Analytic and Algebraic Geometry

edited by
Tadeusz Krasiński
Stanisław Spodzieja

FACULTY OF MATHEMATICS
AND COMPUTER SCIENCE
UNIVERSITY OF ŁÓDŹ

Łódź 2013

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE
UNIVERSITY OF ŁÓDŹ

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COVER DESIGN
Michał M. Jankowski

Publication reviewed

Printed directly from camera-ready materials provided to the Łódź University Press by Department of Analytical Functions and Differential Equations

First Edition. W.06395.13.0.K

ISBN (wersja drukowana) 978-83-7969-017-6
ISBN (ebook) 978-83-7969-243-9

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tel. (42) 665 58 63, faks (42) 665 58 62

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Preface

Annual Conferences in Analytic and Algebraic Geometry have been organized by Faculty of Mathematics and Computer Science of the University of Łódź since 1980. Until now, proceedings of these conferences (mainly in Polish) have comprised educational materials describing current state of a branch of mathematics, new approaches to known topics, and new proofs of known results (see the Internet page: http://konfrogi.math.uni.lodz.pl/).

The subject of the present volume include new results and survey articles concerning real and complex algebraic geometry, singularities of curves and hypersurfaces, invariants of singularities (the Milnor number, degree of $\mathcal{C}^0$-sufficiency), algebraic theory of derivations and others topics.

One remarkable element of this collection is an English translation of the Polish version, published in proceedings of the above mentioned conferences, of an article by Stanisław Łojasiewicz (1926-2002) devoted to the famous Hironaka theorem on resolution of singularities. It contains his original approach to the problem in the case of curves and coherent analytic sheaves on 2-dimensional manifolds. This interesting article has not yet been available in English. Additionally, we add a photo portrait of him and the facsimile of one page of his original handwritten manuscript.

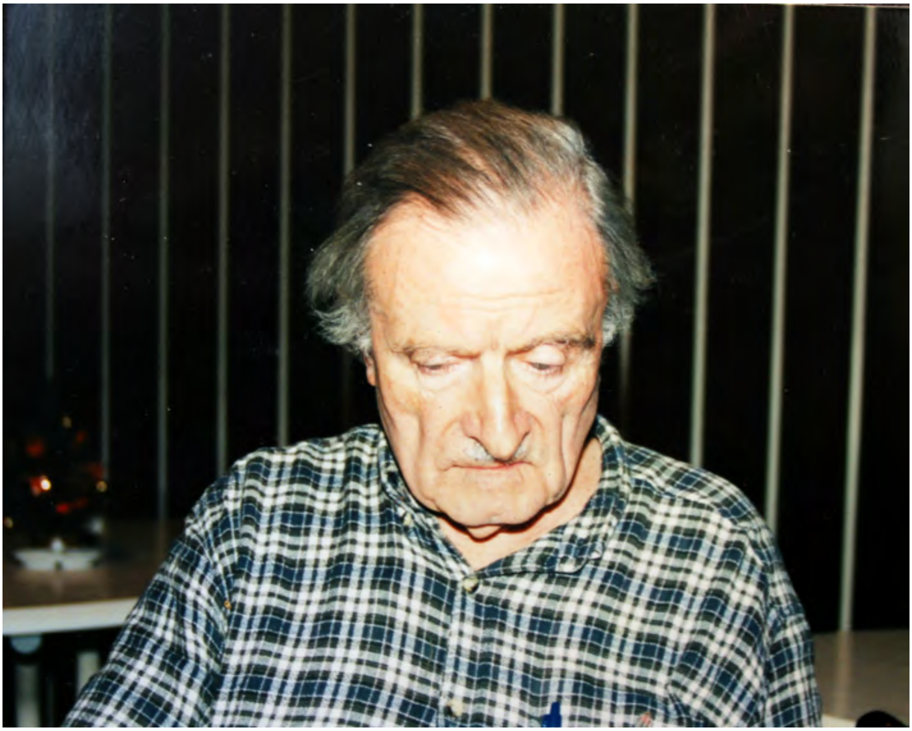
We would like to thank Arkadiusz Płoski for the help in preparing the volume, Michał Jankowski for designing the cover, referees for preparing reports of the articles and all participants of the Conferences for their good humor and enthusiasm in doing mathematics.

