes the proof. □

Over the complex numbers, we can only show the following bound, which takes into account also quadruple points.

Theorem 2.5. *Let $\mathcal{P} = \{P_1, ..., P_n\}$ be a point configuration in the complex projective plane such that no subset of $\frac{2n}{3}$ is collinear. Then*

$$\ell_2 + \ell_3 + \ell_4 \geq \frac{n(n+15)}{18}.$$

Proof. Using Bojanowski's inequality we have

$$l_3 + \frac{3}{4}l_3 \geq n + \sum_{r \geq 5}\frac{r^2-4r}{4}l_r.$$

Now we need to observe that for $r \geq 5$ one has

$$\frac{r^2-4r}{4} \geq \frac{1}{8} \cdot \frac{r^2-r}{2},$$

and using the combinatorial count

$$\binom{n}{2} = l_2 + 3l_3 + 6l_4 + \sum_{r\geq 5}\binom{r}{2}l_r$$

we obtain that

$$l_2 + \frac{3}{4}l_3 \geq n + \frac{1}{8}\left(\binom{n}{2} - l_2 - 3l_3 - 6l_4\right).$$

Simple manipulations give

$$\frac{9}{8}(l_2 + l_3 + l_4) \geq \frac{9}{8}l_2 + \frac{9}{8}l_3 + \frac{6}{8}l_4 \geq \frac{n(n+15)}{16},$$

so finally we obtain

$$l_2 + l_3 + l_4 \geq \frac{n(n+15)}{18}.$$

□

3. Simplicial line arrangements

Definition 3.1. Let $\mathcal{A} = \{H_1, ..., H_d\}$ be a central arrangement of $d \geq 3$ hyperplanes in $\mathbb{R}^3$ (so it provides an arrangement of lines in the real projective plane). We say that $\mathcal{A}$ is simplicial if each connected components of the complement of $\mathcal{A}$ in $\mathbb{R}^3$ is a simplicial cone.

It is well-known, by Melchior's result, that $\mathcal{A}$ is a simplicial line arrangement if and only if the following equality holds

$$t_2 = 3 + \sum_{r\geq 4}(r-3)t_r.$$

We will also need the following folklore result on the multiplicity of an *irreducible* simplicial line arrangement in the real projective plane (i.e., the maximal multiplicity of singular points).

Definition 3.2. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be central arrangements in $\mathbb{K}^\ell$ and $\mathbb{K}^m$, where $\mathbb{K}$ is any field, with defining polynomials $Q_1(x_1, ..., x_\ell)$ and $Q_2(x_1, ..., x_m)$, respectively. The product arrangement $\mathcal{A}_1 \times \mathcal{A}_2$ is the arrangement in $\mathbb{K}^{\ell+m} = \mathbb{K}^\ell \times \mathbb{K}^m$ with defining polynomial

$$Q(x_1, ..., x_{\ell+m}) = Q_1(x_1, ..., x_\ell) \cdot Q_2(x_{\ell+1}, ..., x_{\ell+m}).$$

We say that a central arrangement $\mathcal{A}$ is *irreducible* if $\mathcal{A}$ cannot be expressed as a product arrangement.

Theorem 3.3 (Folklore)**.** *Let $\mathcal{A} \subset \mathbb{P}^2_{\mathbb{R}}$ be an irreducible simplicial line arrangement, then the multiplicity of $\mathcal{A}$ is $\leq \frac{d}{2}$.*

An interested reader might want to consult [8, Proposition 2.1] for a modern proof of the above result.

We would like to add the following observation to the above list of constraints.

Proposition 3.4. *Let $\mathcal{A}$ be an irreducible simplicial arrangement in the real projective plane, then*

$$t_3 + t_4 + t_5 \geq d - 3.$$

Proof. By Melchior's result,

$$t_2 = 3 + \sum_{r\geq 3}(r-3)t_r$$

and we can plug it into Bojanowski's inequality obtaining

$$3 + \sum_{r\geq 4}(r-3)t_r + \frac{3}{4}t_3 \geq d + \sum_{r\geq 4}\left(\frac{r^2-4r}{4}\right)t_r.$$

It leads to

$$3t_3 \geq 4(d-3) + \sum_{r\geq 4}\Big(r^2-8r+12\Big)t_r = 4(d-3) - 4t_4 - 3t_5 + \sum_{r\geq 6}\Big(r^2-8r+12\Big)t_r.$$

Then we have

$$3t_3 + 4t_4 + 3t_5 \geq 4(d-3),$$

which completes the proof. □

4. Combinatorics and the freeness of line arrangements

Let $\mathcal{A} = \{H_1, ..., H_n\}$ be an essential and central hyperplane arrangement in $\mathbb{C}^3$, it means that $H_i = V(\ell_i)$ for homogeneous linear form ℓ_i and $\bigcap_{i=1}^n H_i = 0 \in \mathbb{C}^3$ – the last condition tells us that $\mathcal{A}$ also defines an arrangement of lines in $\mathbb{P}^2_{\mathbb{C}}$. The main combinatorial object which can be associated with $\mathcal{A}$ is the intersection lattice $L_{\mathcal{A}}$ – it consists of the intersections of the elements of $\mathcal{A}$, ordered by reverse inclusion. In this setting, $\mathbb{C}^3$ is the lattice element $\tilde{0}$ and the rank one elements of $L_{\mathcal{A}}$ are the planes. In this section we denote by S the polynomial ring $\mathbb{C}[x, y, z]$,

Definition 4.1. The Möbius function $\mu : L_{\mathcal{A}} \to \mathbb{Z}$ is defined as

$$\mu(\tilde{0}) = 1,$$
$$\mu(t) = -\sum_{s<t}\mu(s), \text{ if } \tilde{0} < t.$$

Definition 4.2. The Poincaré and the characteristic polynomials of $\mathcal{A}$ are defined as

$$\pi(\mathcal{A}, t) = \sum_{x\in L_{\mathcal{A}}}\mu(x)\cdot(-t)^{\mathrm{rank}(x)}, \text{ and } \chi(\mathcal{A}, t) = t^{\mathrm{rank}(\mathcal{A})}\pi\left(\mathcal{A}, \frac{-1}{t}\right).$$

Definition 4.3. The module of $\mathcal{A}$-derivations is the submodule of $\mathrm{Der}_{\mathbb{C}}(S)$ consisting of vector fields tangent to $\mathcal{A}$, namely

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}_{\mathbb{C}}(S) \,|\, \theta(\ell_i) \in \langle \ell_i\rangle \text{ for all } \ell_i \text{ with } \mathrm{Zeros}(\ell_i) \in \mathcal{A}\}.$$

Definition 4.4. An arrangement is free when $D(\mathcal{A})$ is a free S-module.

Theorem 4.5 (Terao's factorization)**.** *If $D(\mathcal{A})$ is free, then*

$$\pi(\mathcal{A}, t) = (1+t)(1+a_1 t)(1+a_2 t).$$

Now we would like to present the main result for this section.

Theorem 4.6. *Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of d lines with $t_r = 0$ for $r > \frac{2d}{3}$. Assume that $\mathcal{L}$ is free, then*

$$\sum_{r\geq 2}(r-4)^2 t_r \geq 12.$$

Proof. Let us recall that for an arrangement of lines $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ the Poincaré polynomial has the following form

$$\pi(\mathcal{L}, t) = 1 + dt + \Big(\sum_{r\geq 2}(r-1)t_r\Big)t^2 + \Big(\sum_{r\geq 2}(r-1)t_r + 1 - d\Big)t^3,$$

which follows from simple calculations using the Möbius function – for each line $\ell_i \in \mathcal{L}$ we have that $\mu(\ell) = -1$, and for each point $P \in L(\mathcal{L})$ of multiplicity r we have $\mu(P) = r-1$. Since $\mathcal{L}$ is central, then $(1+t)$ divides $\pi(\mathcal{L}, t)$, which follows from the fact that the Euler derivation is always an element of $D(\mathcal{L})$ [5, Section 8.1], and it leads to the following presentation

$$\pi(\mathcal{L}, t) = (1+t)\Big(1 + (d-1)t + \Big(\sum_{r\geq 2}(r-1)t_r + 1 - d\Big)t^2\Big).$$

Now the freeness of $\mathcal{L}$ implies that

$$(d-1)^2 - 4\cdot\Big(\sum_{r\geq 2}(r-1)t_r - d + 1\Big) = d^2 + 2d - 3 - 4\sum_{r\geq 2}(r-1)t_r \geq 0.$$

By the standard combinatorial count

$$d(d-1) = \sum_{r\geq 2} r(r-1)t_r$$

one obtains

$$3d + \sum_{r\geq 2}\Big(r^2 - 5r + 4\Big)t_r \geq 3. \tag{$\star$}$$

Using Bojanowski's inequality, we get

$$-\sum_{r\geq 2}\Big(\frac{r^2}{4} - r\Big)t_r \geq d$$

and this leads us to

$$-3\sum_{r\geq 2}\Big(\frac{r^2}{4} - r\Big)t_r + \sum_{r\geq 2}\Big(r^2 - 5r + 4\Big)t_r \geq 3,$$

and we finally obtain

$$\sum_{r\geq 2}(r-4)^2 t_r \geq 12.$$

□

Our result gives us some insights in the context of free line arrangements with small number of lines. Assume that we want to find a free arrangement of $d \geq 6$ lines having only triple points. Our inequality implies that $t_3 \geq 12$, and we know that the dual Hesse arrangement of $d = 9$ lines with $t_3 = 12$ is free, so our lower bound is sharp.

The next result of the section gives a lower bound on the number of double and triple points for free line arrangements.

Proposition 4.7. *Let $\mathcal{L}$ be a free arrangement of d lines such that $t_r = 0$ for $r \geq \frac{2d}{3}$. Then*

$$2t_2 + t_3 \geq d + 3.$$

Proof. Since $\mathcal{L}$ is free, we can use condition $(\star)$, namely

$$3d - 3 + \sum_{r\geq 2}(r^2 - 5r + 4)t_r \geq 0$$

since the Poincaré polynomial splits over the integers. This leads to

$$2t_2 + 2t_3 \leq 3d - 3 + \sum_{r\geq 5}(r^2 - 5r + 4)t_r \leq 3d - 3 + \sum_{r\geq 5}(r^2 - 4r)t_r.$$

Using Bojanowski's inequality

$$4t_2 + 3t_3 - 4d \geq \sum_{r\geq 5}(r^2 - 4r)t_r$$

we obtain

$$2t_2 + 2t_3 \leq 3d - 3 - 4d + 4t_2 + 3t_3,$$

so finally we get

$$d + 3 \leq 2t_2 + t_3,$$

which completes the proof. □

Observe that the above inequality is sharp for several free arrangements of lines, the simplest one is a star-configuration of $d = 3$ lines with 3 double points.

5. (n_k)-configurations in the complex projective plane

Definition 5.1. Let $\mathcal{L} \subset \mathbb{P}^2_{\mathbb{C}}$ be an arrangement of $n \geq 4$ lines, then $\mathcal{L}$ is called (n_k)-configuration if it consists of exactly n points of multiplicity k and we have exactly n lines in the arrangement with the property that on each line we have exactly k points of multiplicity k.

Let us observe here that usually one defines (n_k)-configurations as objects in the real projective plane, and we distinguish geometrical and topological configurations, i.e., geometrical are those which can be realized as straight lines, topological are those that can be realized with use of pseudolines. Let us recall here that a pseudoline is a simple closed curve in $\mathbb{P}^2_{\mathbb{R}}$ such that its removal does not cut $\mathbb{P}^2_{\mathbb{R}}$ in two connected components. The main open problem in this subject is to determine all those (n_k)-configurations which are geometrically realizable. Since the case of (n_3)-configurations is completely characterized, and for (n_4)-configurations the only open case is when $n = 23$ due to an interesting results by Cuntz [4], so we assume from now on that $k \geq 5$. We will follow the last section from [1].