Finally, we would like to thank Stanisław Łojasiewicz jr and Anna Ostoja-Łojasiewicz, the heirs of Stanisław Łojasiewicz, for having agreed to include his article into this volume.

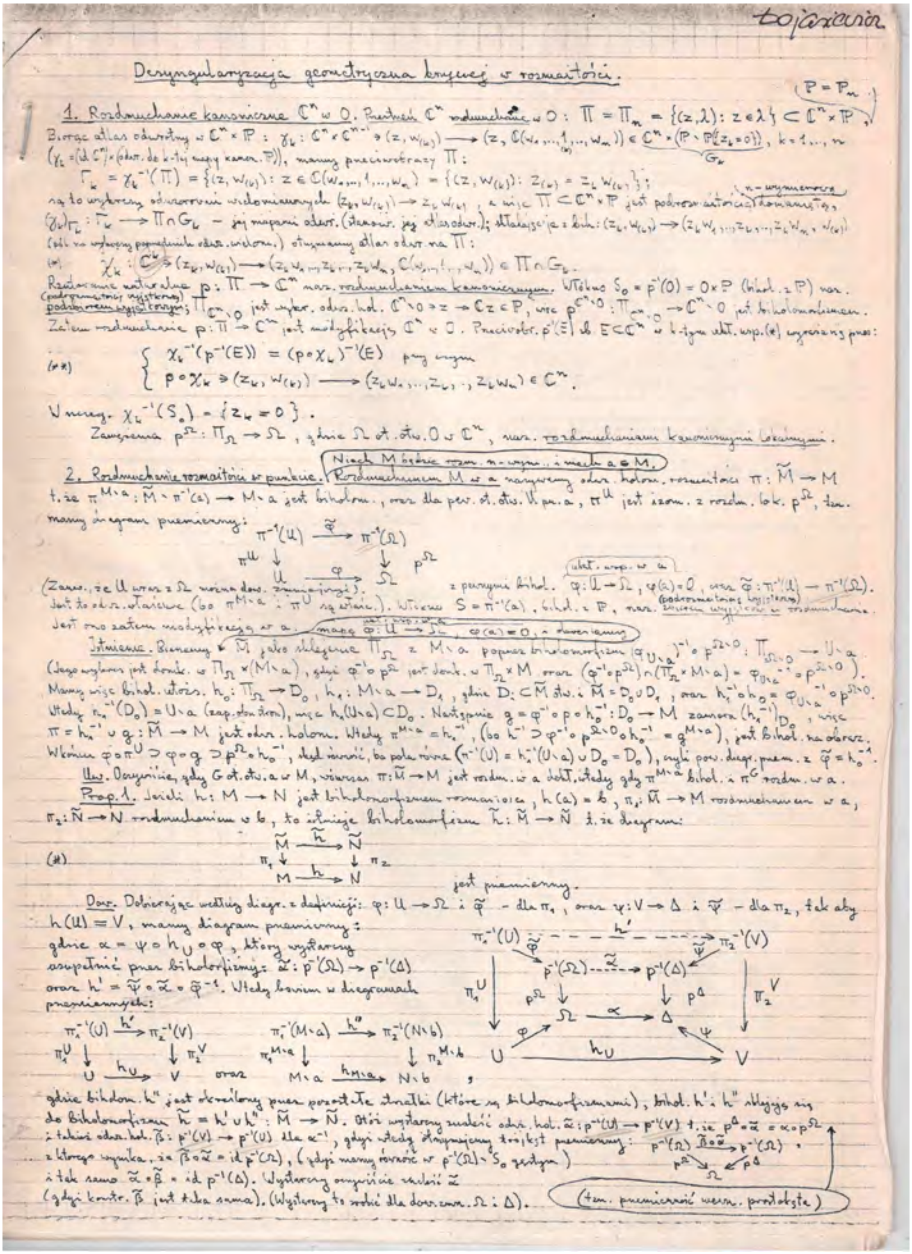
We dedicate the whole volume to the memory of Stanisław Łojasiewicz.

Tadeusz Krasiński
Stanisław Spodzieja

November 2013, Łódź



Stanisław Łojasiewicz (9 X 1926 – 14 XI 2002)
(The photo was taken by Przemysław Skibiński in 2000)



Łojasiewicz

Desyngularyzacja geometryczna krzywej w rozmaitości.

The facsimile of the first page of the Polish handwritten version of the article (1988) by Stanisław Łojasiewicz, translated in this volume.

Analytic and Algebraic Geometry
Łódź University Press 2013, 11 – 32

GEOMETRIC DESINGULARIZATION OF CURVES IN MANIFOLDS *) **)

STANISŁAW ŁOJASIEWICZ

1. Introduction

The article does not pretend to any originality. In the literature there exists a number of descriptions of desingularizations in the case of curves. Deciding for this description the author think it is worth looking in details into this fascinating topic in an easily accessible case, namely – in the effects of multi blowings-up for curves in manifolds and for coherent sheaves on 2-dimensional manifolds.

All the needed facts from analytic geometry can be find in the author's books [L1], [L2].

2. The canonical blowing-up of $\mathbb{C}^n$ at 0

The *blow-up of* $\mathbb{C}^n$ *at* 0 is

$$\Pi = \Pi_n = \{(z, \lambda) : z \in \lambda\} \subset \mathbb{C}^n \times \mathbb{P}, \quad \mathbb{P} = \mathbb{P}_{n-1}.$$

Taking the inverse atlas for $\mathbb{C}^n \times \mathbb{P}$

$$\gamma_k : \mathbb{C}^n \times \mathbb{C}^{n-1} \ni (z, w_{(k)}) \mapsto$$
$$(z, \mathbb{C}(w_1, ..., \underset{(k)}{1}, ..., w_n)) \in \mathbb{C}^n \times \{\mathbb{P} \setminus \mathbb{P}(\{z_k = 0\}))= G_k,\ k = 1, ..., n,$$

2010 *Mathematics Subject Classification.* Primary 32Sxx, Secondary 14Hxx.
Key words and phrases. Resolution of singularities, curve, blowing-up, coherent analytic sheaf.

*) This article was published (in Polish) in the proceedings of X^{th} Workshop on Theory of Extremal Problems (1989) and has never appeared in translation elsewhere. To honor this outstanding mathematician (who passed away in 2002) this article was translated into English (by T. Krasiński) in order to make it accesible to the mathematical community.

**) The translator thanks Dinko Pervan (an Erasmus student from Croatia) for preparing the article in TeX and W. Kucharz, A. Płoski and Sz. Brzostowski for improving the English text.

(that is $\gamma_k = (\mathrm{id}\,\mathbb{C}^n) \times$ (inverse mapping to the k-th canonical map on $\mathbb{P}$)), we have the inverse images of Π

$$\Gamma_k = \gamma_k^{-1}(\Pi) = \{(z, w_{(k)}) : z \in \mathbb{C}(w_1, ..., 1, ..., w_n)\} = \left\{(z, w_{(k)}) : z_{(k)} = z_k w_{(k)}\right\};$$

they are graphs of the polynomial mappings $(z_k, w_{(k)}) \to z_k w_{(k)}$, whence $\Pi \subset \mathbb{C}^n \times \mathbb{P}$ is an n-dimensional closed submanifold, $(\gamma_k)_{\Gamma_k} : \Gamma_k \to \Pi \cap G_k$ – its inverse maps (they give an inverse atlas on Π); composing them with biholomorphisms: $(z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n, w_{(k)})$ (domains onto the graphs of the preceding polynomial mappings) we obtain an inverse atlas on Π

$$(*) \qquad \chi_k : \mathbb{C}^n \ni (z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n, \mathbb{C}(w_1, ..., 1, ..., w_n)) \in \Pi \cap G_k.$$