If we assume that $\mathcal{PL}$ is an (n_k)-configuration topologically realizable (i.e., is a pseudoline configuration) in the real projective plane, then we have the following Shnurnikov's inequality [14]:

$$t_2 + \frac{3}{2}t_3 \geq 8 + \sum_{r\geq 4}(2r - 7.5)t_r,$$

provided that $t_n = t_{n-1} = t_{n-2} = t_{n-3} = 0$. Using a local deformation argument for $\mathcal{PL}$ we can assume that our configuration has only k-fold and double points, so we have the following quantities:

$$t_k = n, \quad t_2 = \frac{n(n-1)}{2} - n \cdot \frac{k(k-1)}{2}.$$

Plugging this into Shnurnikov's inequality we obtain that

$$\frac{n(n-1)}{2} - n \cdot \frac{k(k-1)}{2} - (2k - 7.5)n - 8 \geq 0,$$

and this is a necessary condition for the existence of topological (n_k)-configurations. If we restrict our attention to $k = 6$, then we can easily see that there are no (n_6)-configurations if $n \leq 40$.

Assume now that $\mathcal{L}$ is a complex geometric (n_k)-configuration with the property that it has only double and k-fold points. Using Bojanowski's inequality we see that the following condition is necessary:

$$n^2 - n \cdot \left(\frac{3k^2 - 6k + 6}{2}\right) \geq 0,$$

so there are no such arrangements if we have

$$n \leq \frac{3k^2 - 6k - 4}{2}.$$

If we restrict our attention to $k = 6$, then the first non-trivial case is $n = 39$, and this is an extremely important open problem. If such a configuration exists, then we will be able to construct a new example of complex compact ball-quotient surface via Hirzebruch's construction, i.e., a minimal desingularization of the abelian cover of the complex projective plane branched along complex (39_6)-configuration. It is worth emphasizing here that ball-quotient surfaces constructed with use of abelian

covers are rather rare, and it would be very interesting to know whether we can construct a new example of such surfaces with use of line arrangements.

It seems to be quite difficult to decide whether the above (39_6)-configuration can potentially exists, and it is extreme from a viewpoint of the Bojanowski's inequality (it provides the equality). On the other hand, we can formulate the following problem.

Problem 5.2. Is it possible to construct complex (39_6)-configuration?

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Pedagogical University of Cracow, Department of Mathematics, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: `piotrpkr@gmail.com`

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A FEW INTRODUCTORY REMARKS ON LINE ARRANGEMENTS

JUSTYNA SZPOND

Abstract. Points and lines can be regarded as the simplest geometrical objects. Incidence relations between them have been studied since ancient times. Strangely enough our knowledge of this area of mathematics is still far from being complete. In fact a number of interesting and apparently difficult conjectures has been raised just recently. Additionally a number of interesting connections to other branches of mathematics have been established. This is an attempt to record some of these recent developments.

1. Introduction

In this note we consider arrangements of lines in the projective plane $\mathbb{P}^2$ coming from a standard construction over a field $\mathbb{K}$. That means, the points in the plane represent one dimensional linear subspaces of a 3-dimesional vector space over $\mathbb{K}$. For the most of the material presented here it is irrelevant what properties the field $\mathbb{K}$ enjoys. We will clearly mark the spots where this becomes important.

Definition 1 (Arrangement)**.** An arrangement of lines is a finite set of at least two mutually distinct lines in the projective plane.

The points where lines from a given arrangement intersect are of special interest.

Definition 2 (Singular points of an arrangement)**.** We say that a point $P \in \mathbb{P}^2$ is a *singular point* of an arrangement $\mathcal{L}$, if there are at least 2 lines in $\mathcal{L}$, which pass through P.
If P is a singular point of an arrangement $\mathcal{L}$, then its *multiplicity* is the number of lines in $\mathcal{L}$ passing through P.

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Note that the notions of a singular point and multiplicity agree with corresponding notions in algebraic geometry, if we look at an arrangement $\mathcal{L}$ as the divisor $\sum_{L\in\mathcal{L}} L$.

Given an arrangement $\mathcal{L}$, for an integer $r \geq 2$, we denote by t_r the number of points of multiplicity r. Since in the projective plane any pair of lines has an intersection point, we have the following fundamental combinatorial equality

$$\binom{d}{2} = \sum_{r\geq 2} t_r \binom{r}{2}, \tag{1}$$

where d is the number of lines in $\mathcal{L}$.

The numbers $t_2, \ldots, t_d$ are basic numerical invariants of an arrangement. We consider them as coordinates of a vector $T = (t_2, t_3, \ldots, t_d)$, which we call the *$T$-vector* of an arrangement. The following natural question has motivated a lot of research, nevertheless it remains widely open.

Problem 3 (Geometrical realisability of T-vectors)**.** Decide which $(d-1)$-tuples of integers $a_2, a_3, \ldots a_d$ arise as T-vectors of line arrangements.

The equality (1) is just one of many constrains for a $(d-1)$-tuple to be a T-vector. We refer to [15] for a number of additional constrains of similar nature.

Of course in some cases, there is an easy answer to Problem 3.

Example 4 (Star configuration)**.** A $d-1$-tuple $(\binom{d}{2}, \underbrace{0, \ldots, 0}_{d-2})$ is a T-vector of an arrangement of d *general* lines in $\mathbb{P}^2$. This means that there are just double points (points of multiplicity 2) in the arrangement, i.e., each pair of lines intersects in point but there are no additional incidences. Such general arrangements are also called *star configurations*, see [20] for an extensive account of this concept.

On the other extreme we the following two arrangements.

Example 5 (Pencil and near-pencil)**.** For any $d \geq 2$ the array

$$(\underbrace{0, 0, \ldots, 0}_{d-2}, 1)$$

is a T-vector of a *pencil*, i.e., an arrangement where all lines pass through the same point, i.e., there is just one singular point of multiplicity d.
Similarly, the array

$$(d-1, \underbrace{0, 0, \ldots, 0}_{d-4}, 1, 0)$$

is a T-vector of a *near-pencil*. This is an arrangement where all but one line pass through the same point P, which consequently has multiplicity $d-1$. The remaining line intersects lines passing through P in altogether $d-1$ double points.

A specific $(d-1)$-tuple for which it is not known if it comes up as a T-vector of a line arrangement, has $12^2+12=156$ entries, all of which are 0, with the exception of t_{13}, which is supposed to be 157. It is expected that this *is not* a T-vector of a line arrangement. More specifically, this is the first instance, where the existence of a finite projective plane (fpp in short) of certain order (here order 12) is not known, see [31] for a survey on fpps.

A highly non-trivial constrain in Problem 3 is the following theorem due to Erdös and de Bruijn [12].

Theorem 6 (Erdös and de Bruijn)**.** *Let $\mathcal{L}$ be an arrangement of d lines, which is not a pencil. Then*

$$\sum_{r\geq 2} t_r \geq d. \tag{2}$$

Moreover, there is equality in (2) *if and only if $\mathcal{L}$ is either a near-pencil or it consists of all lines in a finite projective plane.*

2. Arrangements in the real projective plane

In this section we study line arrangements in the projective plane $\mathbb{P}^2(\mathbb{R})$. The central result here is that if $\mathcal{L}$ is not a pencil, then it must be

$$t_2 > 0.$$

In fact much more is known. The story begins with a question raised in 1821 by Jackson in a recreational mathematics collection [29]. We recall first its original formulation:

"Your aid I want nine trees to plant
In rows just half a score;
And let there be in each row three
Solve this: I ask no more".

Due to this formulation this problem is now known as the orchard problem. We present its two mathematical versions and then pass to generalizations.

Problem 7 (Orchard problem)**.**

a) In the original version Jackson asks if it is possible to put 9 points in the plane in such a way, that on any line through a pair of these points there is an additional point.
b) Passing to the dual version, i.e., exchanging the role of points and lines, we get a version directly related to Problem 3: Is there an arrangement of 9 real lines such that they intersect only in triple points? In other words, is the 8-tuple of integers with $t_3=12$ and all other entries zero a T-vector of a real line arrangement?

The Orchard Problem remained unsolved and went forgotten, before it was rediscovered and generalized by Sylvester in 1893 [38].

Problem 8 (Sylvester Problem)**.** Is the pencil the only arrangement of real lines which has no double points?

Sylvester interest in this question was probably motivated by a discovery of Hesse [26]. He found a non-trivial line arrangement with no double points over the complex numbers. More precisely, Hesse studied the problem in the dual version and found that 3-torsion points on an elliptic curve give rise to an interesting arrangement of 12 lines. We present his discovery in a version fitting better this notes, see [2] for an excellent account on this and related constructions.

Example 9 (Dual Hesse arrangement)**.** Let $\mathcal{L}$ be the arrangement of 9 lines defined in the complex projective plane with homogeneous coordinates $(x : y : z)$ by linear factors of the polynomial

$$(x^3 - y^3)(y^3 - z^3)(z^3 - x^3).$$

Then $t_3(\mathcal{L}) = 12$.

Remark 10. Note that the dual Hesse arrangement is a member of very interesting family of Fermat arrangement, see [40] for a recent survey on this kind of arrangements.

It took almost 50 years to answer Sylvester's question. In fact it went again forgotten and got revived by Erdös. The answer was given by another Hungarian mathematician, Tibor Gallai during the World War II and thus remained unpublished, see citation in [19].

Theorem 11 (Sylvester and Gallai)**.** *The only arrangements of lines in the real projective plane with $t_2 = 0$ are pencils.*

Over the years this theorem has been reproved in various manners. A proof attributed to Kelly by Coxeter in [8] is considered to be probably the most elegant one. For this reason it made the way to the famous "Proofs from The Book" by Aigner and Ziegler [1].

However, it was another proof which attracted a lot of attention and triggered new research directions. It is due to Melchior [34] and it provides strong numerical constrains to our central Problem 3. His method was to use in a clever way Euler's formula.

Theorem 12 (Melchior inequality)**.** *Let $\mathcal{L}$ be an arrangement of lines in the real projective plane which is not a pencil, then the following inequality holds:*

$$t_2 \geq 3 + \sum_{r \geq 3} (r - 3)t_r.$$

Melchior's result shows that a non-trivial real line arrangement must not only have positive t_2 but in fact there must be at least 3 double points. Interesting generalizations of Melchior's inequality in the realms of complex line arrangements

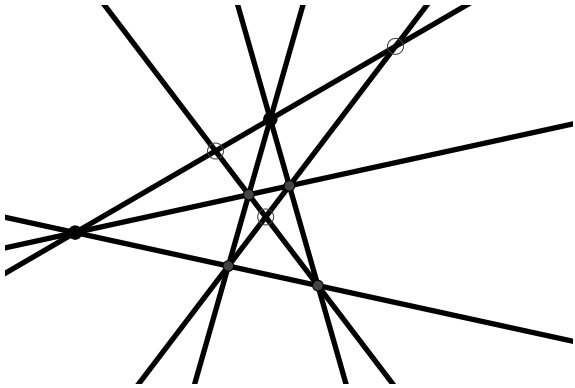

FIGURE 1. Kelly Moser arrangement

have been obtained by Hirzebruch [27], see also [39] and the article by Piotr Pokora in this volume [36].

It is then natural to wonder how many double points must there be and if this might depend on the number of lines. This problem became known as Dirac's conjecture.

Conjecture 13 (Dirac, 1951)**.** *Let $\mathcal{L}$ be an arrangement of d lines, not a pencil. If d is sufficiently large, then there must be*

$$t_2 \geq \left\lfloor \frac{d}{2} \right\rfloor .$$

The reason for the assumption that d is sufficiently large are the following two arrangements, which are in fact the only known arrangements where it is necessary to round down the term on the right in Dirac's conjecture.

Example 14 (Kelly and Moser)**.** Taking a complete quadrilateral with all its diagonals we get an arrangement of 7 lines with only 3 ordinary points. This are the points indicated by empty circles in Figure 1. All other singular points of this arrangement have multiplicity 3.

The next example is easier to explain in the dual form. In this setting we are interested in a finite set $\mathcal{P}$ of points in $\mathbb{P}^2$ and all lines determined by pairs of points in $\mathcal{P}$. The multiplicity of a line L is then the number of points in $\mathcal{P}$ which lie on it. This is in agreement with the multiplicity of the corresponding point L' in the dual projective plane since the points on L correspond to lines passing through L'. Our description is borrowed from Crowe and McKee survey [9].

Example 15 (McKee arrangement)**.** Let A, B, C, D, E be vertices of a regular pentagon. Let C', D', E' be images of C, D, E respectively under reflection along the line determined by points A and B. Let M be the midpoint of the segment AB. Finally, let I be the point at infinity on the line determined by C and D, let

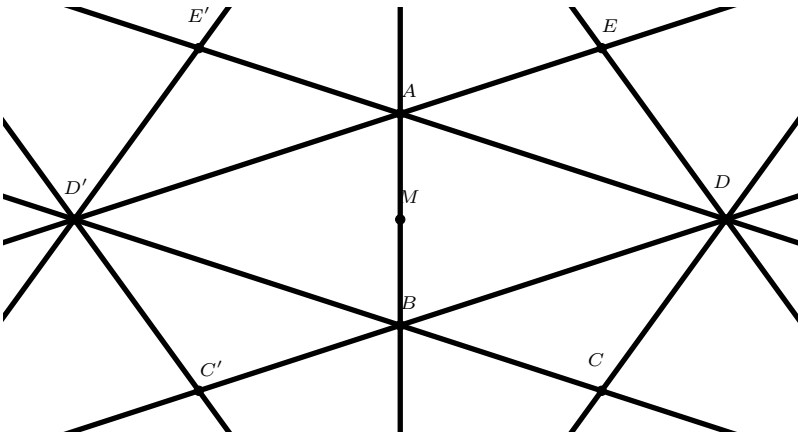

FIGURE 2. McKee configuration of points

K be the point at infinity on the line determined by D and E and let J and L be points at infinity in the direction of x and y axis respectively, see Figure 2 for points and most relevant lines in the affine part, the 4 points at the infinity are not visible. Let $\mathcal{P} = \{A, B, C, D, E, C', D', E', I, J, K, L, M\}$, then the only lines of multiplicity 2 with respect to $\mathcal{P}$ are those determined by the following pairs of points:

$$AJ,\ BJ,\ DL,\ D'L,\ MI,\ MK.$$

Dirac's conjecture has been proved in steps. In 1951 Motzkin [35] showed that for non pencil arrangements $t_2 \geq \sqrt{2d-2}$. In 1958 Kelly and Moser [30] showed that, apart of a pencil, it is always

$$t_2 \geq \frac{3}{7}d.$$

In 1993 Csima and Sawyer [10] showed that, apart of pencils and Example 14, it is always

$$t_2 \geq \frac{6}{13}d.$$

The final step so far has been made by Green and Tao [21]. They proved that Dirac's conjecture holds for d sufficiently large. Their proof allows for an effective estimate on what "sufficiently many means". The number one gets is in the magnitude of 10^5 and thus far away from 13. It is expected that the conjecture, without rounding down the number on the right, holds true for all $d \geq 14$.

We want to conclude this part with short glimpse at recent developments for complex line arrangements. We need first to define one more concept.

Definition 16 (Supersolvable arrangement)**.** We say that an arrangement $\mathcal{L}$ is *supersolvable*, if there exists a singular point of $\mathcal{L}$ which is connected by a line from

$\mathcal{L}$ to any other singular point of $\mathcal{L}$. A point with this property is called a *modular point*.

Example 17. A pencil and a near-pencil are supersolvable with all their singular points being modular.
A star configuration (see Example 4) of 4 or more lines is not supersolvable.

Whereas Example 9 shows that it might happen that $t_2 = 0$ for a non-trivial arrangement of lines in the complex projective planes, it is worth to point out that there are not too many such arrangement known. Moreover, none of known examples is supersolvable, see recent arXiv posting by Hanumanthu and Harbourne [23]. Thus it is reasonable to make the following conjecture, which parallels Sylvester's problem.

Conjecture 18 (Hanumanthu and Harbourne, 2019)**.** *Let $\mathcal{L}$ be an arrangement of complex lines which is supersolvable and which is not a pencil. Then $t_2 > 0$.*

This Conjecture is, of course, the first step towards the following bold conjecture by Anzis and Tohaneanu.

Conjecture 19 (Anzis and Tohaneanu, 2015)**.** *Let $\mathcal{L}$ be an arrangement of s complex lines which is supersolvable and which is not a pencil. Then $t_2 \geq \frac{s}{2}$.*

3. Line arrangements and the containment problem

In this section our story turns back to algebra and geometry. Building upon ideas of Swanson [37], Ein, Lazarsfeld and Smith [17] in characteristic zero and Hochster and Huneke [28] in positive characteristic (followed recently by Ma and Schwede [33] in mixed characteristic) proved that for a non-trivial homogeneous ideal I in a polynomial ring $R = \mathbb{K}[x_0, \ldots, x_N]$ there is always the containment

$$I^{(m)} \subset I^r \tag{3}$$

provided $m \geq Nr$. Here $I^{(m)}$ stays for the symbolic power of I, which is defined by

$$I^{(m)} = \bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} (I^m R_{\mathfrak{p}} \cap R).$$

Of course there are ideals, an extremal example coming from ideals generated by a regular sequence, where the containment in (3) holds for m much smaller than the bound provided above (it is well-known and easy to check that m must be at least r in order that (3) holds). On the other hand examples showing that one indeed needs $m \geq Nr$ have been discovered only recently and such examples are rather rare.

Since $I^1 = I$ and $I^{(m)} \subset I$ for any $m \geq 1$, the first nontrivial containment in (3) is that of $I^{(4)}$ in I^2 for an ideal I defining points in the projective plane $\mathbb{P}^2(\mathbb{K})$. Around 2000 Huneke asked if for an ideal of points in $\mathbb{P}^2$ one has

$$I^{(3)} \subset I^2 \tag{4}$$

and he asked for a simple proof of this fact. The containment in (4) has been verified in many interesting cases, see [5], [6], [24]. However Dumnicki, Szemberg and Tutaj-Gasińska [16] showed the first example where the containment in (4) fails. This example is delivered by singular points of the dual Hesse arrangement, i.e., of our Example 9.

Theorem 20 (Dumnicki and Szemberg and Tutaj-Gasińska)**.** *For the ideal I of points*

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:1), & P_5 = (1:1:\varepsilon), & P_6 = (1:1:\varepsilon^2), \\ P_7 = (1:\varepsilon:1), & P_8 = (1:\varepsilon:\varepsilon), & P_9 = (1:\varepsilon:\varepsilon^2), \\ P_{10} = (1:\varepsilon^2:1), & P_{11} = (1:\varepsilon^2:\varepsilon), & P_{12} = (1:\varepsilon^2:\varepsilon^2) \end{array}$$

the containment $I^{(3)} \subset I^2$ fails.

It has been quickly realized that singular points of other line arrangements also provide non-containment examples, see [25], [14], [3]. In this way also non-containment examples over the reals [11] and over the rationals [32] have been identified.

Interestingly all non-containment results identified so far in characteristic zero, show only that one needs the symbolic power 4 or higher in order to ensure the containment

$$I^{(m)} \subset I^2.$$

Passing to the third ordinary power and keeping I the ideal of points in $\mathbb{P}^2$, it is not even known if

$$I^{(5)} \subset I^3$$

might fail.

There is in fact a much more general conjecture due to Harbourne.

Conjecture 21 (Harbourne)**.** *Let I be a radical ideal in the ring of polynomials $R = \mathbb{K}[x_0, \ldots, x_N]$. Let c be the big height of I. Then for all integers $r \geq 1$,*

$$I^{(cr-c+1)} \subset I^r. \tag{5}$$

If I is an ideal of points, then its big height is equal N. Hence for $N = 2$ and $r = 2$ we are in the old containment $I^{(3)} \subset I^2$, which is false in general, as we know. However, in the case of complex numbers the conjecture can be potentially saved adding an important additional requirement that the containment in (5) holds for **all n sufficiently large**. Recent paper by Grifo, Huneke and Mukundan [22] puts Conjecture 21 in the asymptotic perspective.

4. Line arrangements and unexpected curves

In recent years we have witnessed extremely interesting developments in the theory of positivity of linear systems. In this presentation we restrict only to objects in the projective plane, which is just a piece of theory growing at awesome

speed. On the other hand, it is the projective plane where the story begins with the ground-breaking article by Cook II, Harbourne, Migliore and Nagel [7]. We follow their approach.

Definition 22 (Unexpected plane curves)**.** We say that a finite set Z of reduced points in $\mathbb{P}^2$ *admits an unexpected curve* of degree $m+1$, if for a general point P, the fat point scheme mP (i.e. defined by the ideal $I(P)^m$) fails to impose independent conditions on the linear system of curves of degree m vanishing at all points of Z. In other words, Z admits an unexpected curve of degree $m+1$ if

$$h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes I(Z+mP)) \ > \ \max\left\{h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d) \otimes I(Z)) - \binom{m+1}{2},\ 0\right\}.$$

It is clear that a single fat point imposes independent conditions on the *complete* linear systems of curves of fixed degree. Thus the empty set Z does not admit any unexpected curves. It came as a surprise that such non-empty sets do exist. The first example, which was the main motivation for [7], was discovered by Di Gennaro, Ilardi and Valles [13, Proposition 7.3]. This example comes from a reflexive line arrangement B_3, we refer to [40] for an extensive account on this kind of arrangements. Here we content ourselves with the definition of a root system and explicit equations of arrangement lines.

Definition 23 (Root system)**.** A *root system* is a finite collection R of vectors in an affine space V (in our case it will be $\mathbb{R}^3$) such that

- the elements in R span V;
- for each $\alpha \in R$, the vector $-\alpha$ is in R and no other multiple of α is there;
- for each α and β in R, the vector $s_\alpha(\beta)$ is also in R (here $s_\alpha(v) = v - 2\frac{\langle \alpha, v\rangle}{\langle \alpha, \alpha\rangle}\alpha$ is the reflection in the hyperplane perpendicular to α);
- for each α and β in R, the number $2\frac{\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle}$ is an integer.

Example 24 (The B_3 arrangement of lines)**.** The linear factors of the polynomial

$$xyz(x^2-y^2)(y^2-z^2)(z^2-x^2)$$

define an arrangement of lines in $\mathbb{P}^2$ which are reflections in the Weyl group generated by a B_3 root system. Passing to the dual setting we identify the projectivized B_3 root system consisting of points

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:0), & P_5 = (1:-1:0), & P_6 = (1:0:1), \\ P_7 = (1:0:-1), & P_8 = (0:1:1), & P_9 = (0:1:-1). \end{array}$$

Theorem 25 (Di Gennaro, Ilardi, Valles)**.** *Let Z be the set of* 9 *points defined in Example 24. Then Z admits an unexpected curve of degree* 4 *with a point of multiplicity* 3*.*

Explicit equations of curves whose existence is guaranteed by Theorem 25 have been found by Bauer, Malara, Szemberg and the author in [4]. Farnik, Galuppi,

Sodomaco and Trok showed in [18] that this is the unique unexpected quartic curve up to projective change of coordinates.

The set Z in Theorem 25 arises as dual points of an interesting arrangement of lines. Similarly as in Section 3 sets admitting unexpected curves arise also as singular points of certain line arrangements.