The canonical projection $p : \Pi \to \mathbb{C}^n$ is called the canonical blowing-up. The fiber $S_0 = p^{-1}(0) = 0 \times \mathbb{P}$ (biholomorphic to $\mathbb{P}$) is called the exceptional set (the exceptional submanifold); $\Pi_{\mathbb{C}^n \setminus 0}$ is the graph of the holomorphic mapping $\mathbb{C}^n \setminus 0 \ni z \to \mathbb{C}z \in \mathbb{P}$, whence $p^{\mathbb{C}^n \setminus 0} : \Pi_{\mathbb{C}^n \setminus 0} \to \mathbb{C}^n \setminus 0$ is a biholomorphism. Hence the blowing-up $p : \Pi \to \mathbb{C}^n$ is a modification of $\mathbb{C}^n$ at 0. The inverse image $p^{-1}(E)$ of a set $E \subset \mathbb{C}^n$ in the k-th coordinate system $(*)$ can be expressed by

$$(**) \qquad \begin{cases} \chi_k^{-1}(p^{-1}(E)) = (p \circ \chi_k)^{-1}(E) \text{ where} \\ p \circ \chi_k \ni (z_k, w_{(k)}) \to (z_k w_1, ..., z_k, ..., z_k w_n) \in \mathbb{C}^n. \end{cases}$$

In particular $\chi_k^{-1}(S_0) = \{z_k = 0\}$.

The restrictions $p^\Omega : \Pi_\Omega \to \Omega$, where Ω is an open neighbourhood of 0 at $\mathbb{C}^n$, are called the local canonical blowings-up.

3. The blowing-up of a manifold at a point

Let M be an n-dimensional manifold and $a \in M$. A blowing-up of M at the point a is a holomorphic mapping of manifolds $\pi : \bar{M} \to M$ such that $\pi^{M \setminus a} : \bar{M} \setminus \pi^{-1}(a) \to M \setminus a$ is a biholomorphism and for an open neighbourhood U of a, the mapping π^U is isomorphic to a local canonical blowing-up p^Ω i.e. we have a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\bar{\phi}} & p^{-1}(\Omega) \\ \Big\downarrow{\scriptstyle \pi^U} & & \Big\downarrow{\scriptstyle p^\Omega} \\ U & \xrightarrow{\phi} & \Omega \end{array}$$

for some biholomorphisms $\phi : U \to \Omega$, $\phi(a) = 0$ and $\bar{\phi} : \pi^{-1}(U) \to p^{-1}(\Omega)$. (Notice that U and Ω can be abitrarily diminished). π is a proper mapping (because $\pi^{M \setminus a}$ and π^U are proper). The fiber $S = \pi^{-1}(a)$, biholomorphic to $\mathbb{P}$, is called the exceptional set (the exceptional submanifold) of the blowing-up. Thus π is a modification of M at a.