Theorem 26 (Dual Hesse arrangement and an unexpected quintic)**.** *Let Z be the set of singular points of the dual Hesse arrangement defined in Example 9. Then Z admits an unexpected curve of degree* 5 *with a point of multiplicity* 4*.*

Proof. The points in Z are the following 12 points:

$$\begin{array}{lll} P_1 = (1:0:0), & P_2 = (0:1:0), & P_3 = (0:0:1), \\ P_4 = (1:1:1), & P_5 = (1:1:\varepsilon), & P_6 = (1:1:\varepsilon^2), \\ P_7 = (1:\varepsilon:1), & P_8 = (1:\varepsilon:\varepsilon), & P_9 = (1:\varepsilon:\varepsilon^2), \\ P_{10} = (1:\varepsilon^2:1), & P_{11} = (1:\varepsilon^2:\varepsilon), & P_{12} = (1:\varepsilon^2:\varepsilon^2). \end{array}$$

The ideal $I(Z)$ is an almost complete intersection ideal generated by

$$x(y^3 - z^3),\ y(z^3 - x^3),\ z(x^3 - y^3),$$

see [16].

Let $P = (a : b : c)$ be a general point in $\mathbb{P}^2$. Then it is easy to check that the curve defined by the polynomial

$$\begin{aligned} Q_P(x:y:z) =& cxy((2b^3 + c^3)(z^3 - x^3) + (2a^3 + c^3)(y^3 - z^3)) \\ & bxz((2a^3 + b^3)(y^3 - z^3) + (2c^n + b^n)(x^3 - y^3)) \\ & ayz((2b^3 + a^3)(z^3 - x^3) + (2c^3 + a^3)(x^3 - y^3)) \\ & - 6a^2bcx^2(y^3 - z^3) - 6ab^2cy^2(z^3 - x^3) - 6abc^2z^2(x^3 - y^3) \end{aligned} \tag{6}$$

satisfies the assertions of the theorem. □

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Pedagogical University of Cracow, Department of Mathematics, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: szpond@up.krakow.pl

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RINGS AND FIELDS OF CONSTANTS OF CYCLIC FACTORIZABLE DERIVATIONS

JANUSZ ZIELIŃSKI

Abstract. We present a survey of the research on rings of polynomial constants and fields of rational constants of cyclic factorizable derivations in polynomial rings over fields of characteristic zero.

1. Motivations and preliminaries

The first inspiration for the presented series of articles (some of them are joint works with Hegedűs and Ossowski) was the publication [20] of professor Nowicki and professor Moulin Ollagnier. The fundamental problem investigated in that series of articles concerns rings of polynomial constants ([26], [28], [33], [29], [8]) and fields of rational constants ([30], [31], [32]) in various classes of cyclic factorizable derivations. Moreover, we investigate Darboux polynomials of such derivations together with their cofactors ([33]) and applications of the results obtained for cyclic factorizable derivations to monomial derivations ([31]).

Let k be a field. If R is a commutative k-algebra, then k-linear mapping $d : R \to R$ is called a *k-derivation* (or simply a *derivation*) of R if $d(ab) = ad(b) + bd(a)$ for all $a, b \in R$. The set $R^d = \ker d$ is called a *ring* (or an *algebra*) *of constants* of the derivation d. Then $k \subseteq R^d$ and a *nontrivial constant* of the derivation d is an element of the set $R^d \setminus k$. By $k[X]$ we denote $k[x_1, \ldots, x_n]$, the polynomial ring in n variables. If $f_1, \ldots, f_n \in k[X]$, then there exists exactly one derivation $d : k[X] \to k[X]$ such that $d(x_1) = f_1, \ldots, d(x_n) = f_n$.

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More information on derivations one can find in the monographs [6] and [24]. They also present links of derivation theory with the Jacobian Conjecture. Because this still unsettled conjecture one may translate into the language of derivations. Such an equivalent formulation can be found for example in [23], Theorem 5.

There is no general effective procedure for determining the ring of constants of a derivation, although the subject has a long tradition. One of the possible approaches, a certain reduction of the problem, is the Lagutinskii's procedure. Namely, we can associate the factorizable derivation with a given derivation of the polynomial ring over a field of characteristic zero. A derivation $d : k[X] \to k[X]$ is said to be *factorizable* if $d(x_i) = x_i f_i$, where the polynomials f_i are of degree 1 for $i = 1, \ldots, n$. That procedure of association is described for example in [25]. It turns out that in the generic case the problems of determining the fields of rational constants of the initial derivation and its associated derivation are equivalent. Which also gives us a good knowledge on polynomial constants. The challenge is that constants of factorizable derivations are still not sufficiently investigated. We know everything only in the case of number of variables ≤ 3, mainly thanks to the papers by Moulin Ollagnier and Nowicki (e.g. [18], [19], [20]). For a greater number of variables there are examined only some special cases such as Lotka-Volterra derivations (e.g. [8], [32]), derivations that appear in the Lagutinskii's procedure applied to Jouanolou derivations ([16]) and factorizable derivations associated with cyclotomic derivations ([21]). For certain other classes of derivations there are indicated some constants without settling whether they are the complete set of generators of the ring of constants (e.g. Itoh [9], Cairó [4]).

The question of determining constants can be equivalently expressed in the language of differential equations. Namely, over an arbitrary field k of characteristic zero, if δ is a derivation of the ring $k[X]$ (respectively: of the field $k(X)$) such that $\delta(x_i) = f_i$ for $i = 1, \ldots, n$, then the set $k[X]^\delta \setminus k$ (respectively: $k(X)^\delta \setminus k$) coincides with the set of all polynomial (respectively: rational) first integrals of a system of ordinary differential equations

$$\frac{dx_i(t)}{dt} = f_i(x_1(t), \ldots, x_n(t)),$$

where $i = 1, \ldots, n$ (for more details we refer the reader to [24], subsection 1.6).

The topic is also linked to invariant theory. Namely, for every connected algebraic group $G \subseteq \mathrm{Gl}_n(k)$, where k is a field of characteristic zero, there exists a derivation d such that $k[X]^G = k[X]^d$ (more information can be found, among others, in [24], subsection 4.2).

From now on k is a field of characteristic zero. We will call a factorizable derivation $d : k[X] \to k[X]$ *cyclic* if $d(x_i) = x_i(A_i x_{i-1} + B_i x_{i+1})$, where $A_i, B_i \in k$ for $i = 1, \ldots, n$ (in the cyclic sense, that is, we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). In particular, a derivation $d : k[X] \to k[X]$ is said to be

Lotka-Volterra derivation with parameters $C_1, \ldots, C_n \in k$ (see e.g. [11], [18], [19]) if

$$d(x_i) = x_i(x_{i-1} - C_i x_{i+1})$$

for $i = 1, \ldots, n$ (in the cyclic sense as above, that is, indices modulo n). The systems under consideration describe a wide range of phenomena and they appear in numerous domains of science such as population biology (inter-species interactions in the predator-prey model) [27], chemistry (oscillations of the concentration of substances in chemical reactions) [14], hydrodynamics (the convective instability in the Bénard problem) [3], plasma physics (the evolution of electrons and ions) [13], laser physics (the coupling of waves) [12], aerodynamics (the interaction of gases in a mixture) [15], economics [4], neural networks [22] and biochemistry [4]. Further motivations and applications are presented in [1], [2], [5] and many more.

The case of Lotka-Volterra derivations in three variables was settled in [18], [19], [20] by Moulin Ollagnier and Nowicki. For example, the existence of nontrivial polynomial constants is determined by the following theorem (here parameters C_i have opposite signs than in the notation above, which is of no account) from [18]:

Theorem 1. ([18], Theorem 1)
The Lotka-Volterra system

$$\begin{cases} d(x) = x(Cy + z) \\ d(y) = y(Az + x) \\ d(z) = z(Bx + y) \end{cases}$$

has a nontrivial polynomial constant if and only if one of the following cases holds:

(i) $ABC + 1 = 0$,

(ii) $-A - \frac{1}{B} = 1$, $-B - \frac{1}{C} = 1$ *and* $-C - \frac{1}{A} = 1$,

(iii) $C = -k_2 - \frac{1}{A}$, $A = -k_3 - \frac{1}{B}$, $B = -k_1 - \frac{1}{C}$ *where, up to a permutation,* (k_1, k_2, k_3) *is one of the following triples:* $(1, 2, 2)$, $(1, 2, 3)$, $(1, 2, 4)$.

The rings of polynomial constants were determined in each of these cases in [20].

The article [25] contains a full description of monomial derivations (that is, derivations which values on variables are monic monomials) in two ([25], Proposition 5.4) and in three variables ([25], Theorem 8.6) with nontrivial rational constants (that is, in the field of rational functions). The results for three variables in the generic case are based on Lotka-Volterra derivations, thoroughly investigated before for three variables mainly in [17]. That complete characterization of all cases for monomial derivations in two and three variables has marked at the same time the limitations of the usefulness of the Lagutinskii's procedure, potentially general, and practically limited by the knowledge of constants of factorizable derivations. It was the motivation and a starting point to deal with determining of constants of cyclic factorizable derivations in $n \geq 4$ variables.

2. Methods and research techniques

In contrast to e.g. Jouanolou derivations ([16], [34]), where one has to show that a constant of a positive degree does not exist, here we also deal with some nontrivial constants. Therefore instead of obtaining a contradiction we have to prove that a constant is a polynomial in given generators. A direct investigation of constants of derivations of considered type came across serious problems. These constants did not subject to the induction on degree, and the calculations turned out to be virtually impossible to perform. Therefore, the idea proved valuable, was to analyze, instead of constants of a derivation d, polynomials φ that fulfill the condition $d(\varphi^A)^A = 0$, where f^A denotes the restriction of a polynomial f to the ring of polynomials in variables with indices in the set A, where $A \subseteq \{1, \ldots, n\}$. Constants of a factorizable derivation d fulfill the condition $d(\varphi^A)^A = 0$, hence we obtained also some properties of these constants, which have been applied in the proofs of main theorems. However the properties of polynomials φ such that $d(\varphi^A)^A = 0$ have turned out to be possible to prove by combinatorial and inductive methods.

Moreover, our frequently used method of investigation of constants was the restriction of these constants to the polynomial ring in a smaller number of variables. And then, after the obtainment of their properties for various subsets of variables, we have tried to merge these data to receive some information about the shape of these initial constants.

Another important method was study of Darboux polynomials of a derivation d. This is a standard procedure in the case of determining rational constants, however much rarer in the case of determining polynomial constants, as here. Particularly important turned out to be characterizations of the coefficients of the cofactors of strict Darboux polynomials.

The next method was an investigation of the leading monomials according to fixed ordering. This approach originates from Gröbner bases theory, although we did not use these bases explicitly. Namely, we tried to establish as precisely as possible the shape of the leading monomials of elements from $k[X]^d$ according to the standard lexicographic ordering (after a convenient choice of the initial variable for this ordering). The aim was to delete the leading monomial using the generators, so as we could apply induction on the ordering.

Moreover, we often employ combinatorial methods, for instance to compare the coefficients of monomials of a given constant.

3. Volterra derivations

A Lotka-Volterra derivation with parameters $C_i = 1$ for all i is called a *Volterra* derivation (see e.g. [2]). The work [26] presents a description of the ring of constants of the Volterra derivation in four variables, which in this case has three algebraically independent generators:

Theorem 2. ([26], Theorem 3.1)
Let $R = k[x_1, \ldots, x_4]$. Let $d : R \to R$ be the derivation of the form

$$d(x_i) = x_i(x_{i-1} - x_{i+1})$$

for $i = 1, \ldots, 4$. Then

$$R^d = k[x_1 + x_2 + x_3 + x_4,\ x_1x_3,\ x_2x_4].$$

In [26] there are also shown numerous facts for n variables. In particular, they concern the restrictions of constants to the polynomial rings in a smaller number of variables. Let $R_{(m)}$ denote the homogeneous component of $R = k[x_1, \ldots, x_n]$ of degree m (since the derivation d is homogeneous, we need only search for homogeneous constants). For $\varphi \in R$ and for each subset $A \subseteq \{1, \ldots, n\}$ denote by φ^A the sum of monomials of the polynomial φ that depend on variables with indices in A, that is, $\varphi^A = \varphi_{|_{x_j=0 \text{ for } j \notin A}}$. As indicated in the previous section, to successfully perform complicated computations, the key idea was to investigate, instead of constants of the derivation d, polynomials φ such that $d(\varphi^A)^A = 0$ for various sets A (constants of the derivation d fulfill that condition, see [26], Corollary 2.8). This allowed to obtain some essential properties, ignoring at the same time in the calculations of a huge number of irrelevant data, greater than for the standard restriction. We quote below two examples of the results obtained in that way.

Proposition 3. ([26], Proposition 2.10)
Let $n \geq 3$. If $\varphi \in R_{(m)}$, $A = \{i, i+1\} \subseteq \{1, \ldots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A = c(x_i + x_{i+1})^m$ for some $c \in k$.

Proposition 4. ([26], Proposition 2.11)
Let $n \geq 4$. If $\varphi \in R_{(m)}$, $A = \{i, i+1, i+2\} \subseteq \{1, \ldots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A \in k[x_i + x_{i+1} + x_{i+2},\ x_ix_{i+2}]$.

In [26] there are also given generalizations of various results also for Lotka-Volterra derivations with arbitrary parameters $C_i \in k$. And as it turned out once again (see: Jouanolou derivations, Hilbert's fourteenth problem), the cases of a small number of variables were more difficult (less independence in the cyclic sense, too high "density" of variables).

The work [28] gives a description of the ring of polynomial constants of the five-variable Volterra derivation.

Theorem 5. ([28], Theorem 4.1)
Let $R = k[x_1, \ldots, x_5]$. Let $d : R \to R$ be the derivation of the form

$$d(x_i) = x_i(x_{i-1} - x_{i+1})$$

for $i = 1, \ldots, 5$. Then

$$R^d = k[\sum_{j=1}^{5} x_j,\ x_1x_3 + x_1x_4 + x_2x_4 + x_2x_5 + x_3x_5,\ x_1x_2x_3x_4x_5].$$

Thus, starting from five variables there appear generators, which are linear forms of the shape: the sum of products of nonconsecutive variables (obviously the sum of all variables is also a trite case of this). It turns out that for n variables the ring of constants of the Volterra derivation is a polynomial ring, which generators are of a such form, plus the product of all variables for n odd (analogously to the case $n = 5$) and two products of variables of the same parity for n even (analogously to the case $n = 4$). It is conjectured in [28] that analogous results as for $n \leq 5$ remain valid also for an arbitrary number of variables, which was confirmed in [7].

In [28] the methods of proofs are based upon obtaining more extensive properties of polynomials φ fulfilling the condition $d(\varphi^A)^A = 0$ for suitable sets A. Amidst these facts was, among others:

Lemma 6. ([28], Lemma 3.1)
Let $n \geq 5$. If $\varphi \in R_{(m)}$, $A = \{i, i+2, i+3\} \subseteq \{1, \ldots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A \in k[x_i,\ x_{i+2} + x_{i+3}]$.

From which we can obtain the following, this time stronger properties of constants of the derivation d:

Lemma 7. ([28], Lemma 3.2)
Let $n \geq 5$. If $\varphi \in R^d_{(m)}$ and $A = \{i, i+2, i+3\} \subseteq \{1, \ldots, n\}$, then $\varphi^A \in k[x_i + x_{i+2} + x_{i+3},\ x_i(x_{i+2} + x_{i+3})]$.

Proposition 8. ([28], Proposition 3.5)
Let $n \geq 5$. If $\varphi \in R^d_{(m)}$ and $A = \{i, i+1, i+2, i+3\} \subseteq \{1, \ldots, n\}$, then $\varphi^A \in k[x_i + x_{i+1} + x_{i+2} + x_{i+3},\ x_i x_{i+2} + x_i x_{i+3} + x_{i+1} x_{i+3}]$.

4. Darboux polynomials and Lotka-Volterra derivations

Results of [33] are of two kinds. First, there are described the cofactors of strict Darboux polynomials of four-variable Lotka-Volterra derivations. Let $R = k[x_1, \ldots, x_n]$. A polynomial $g \in R$ is called *strict* if it is homogeneous and not divisible by the variables $x_1, \ldots, x_n$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we introduce the notation $X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Clearly, every nonzero homogeneous polynomial $f \in R$ has a unique presentation $f = X^\alpha g$, where X^α is a monic monomial and g is a strict polynomial. Recall also that a nonzero polynomial f is said to be a *Darboux polynomial* of a derivation $\delta : R \to R$ if $\delta(f) = \Lambda f$ for some polynomial $\Lambda \in R$. We will call Λ a *cofactor* of f. In the following lemma we give the aforementioned description of cofactors:

Lemma 9. ([33], Lemma 3.2)
Let $n = 4$. Let $g \in R_{(m)}$ be a Darboux polynomial of a Lotka-Volterra derivation d with the cofactor $\lambda_1 x_1 + \ldots + \lambda_4 x_4$. Let $i \in \{1, 2, 3, 4\}$. If g is not divisible by x_i, then $\lambda_{i+1} \in \mathbb{N}$. More precisely, if $g(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_4) = x_{i+2}^{\beta_{i+2}} \overline{G}$ and $x_{i+2} \nmid \overline{G}$, then $\lambda_{i+1} = \beta_{i+2}$ and $\lambda_{i+3} = -C_{i+2}\lambda_{i+1}$.

Consequences of that lemma, which is a fairly technical nature, are more elegant:

Corollary 10. ([33], Corollary 3.3)
Let $n = 4$. If $g \in R_{(m)}$ is a strict Darboux polynomial, then its cofactor is a linear form with coefficients in $\mathbb{N}$.

Lemma 11. ([33], Lemma 3.4)
Let $n = 4$. If $d(f) = 0$ and $f = X^\alpha g$, where g is a strict polynomial, then $d(X^\alpha) = 0$ and $d(g) = 0$.

In other words, for an arbitrary nonzero constant, in the factorization of the above type both the monomial factor and the strict factor are constants, too. Lemma 9 has turned out to be useful for investigation of polynomial constants ([33], [29]) and of rational constants ([30], [31], [32]). That lemma together with its potential generalizations seem crucial in a further study of rational constants.

The second result of [33] was a description of the ring of constants of four-variable Lotka-Volterra derivations in the generic case. It turns out that in such a case the ring of constants is trivial, that is, equal to k ([33], Theorem 5.1 and Corollary 5.2).

Among the methods used, besides the investigation of cofactors of strict Darboux polynomials, the second approach was to generalize results for Volterra derivations from [26] and [28] to cases of arbitrary parameters C_i, for example:

Proposition 12. ([33], Lemma 4.4)
Let $n \geq 3$. If $\varphi \in R_{(m)}$, $A = \{i, i+1\} \subseteq \{1, \ldots, n\}$ and $d(\varphi^A)^A = 0$, then $\varphi^A = a(x_i + C_i x_{i+1})^m$ for some $a \in k$.

The paper [29] gives a description of the rings of constants of four-variable Lotka-Volterra derivations depending on the parameters C_i, except the case when the product $C_1C_2C_3C_4$ is a root of unity not equal to 1 (see [8]). Denote by $\mathbb{N}_+$ the set of positive integers, and by $\mathbb{Q}_+$ the set of positive rationals.

Consider the three sentences:

s_1 : $C_1C_2C_3C_4 = 1$.

s_2 : $C_1, C_3 \in \mathbb{Q}_+$ and $C_1C_3 = 1$.

s_3 : $C_2, C_4 \in \mathbb{Q}_+$ and $C_2C_4 = 1$.

In case s_2 let $C_1 = \frac{p}{q}$, where $p, q \in \mathbb{N}_+$ and $\gcd(p, q) = 1$. In case s_3 let $C_2 = \frac{r}{t}$, where $r, t \in \mathbb{N}_+$ and $\gcd(r, t) = 1$. Denote by $\neg s_i$ the negation of the sentence s_i. We assume that $C_1C_2C_3C_4$ is not nontrivial root of unity.

Theorem 13. ([29], Theorem 5.1)
Let $R = k[x_1, x_2, x_3, x_4]$ and $d : R \to R$ be a derivation of the form

$$d = \sum_{i=1}^{4} x_i(x_{i-1} - C_i x_{i+1}) \frac{\partial}{\partial x_i},$$

where $C_1, C_2, C_3, C_4 \in k$. Then the ring of constants of d is always finitely generated over k with at most three generators. In each case it is a polynomial ring, more precisely:

(1) *if* $s_1 \wedge \neg s_2 \wedge \neg s_3$, *then* $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4]$,

(2) *if* $\neg s_1 \wedge \neg s_2 \wedge \neg s_3$, *then* $R^d = k$,

(3) *if* $\neg s_1 \wedge \neg s_2 \wedge s_3$, *then* $R^d = k[x_2^t x_4^r]$,

(4) *if* $\neg s_1 \wedge s_2 \wedge \neg s_3$, *then* $R^d = k[x_1^q x_3^p]$,

(5) *if* $s_1 \wedge \neg s_2 \wedge s_3$, *then* $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_2^t x_4^r]$,

(6) *if* $s_1 \wedge s_2 \wedge \neg s_3$, *then* $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^q x_3^p]$,

(7) *if* $s_2 \wedge s_3$, *then* $R^d = k[x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^q x_3^p,\ x_2^t x_4^r]$.

Techniques of proofs used in [29] are extension of the methods from the former articles. For example, a useful tool is:

Lemma 14. ([29], Lemma 3.1)
Let $n \geq 4$, $\varphi \in R_{(m)}$, $i \in \{1, \ldots, n\}$ *and* $A = \{i, i+1, i+2\}$. *Let* $C_i \in \mathbb{Q}_+$ *and* $C_i = \frac{p}{q}$, *where* $p, q \in \mathbb{N}_+$ *and* $\gcd(p, q) = 1$. *If* $d(\varphi^A)^A = 0$, *then* $\varphi^A \in k[x_i + C_ix_{i+1} + C_iC_{i+1}x_{i+2},\ x_i^q x_{i+2}^p]$.

While the complementary to the Lemma 14 case of $C_i \notin \mathbb{Q}_+$ had already been resolved in [33] (Lemma 4.5).

5. Fields of rational constants

The article [30] explores the fields of rational constants, that is, constants belonging to the field of rational functions. Recall that for any derivation $\delta : k[X] \to k[X]$ of the polynomial ring in n variables there exists exactly one derivation $\bar{\delta} : k(X) \to k(X)$ of the field of rational functions in n variables such that $\bar{\delta}_{|k[X]} = \delta$. By a *rational constant* of the derivation $\delta : k[X] \to k[X]$ we mean the constant of its corresponding derivation $\bar{\delta} : k(X) \to k(X)$. The rational constants of δ form a field. For simplicity, we write δ instead of $\bar{\delta}$. In [30] it is shown the following theorem:

Theorem 15. ([30], Theorem 2)
If d *is the four-variable Volterra derivation, then*

$$k(X)^d = k(x_1 + x_2 + x_3 + x_4, x_1x_3, x_2x_4).$$

Thus, in this case $k(X)^d$ is the field of fractions of the ring $k[X]^d$ (which is not true in general, see e.g. [31], Example 1).

Theorem 15 was proved using an analysis of the cofactors of strict Darboux polynomials. In particular, it was shown ([30], Lemma 2) that every strict Darboux polynomial of the four-variable Volterra derivation is also a constant of this derivation.

The field of rational constants of a four-variable Lotka-Volterra derivation in the generic case was determined in [31]. Namely, it was demonstrated that in such a

generic case nontrivial rational constants do not exist ([31], Theorem 2). It was also determined the field of rational constants in some another case ([31], Theorem 2 again). In both cases these fields of constants are the fields of fractions of the rings of constants.

An important aspect of the results obtained are their applications. Therefore, in [31] there were investigated, similarly to [25], monomial derivations, but this time in four variables. Recall that a derivation $\delta : k(X) \to k(X)$ is monomial if

$$\delta(x_i) = x_1^{\beta_{i1}} \cdots x_n^{\beta_{in}}$$

for $i = 1, \ldots n$, where each β_{ij} is an integer. It is shown how one can determine its rational constants using two tools, which are a description of strict Darboux polynomials ([33], Lemma 3.2) and so far proven results on constants of Lotka-Volterra derivations. This was demonstrated on the following example of a class of monomial derivations depending on four natural parameters s_i.

Theorem 16. ([31], Theorem 5)
Let $s_1, \ldots, s_4 \in \mathbb{N}_+$, where $(s_1, s_3) \neq (1, 1)$ and $(s_2, s_4) \neq (1, 1)$. Let $D : k(X) \to k(X)$ be a derivation of the form

$$D(x_i) = x_{i-1}^{s_{i-1}+1} x_i^{s_i+1} x_{i+2}^{s_{i+2}}$$

for $i = 1, \ldots, 4$ (in the cyclic sense). Then $k(X)^D = k$.

As we remember, a factorizable derivation $d : k[X] \to k[X]$ is called cyclic if $d(x_i) = x_i(A_i x_{i-1} + B_i x_{i+1})$, where $A_i, B_i \in k$ for $i = 1, \ldots, n$ (and we adhere to the convention that $x_{n+1} = x_1$ and $x_0 = x_n$). Suppose that $A_i \neq 0$ for all i. Consider an automorphism $\sigma : k[X] \to k[X]$ defined by $\sigma(x_i) = A_{i+1}^{-1} x_i$ for $i = 1, \ldots, n$. Then $\Delta = \sigma d \sigma^{-1}$ is also a derivation of the ring $k[X]$. Moreover, f is a nontrivial polynomial (respectively: rational) constant of a derivation d if and only if $\sigma(f)$ is a nontrivial polynomial (respectively: rational) constant of a derivation Δ. Clearly $\sigma^{-1}(x_i) = A_{i+1} x_i$ and $\Delta(x_i) = x_i(x_{i-1} - C_i x_{i+1})$ for $C_i = -B_i A_{i+2}^{-1}$ (we allow $C_i = 0$) and $i = 1, \ldots, n$. We can proceed similarly if $A_i = 0$ for some i but $B_i \neq 0$ for all i.

A characterization of all four-variable Lotka-Volterra derivations with a nontrivial constant in the field of rational functions is given in [32]:

Proposition 17. ([32], Corollary 2)
If d is a four-variable Lotka-Volterra derivation, then $k(X)^d$ contains a nontrivial rational constant if and only if at least one of the following four conditions is fulfilled:

(1) $C_1 C_2 C_3 C_4 = 1$,

(2) $C_1, C_3 \in \mathbb{Q}$ *and* $C_1 C_3 = 1$,

(3) $C_2, C_4 \in \mathbb{Q}$ *and* $C_2 C_4 = 1$,

(4) $C_1 C_2 C_3 C_4 = -1$ *and* $C_i = 1$ *for two consecutive indices* i.

Note that the existence of a nontrivial polynomial constant is equivalent to similar four conditions, wherein in conditions (2) and (3) the set $\mathbb{Q}$ is replaced by $\mathbb{Q}_+$ (a consequence of Theorem 1.2 from [8]).

In many of the cases we can describe the full fields of constants. Namely, consider the sentences:

$\tilde{s}_2$: $C_1, C_3 \in \mathbb{Q}$ and $C_1C_3 = 1$.

$\tilde{s}_3$: $C_2, C_4 \in \mathbb{Q}$ and $C_2C_4 = 1$.

Sentences s_1, s_2, s_3 and numbers p, q, r, t are as in Theorem 13. We define the sentence:

s_4 : $C_1C_2C_3C_4 = -1$ and $C_i = 1$ for two consecutive indices i.

If the sentence s_4 is true we define the polynomial f_4, namely for $C_1 = C_2 = 1$ let

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2C_3x_1x_4 + 2x_2x_3 - 2C_3x_2x_4 + 2C_3x_3x_4,$$

for the other possibilities one has to rotate the indices appropriately.

Theorem 18. ([32], Theorem 4.1)
Let $d : k(X) \to k(X)$ be a four-variable Lotka-Volterra derivation with parameters $C_1, C_2, C_3, C_4 \in k$. Then:

(1) *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge \neg\tilde{s}_3 \wedge \neg s_4$, then $k(X)^d = k$,*

(2) *if $s_1 \wedge \neg\tilde{s}_2 \wedge \neg\tilde{s}_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4)$,*

(3) *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge \neg\tilde{s}_3 \wedge s_4$, then $k(X)^d = k(f_4)$,*

(4) *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge s_3 \wedge \neg s_4$, then $k(X)^d = k(x_2^tx_4^r)$,*

(5) *if $\neg s_1 \wedge s_2 \wedge \neg\tilde{s}_3 \wedge \neg s_4$, then $k(X)^d = k(x_1^qx_3^p)$,*

(6) *if $\neg s_1 \wedge \neg\tilde{s}_2 \wedge s_3 \wedge s_4$, then $k(X)^d = k(f_4, x_2^tx_4^r)$,*

(7) *if $\neg s_1 \wedge s_2 \wedge \neg\tilde{s}_3 \wedge s_4$, then $k(X)^d = k(f_4, x_1^qx_3^p)$,*

(8) *if $s_1 \wedge \neg\tilde{s}_2 \wedge s_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_2^tx_4^r)$,*

(9) *if $s_1 \wedge s_2 \wedge \neg\tilde{s}_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^qx_3^p)$,*

(10) *if $s_2 \wedge s_3$, then $k(X)^d = k(x_1 + C_1x_2 + C_1C_2x_3 + C_1C_2C_3x_4,\ x_1^qx_3^p,\ x_2^tx_4^r)$.*

The proof of the above theorem also uses investigations of the cofactors of strict Darboux polynomials.

6. Rings of constants in n variables

The paper [8] resolves in a complete way the problem of describing the rings of constants of Lotka-Volterra derivations for an arbitrary number of variables. The case $C_i = 1$ for all i was determined in [7]. All other cases are included in the following theorem:

Theorem 19. ([8], Theorem 1.1 and Theorem 1.2)
The ring of constants of Lotka-Volterra derivation in $n \geq 4$ variables is finitely generated over k with at most 3 generators, if there exists i such that $C_i \neq 1$. In every case it is a polynomial ring.

In [8] all of these rings of constants are determined in an effective way depending on n. It is presented in the following Theorems 20 and 25.

Let $f = \sum_{i=1}^{n}(\prod_{j=1}^{i-1} C_j)x_i = x_1 + C_1x_2 + C_1C_2x_3 + \ldots + C_1C_2\cdots C_{n-1}x_n$. Moreover, consider nonempty subsets $\mathcal{A} \subseteq \mathbb{Z}_n$ of integers mod n closed under $i \mapsto i+2$. If n is odd then $\mathcal{A} = \mathbb{Z}_n$, if n is even we have two additional subsets $\mathcal{E} = \{2i \mid i \leq n/2\}$ and $\mathcal{O} = \{2i-1 \mid i \leq n/2\}$. For a given $\mathcal{A}$ we define a polynomial $g_{\mathcal{A}}$ if there exist $\theta_i \in \mathbb{N}_+$ for $i \in \mathcal{A}$, such that $\theta_{i+2} = C_i\theta_i$. We can choose the set of θ_i coprime, then that numbers are uniquely determined. Then let $g_{\mathcal{A}} = \prod_{i\in\mathcal{A}} x_i^{\theta_i}$.

Theorem 20. ([8], Theorem 1.1)
Let $n > 4$ and let there exist i such that $C_i \neq 1$. Then the number of generators of the ring of constants of the Lotka-Volterra derivation with parameters $C_1, \ldots, C_n$ is equal to:

- *0 if $\prod C_i \neq 1$ and no $g_{\mathcal{A}}$ is defined;*
- *3 if n is even and both $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ are defined;*
- *2 if n is odd and $g_{\mathbb{Z}_n}$ is defined, or n is even and $\prod C_i = 1$ but only one of $g_{\mathcal{E}}$ and $g_{\mathcal{O}}$ is defined;*
- *1 in all other cases.*

The generators are always those polynomials $g_{\mathcal{A}}$ that are defined together with f if $\prod C_i = 1$.

To prove the theorem above, we have to show that the aforementioned generators are constants, which is a quick calculation, that these generators are algebraically independent, which can be shown by standard methods using the Jacobian, and that there are no constants not belonging to the polynomial ring with generators given above, which is practically entire difficulty of the proof. In order to establish this last condition we tried to describe as precisely as possible the shape of the leading monomials of polynomial constants according to the standard lexicographic ordering on monomials of a fixed degree. To then be able to eliminate such a leading monomial using generators and to be able to apply the induction (on the aforementioned ordering). This is achieved by several auxiliary facts. We quote below a selection of them.

Assume $C_n \neq 1$. Consider the standard lexicographic ordering. Suppose that h is a counterexample to Theorem 20 with the smallest leading monomial according to the ordering under consideration. Let $m_1 = \prod_{i=1}^{n} x_i^{\alpha_i}$ be that leading monomial. Let $M(h)$ denote the set of monomials occurring in h with a nonzero coefficient.

Proposition 21. ([8], Proposition 2.5)
Suppose $m = \prod_{i=1}^{n} x_i^{\gamma_i}$ is a monomial and r is a positive integer with the following properties:

(1) $\gamma_n = \alpha_n$,
(2) $\gamma_{2i-1} = \alpha_{2i-1}$ *for* $1 \leq i \leq r$,
(3) $\gamma_{2i} = C_{2i-2}\gamma_{2i-2}$ *for* $1 \leq i \leq r-1$,
(4) $\gamma_{2r} \neq C_{2r-2}\gamma_{2r-2}$.

Then $m \notin M(h)$.

Note that the above proposition implies that the even-indexed exponents of m_1 are uniquely determined by α_n. The odd-indexed exponents are determined only up to a certain extent, as described in the following proposition.

Proposition 22. ([8], Proposition 2.7)
Suppose $m = \prod_{i=1}^{n} x_i^{\gamma_i} \in M(h)$ is a monomial and $r < n/2$ is a positive integer with the following properties:

(1) $\gamma_n = \alpha_n$ (*or* $C_n = 0$),
(2) $\gamma_{2i-1} = \alpha_{2i-1}$ *for* $1 \leq i \leq r$,
(3) $\gamma_{2i} = \alpha_{2i}$ *for* $1 \leq i \leq r$.

Then there exists a nonnegative integer β_{2r-1} such that $C_{2r-1}(\gamma_{2r-1} - \beta_{2r-1}) = \gamma_{2r+1}$ and $m' = m(x_{2r}/x_{2r-1})^{\beta_{2r-1}} \in M(h)$. In particular, there exist nonnegative integers β'_{2i-1} such that $C_{2i-1}(\alpha_{2i-1} - \beta'_{2i-1}) = \alpha_{2i+1}$ for $1 \leq i < n/2$.

The next result enables further reductions and some kind of substitutions of monomials in a constant h and, on the other hand, forces certain conditions on the parameters C_i.

Proposition 23. ([8], Corollary 2.9)
Suppose $C_n \neq 0$ and $m = \prod x_i^{\gamma_i} \in M(h)$ is such that $\gamma_n = \alpha_n$, $\gamma_1 = \alpha_1$ and $\gamma_2 = \alpha_2 = C_n\alpha_n$. Then $l = \gamma_1 - C_{n-1}\gamma_{n-1}$ is a nonnegative integer and $m' = m(x_n/x_1)^l \in M(h)$. In particular, $\alpha_1 - C_{n-1}\alpha_{n-1}$ is a nonnegative integer.

It also turns out that some types of constants may occur only under certain conditions on the product of parameters C_i.

Proposition 24. ([8], Proposition 2.10)
Let $h, g \in k[X]^d$, where g is a monomial. If the leading monomial of h is $m_1 = x_1^s g$ with $s > 0$, then $C_1C_2\cdots C_n$ is s-th root of unity, in particular, all $C_i \neq 0$. If further $n > 4$, then $C_1C_2\cdots C_n = 1$.

We now proceed to the case $n = 4$. In this case it may occur some generator of the ring of constants, which does not occur for $n > 4$. Namely, if $C_1C_2C_3C_4 = -1$ and for two consecutive indices i we have $C_i = 1$ (see already [26], Proposition 4.4, condition (4)). Without loss of generality, if $C_1 = C_2 = 1$ and $C_4 = -1/C_3$, then that generator is equal to

$$f_4 = x_1^2 + x_2^2 + x_3^2 + C_3^2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2C_3x_1x_4 + 2x_2x_3 - 2C_3x_2x_4 + 2C_3x_3x_4.$$

The following theorem describes the ring of constants for 4 variables.

Theorem 25. ([8], Theorem 1.2)
Assume $n = 4$ and let there exist i such that $C_i \neq 1$. Then the number of generators of the ring of constants of the Lotka-Volterra derivation with parameters $C_1, \ldots, C_n$ is equal to:

- 0 *if* $\prod C_i \neq 1$ *and none of* $g_{\mathcal{O}}$, $g_{\mathcal{E}}$, f_4 *is defined;*
- 3 *if both* $g_{\mathcal{E}}$ *and* $g_{\mathcal{O}}$ *are defined;*
- 2 *if* $\prod C_i = 1$ *but only one of* $g_{\mathcal{E}}$ *and* $g_{\mathcal{O}}$ *is defined or one of parameters* C_i *is equal to* -1 *and the other three are equal to* 1*;*
- 1 *in all other cases.*

The generators are always those polynomials $g_{\mathcal{A}}$ that are defined together with f_4 if $\prod C_i = -1$ and two consecutive parameters are equal to 1 *or together with f if $\prod C_i = 1$.*

The case of 4 variables has turned out to be the most difficult to prove. Already in the previously cited facts occurred some distinctions for this case (see e.g. Proposition 24). Also needed were results specific to the four variables, for example:

Proposition 26. ([8], Proposition 2.11)
Let $n = 4$ and $h, g \in k[X]^d$, where g is a monomial. Assume either every C_i is positive rational, or C_4 is not. If the leading monomial of h is $m_1 = x_1^s g$ with $s > 0$, then $C_1C_2C_3C_4 = \pm 1$. If $C_1C_2C_3C_4 = -1$, then $C_2 = 1$ and at least one of $C_1 = 1$ or $C_3 = 1$ also holds.

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Nicolaus Copernicus University, Faculty of Mathematics and Computer Science, ul. Chopina 12/18, 87-100 Toruń, Poland

E-mail address: ubukrool@mat.uni.torun.pl

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A FAMILY OF HYPERBOLAS ASSOCIATED TO A TRIANGLE

MACIEJ ZIĘBA

Abstract. In this note, we explore an apparently new one parameter family of conics associated to a triangle. Given a triangle we study ellipses whose one axis is parallel to one of sides of the triangle. The centers of these ellipses move along three hyperbolas, one for each side of the triangle. These hyperbolas intersect in four common points, which we identify as centers of incircle and the three excircles of the triangle. Thus they belong to a pencil of conics. We trace centers of all conics in the family and establish a surprising fact that they move along the excircle of the triangle. Even though our research is motivated by a problem in elementary geometry, its solution involves some non-trivial algebra and appeal to effective computational methods of algebraic geometry. Our work is illustrated by an animation in Geogebra and accompanied by a Singular file.

1. Introduction

Let $A = (a_1, a_2), B = (b_1, b_2), C = (c_1, c_2)$ be non-collinear points in the real affine plane. Their coordinates satisfy thus the condition

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} \neq 0$$

In [8] we proved the following result.

Theorem 1.1. *We consider ellipses whose one axis is parallel to a fixed side of the triangle (the line containing its two vertices). Then*

- *such ellipses form a one dimensional family;*
- *their centers move along a hyperbola which passes through the incenter S and the three excenters A', B', C' of the triangle.*

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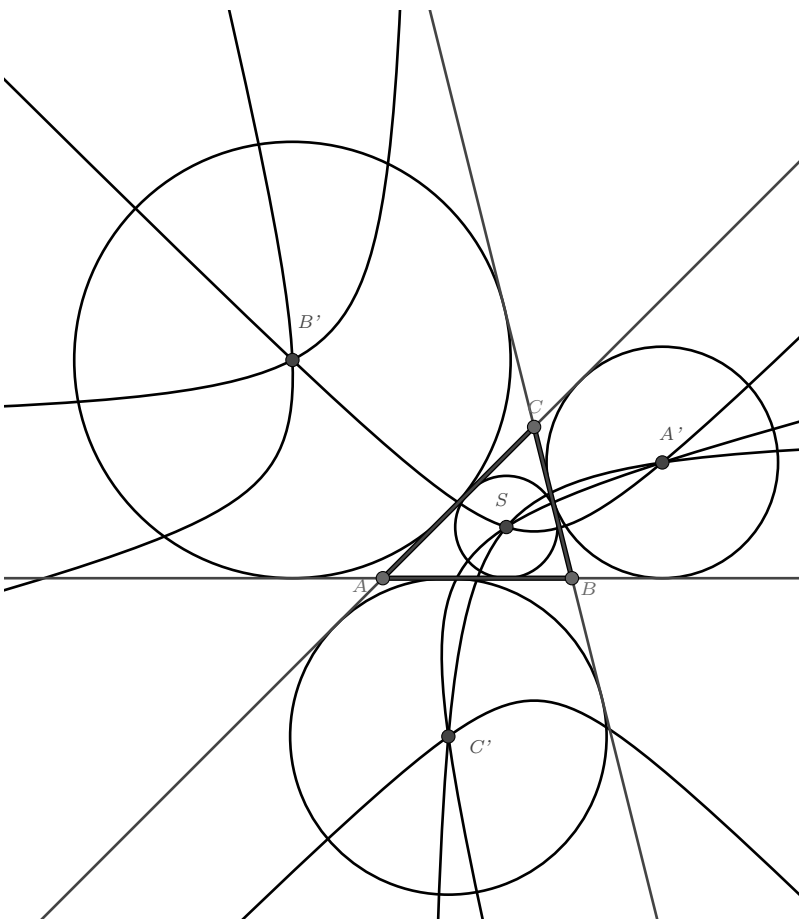
Remark 1.2. By a center of a conic, we understand its center of symmetry. If the conic is degenerate and consists of two intersecting lines then its center is the intersection point of both lines. If the lines are parallel, then we declare the corresponding point at the infinity as their center. We are aware of the fact that for two parallel lines there are infinitely many centers of symmetry but it is convenient and consistent with our approach to declare the point they share at infinity as their center.

Corollary 1.3. *It follows immediately from Theorem 1.1 that taking the three hyperbolas corresponding to each of the sides of the triangle, they all belong to a pencil of conics determined by points S, A', B' and C'. This is illustrated in Figure 1.*

FIGURE 1.



Remark 1.4. Note that another three hyperbolas associated to a triangle have been identified in 1957 by Court. However his construction is not related to ours.

There are well-known formulas we allow us to compute coordinates of points S, A', B' and C' explicitly:

$$S = \left(\frac{a_1\widetilde{a} + b_1\widetilde{b} + c_1\widetilde{c}}{\widetilde{a} + \widetilde{b} + \widetilde{c}}, \frac{a_2\widetilde{a} + b_2\widetilde{b} + c_2\widetilde{c}}{\widetilde{a} + \widetilde{b} + \widetilde{c}} \right),$$
$$A' = \left(\frac{-a_1\widetilde{a} + b_1\widetilde{b} + c_1\widetilde{c}}{-\widetilde{a} + \widetilde{b} + \widetilde{c}}, \frac{-a_2\widetilde{a} + b_2\widetilde{b} + c_2\widetilde{c}}{-\widetilde{a} + \widetilde{b} + \widetilde{c}} \right),$$
$$B' = \left(\frac{a_1\widetilde{a} - b_1\widetilde{b} + c_1\widetilde{c}}{\widetilde{a} - \widetilde{b} + \widetilde{c}}, \frac{a_2\widetilde{a} - b_2\widetilde{b} + c_2\widetilde{c}}{-\widetilde{a} - \widetilde{b} + \widetilde{c}} \right),$$
$$C' = \left(\frac{a_1\widetilde{a} + b_1\widetilde{b} - c_1\widetilde{c}}{\widetilde{a} + \widetilde{b} - \widetilde{c}}, \frac{a_2\widetilde{a} + b_2\widetilde{b} - c_2\widetilde{c}}{\widetilde{a} + \widetilde{b} - \widetilde{c}} \right),$$

where

$$\widetilde{a} = \sqrt{(b_1 - c_1)^2 + (b_2 - c_2)^2},$$
$$\widetilde{b} = \sqrt{(a_1 - c_1)^2 + (a_2 - c_2)^2},$$
$$\widetilde{c} = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

However, since the formulas involve taking roots, they are not so convenient for symbolic computations. We circumvent this difficulty in the next section.

2. Conics determined by four points.

Let $Z = \{P_1, \ldots, P_4\}$ be a set of four points in the plane such that no three of them are collinear. For a subset $V \subset \mathbb{R}^2$ we denote by $I(V)$ its saturated ideal, i.e., the ideal in the polynomial ring $\mathbb{R}[x, y]$ consisting of all polynomials vanishing at all points of V. Then we have

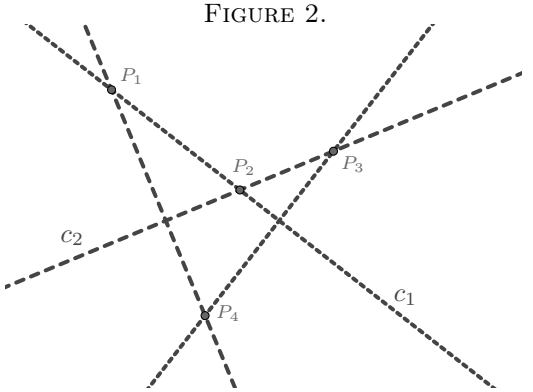
$$I(Z) = I(P_1) \cap \ldots \cap I(P_4).$$

Using the geometry of Z, it is easy to determine generators of $I(Z)$. To this end note that $I(P_i \cup P_j)$ for $i \neq j$ contains a unique (up to a multiplicative factor) element $\ell_{i,j}$ of degree 1 (namely the equation of the line through P_i and P_j). Then, with $c_1 = \ell_{1,2} \cdot \ell_{3,4}$ and $c_2 = \ell_{1,3} \cdot \ell_{2,4}$ we have

$$I(Z) = \langle c_1,\ c_2 \rangle.$$

Geometrically, the set Z is then the intersection of conics c_1 and c_2. This is illustrated in Figure 2.

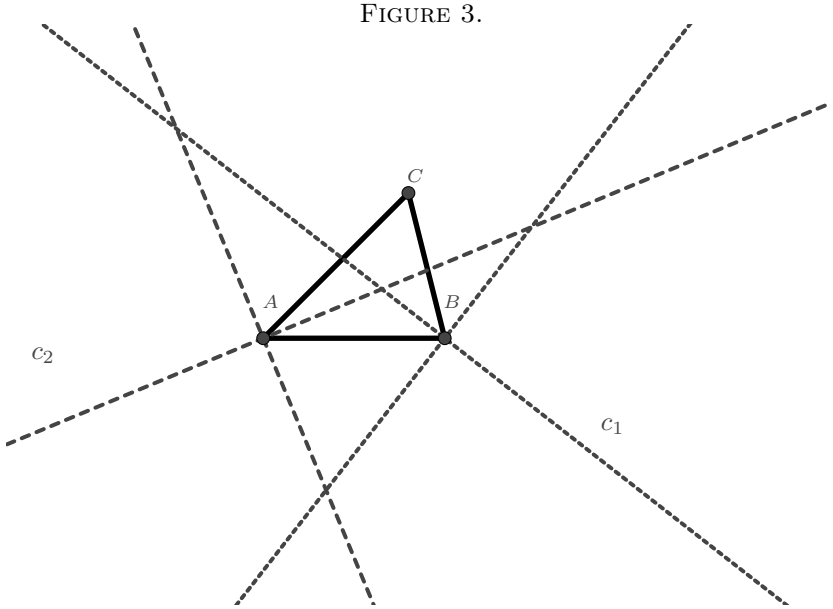
FIGURE 2.



Note that also $c_3 = \ell_{1,4} \cdot \ell_{2,3}$ is an element of $I(Z)$. It can be written down as a linear combination of c_1 and c_2. In the linear system of all conics determined by Z there are exactly three degenerate conics c_1, c_2 and c_3.

Turning back to our situation, we consider $Z = \{S, A', B', C'\}$. Then the lines joining pairs of points in Z have additional geometric meaning: They are either bisectors of angles of the triangle or bisectors of its exterior angles. This is depicted in Figure 3.

FIGURE 3.



Since we are interested in the union of the bisector of an angle of a triangle and the bisector of the exterior angle rather than one of these lines separately, by a slight abuse of the language, we introduce the folowing notion.

Definition 2.1. Let ℓ_1 and ℓ_2 be two distinct lines intersecting at a point P. The *bibisector* of ℓ_1 and ℓ_2 is the union of bisectors of angles formed by the two lines. We denote the bibisector by $\text{bibi}(\ell_1, \ell_2)$, or if there is no ambiguity about the lines just by $\text{bibi}(P)$.

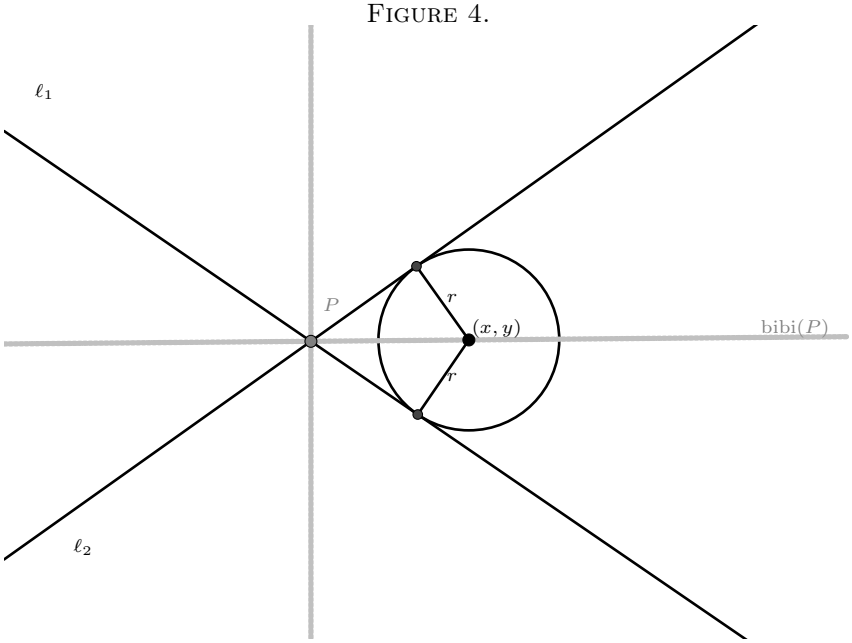
The next Lemma shows how surprisingly easy it is to derive the equation of $\text{bibi}(\ell_1, \ell_2)$ out of equations of lines ℓ_1 and ℓ_2.

Lemma 2.2. *Let ℓ_1 be given by the equation $Ax + By + C = 0$ and let ℓ_2 be given by $\widetilde{A}x + \widetilde{B}y + \widetilde{C} = 0$. Then the bibisector of ℓ_1 and ℓ_2 is given by*

$$\frac{(Ax + By + C)^2}{A^2 + B^2} = \frac{(\widetilde{A}x + \widetilde{B}y + \widetilde{C})^2}{\widetilde{A}^2 + \widetilde{B}^2}. \tag{1}$$

Proof. A geometric property of the bibisector is that it consists of points equidistant to both lines. In other words, we are looking for the locus of points (x, y) subject to the condition for certain $r > 0$, the circle centered at (x, y) is tangent to lines ℓ_1 and ℓ_2, see Figure 4.

FIGURE 4.



A point (x, y) is equidistant to lines ℓ_1 and ℓ_2 if and only if its coordinates satisfy (1) and we are done. As expected, the equation is quadratic in x and y. □

Corollary 2.3. *In the set up of the triangle ABC, Lemma 2.2 we obtain*

$$\operatorname{bibi}(A): \frac{\left((a_2-b_2)\,x+(b_1-a_1)\,y+a_1\,b_2-a_2\,b_1\right)^2}{(a_2-b_2)^2+(b_1-a_1)^2} - \frac{\left((a_2-c_2)\,x+(-a_1+c_1)\,y+a_1\,c_2-a_2\,c_1\right)^2}{(a_2-c_2)^2+(a_1-c_1)^2}$$

$$\operatorname{bibi}(B): \frac{\left((b2-a2)\,x+(a1-b1)\,y+a2\,b1-a1\,b2\right)^2}{(b2-a2)^2+(a1-b1)^2} - \frac{\left((b2-c2)\,x+(-b1+c1)\,y+b1\,c2-b2\,c1\right)^2}{(b2-c2)^2+(-b1+c1)^2}$$

Corollary 2.4. *Since $Z=\{S,A',B',C'\}$ has two generators in degree 2, every element $f\in(I(Z))_2$ can be written as*

$$f = s\cdot\operatorname{bibi}(A)+t\cdot\operatorname{bibi}(B)$$

for some real numbers s,t.

3. Main result

We begin with a Lemma which provides coordinates of the center of a conic.

Lemma 3.1. *Let $g(x,y)=ax^2+by^2+c+2dxy+2ex+2fy$ be a polynomial of degree 2 in an affine real plane. We assume $ab-d^2\neq 0$, i.e., we assume that the set of zeroes of g is not a parabola. Then g describes either an ellipse, if $d^2-ab<0$ or a hyperbola if $d^2-ab>0$. In both cases the curves could be degenerate but in both cases they poses a center of symmetry. More precisely, the point*

$$S=\left(\tfrac{df-be}{ab-d^2},\ \tfrac{de-af}{ab-d^2}\right). \tag{2}$$

is the center of the conic $\{\ g=0\ \}$.

Proof. Since the proof is elementary but also rather technical and lengthy, we refer to [6, Section 6.3] for details. □

From now on, it is convenient to work with projective coordinates, rather than with affine. In particular, this approach allows us to express coordinates of the center of a conic by polynomials in the coefficients of the conic, rather than by rational functions of these coefficients. Indeed, we have in (2)

$$S=(df-be\ :\ de-af\ :\ ab-d^2). \tag{3}$$

We shall need also the following property of a circle viewed as a complex projective conic.

Lemma 3.2. *Let Γ be a circle. Then its complex projective completion contains points $J_1=(1:i:0)$ and $J_2=(1:-i:0)$.*

Proof. Let Γ be given by the equation

$$(x-a)^2+(y-b)^2=r^2,$$

where (a,b) are coordinates of its center and r is the radius. We homogenize the equation with a new variable z and obtain

$$(x-az)^2+(y-bz)^2=r^2z^2. \tag{4}$$

Computing the points at infinity, we insert $z=0$ and get

$$x^2+y^2=0.$$

It is now clear that the points J_1 and J_2 satisfy this equation. □

Remark 3.3. It is easy to see that Lemma 3.2 has an inverse. By this we mean that any complex conic Γ passing through points J_1 and J_2 can be written down in the form of equation (4) for some *complex* numbers a, b and r.

Now we are in the position to state the main result of this note. Animation [9], prepared in Geogebra and available online, illustrates this result.

Theorem 3.4. *Let S, A', B', C' be the incenter and the excenters of a triangle ABC. Let $\mathcal{C}$ be the pencil of conics passing through these 4 points. Then the locus of centers of conics in $\mathcal{C}$ is the excircle of the triangle ABC.*

Proof. According to Corollary 2.4 any element $\mathcal{C}_{(s:t)}$ of $\mathcal{C}$ is defined by an equation of the form

$$f_{(s:t)}=s\,\mathrm{bibi}(A)+t\,\mathrm{bibi}(B),$$

where $(s:t)\in\mathbb{P}^1$, $\mathrm{bibi}(A)$ and $\mathrm{bibi}(B)$ are conics defined in Corollary 2.3.

Using Lemma 3.1 we obtain projective coordinates of the centers $S_{s:t}$ of $\mathcal{C}_{(s:t)}$ expressed as polynomials depending on parameters $(s:t)$. Since the particular formulas are rather obscure, we omit them in this presentation. A motivated reader will easily recover them using any symbolic algebra system. We used Singular.

Eliminating the parameters $(s:t)$ from the equations of the coordinates of $S_{(s:t)}$ and dehomogenizing (i.e. setting $z=1$) we obtain the following quadratic equation in variables x and y.

$$\begin{aligned}
N(x,y)=&(a_1b_2-a_1c_2-a_2b_1+a_2c_1+b_1c_2-b_2c_1)x^2\\
&+(a_1b_2-a_1c_2-a_2b_1+a_2c_1+b_1c_2-b_2c_1)y^2\\
&+(-a_1^2b_2+a_1^2c_2-a_2^2b_2+a_2^2c_2+a_2b_1^2+a_2b_2^2\\
&-a_2c_1^2-a_2c_2^2-b_1^2c_2-b_2^2c_2+b_2c_1^2+b_2c_2^2)x\\
&+(a_1^2b_1-a_1^2c_1-a_1b_1^2-a_1b_2^2+a_1c_1^2+a_1c_2^2\\
&+a_2^2b_1-a_2^2c_1+b_1^2c_1-b_1c_1^2-b_1c_2^2+b_2^2c_1)y\\
&+(-a_1^2b_1c_2+a_1^2b_2c_1+a_1b_1^2c_2+a_1b_2^2c_2-a_1b_2c_1^2-a_1b_2c_2^2\\
&-a_2^2b_1c_2+a_2^2b_2c_1-a_2b_1^2c_1+a_2b_1c_1^2+a_2b_1c_2^2-a_2b_2^2c_1)=0
\end{aligned}$$

Thus N is the equation of a curve of degree 2 which contains all centers of conics in the pencil $\mathcal{C}$.

It remains to check that N defines the excircle of the triangle ABC. To this end we just check that coordinates of points A, B and C satisfy N. We omit easy calculations. Finally we check that also points J_1 and J_2 defined in Lemma 3.2 belong to the zero locus of N. Hence N is a circle passing through A, B and C. But there is just one such circle, namely the excircle of the triangle ABC and we are done. □

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Pedagogical University of Cracow, Department of Mathematics, Podchorążych 2, PL-30-084 Kraków, Poland

E-mail address: `matematyka.maciej@gmail.com`

Łódź University Press
90-131 Łódź, 8 Lindleya St.
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e-mail: ksiegarnia@uni.lodz.pl
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