The existence of blowing-up. We take a chart (a coordinate system) at a: $\phi : U \to \Omega$, $\phi(a) = 0$, and define $\bar{M}$ as a gluing-up of π_Ω with $M\setminus a$ by the biholomorphism $(\phi_{U\setminus a})^{-1} \circ p^{\Omega\setminus 0} : \Pi_{\Omega\setminus 0} \to U\setminus a$. (Its graph is closed in $\Pi_\Omega \times (M\setminus a)$ because $\phi^{-1} \circ p^\Omega$ is a closed set in $\Pi_\Omega \times M$ and $\left(\phi^{-1} \circ p^\Omega\right) \cap (\Pi_\Omega \times M\setminus a) = \phi^{-1}_{U\setminus a} \circ p^{\Omega\setminus 0}$). So we have the identifying biholomorphisms $h_0 : \Pi_\Omega \to D_0$, $h_1 : M\setminus a \to D_1$, where $D_i \subset \bar{M}$, $i = 0, 1$, are open sets, $\bar{M} = D_0 \cup D_1$ and $h_1^{-1} \circ h_0 = \phi^{-1}_{U\setminus a} \circ p^{\Omega\setminus 0}$. Hence $h_1^{-1}(D_0) = U\setminus a$ (the domains of both sides) which implies $h_1(U\setminus a) \subset D_0$. Next $g = \phi^{-1} \circ p \circ h_0^{-1} : D_0 \to M$ contains $(h_1^{-1})_{D_0}$, and hence $\pi = h_1^{-1} \cup g : \bar{M} \to M$ is a holomorphic maping. Then $\pi^{M\setminus a} = h_1^{-1}$ (because $h^{-1} \supset \phi^{-1} \circ p^{\Omega\setminus 0} \circ h_0^{-1} = g^{M\setminus a}$) is a biholomorphism on the image. At last, $\phi \circ \pi^U \supset \phi \circ g \supset p^\Omega \circ h_0^{-1}$ which implies the equality, because the domains are equal ($\pi^{-1}(U) = h_1^{-1}(U\setminus a) \cup D_0 = D_0$), whence the above diagram is commutative with $\bar{\phi} := h_0^{-1}$.

Remark 1. *Obviously, if G is an open neighbourhood of a at M then $\pi : \bar{M} \to M$ is a blowing-up at a if and only if $\pi^{M\setminus a}$ is a biholomorphism and π^G is a blowing-up at a.*

Proposition 1. *If $h : M \to N$ is a biholomorphism of manifolds, $h(a) = b$, $\pi_1 : \bar{M} \to M$ is a blowing-up at a, $\pi_2 : \bar{N} \to N$ a blowing-up at b, then there exists a biholomorphism $\bar{h} : \bar{M} \to \bar{N}$ such that the diagram*

$$
\begin{tikzcd}
\bar{M} \arrow[r, "\bar{h}"] \arrow[d, "\pi_1"'] & \bar{N} \arrow[d, "\pi_2"] \\
M \arrow[r, "h"] & N
\end{tikzcd}
\qquad (\#)
$$

is commutative

Dowód. Choosing by definition: $\phi : U \to \Omega$ and $\bar{\phi}$ - for π_1, and $\psi : V \to \Delta$ and $\bar{\psi}$ - for π_2, such that $h(U) = V$, we have a commutative diagram

$$\begin{tikzcd}
\pi_1^{-1}(U) \arrow[rrr, dashed, "h'"] \arrow[dr, "\bar{\phi}"] \arrow[ddd, "\pi_1^U"'] & & & \pi_2^{-1}(V) \arrow[dl, "\bar{\psi}"'] \arrow[ddd, "\pi_2^V"] \\
& p^{-1}(\Omega) \arrow[r, dashed, "\bar{\alpha}"] \arrow[d, "p^\Omega"'] & p^{-1}(\Delta) \arrow[d, "p^\Delta"] & \\
& \Omega \arrow[r, "\alpha"] & \Delta & \\
U \arrow[rrr, "h_U"] \arrow[ur, "\phi"] & & & V \arrow[ul, "\psi"']
\end{tikzcd}$$

where $\alpha := \psi \circ h_U \circ \phi^{-1}$, and it suffices to complement it by biholomorphisms: $\bar{\alpha} : p^{-1}(\Omega) \to p^{-1}(\Delta)$ and $h' := \bar{\psi}^{-1} \circ \bar{\alpha} \circ \bar{\phi}$. Then in the commutative diagrams

$$\begin{tikzcd}
\pi_1^{-1}(U) \arrow[r, "h'"] \arrow[d, "\pi_1^U"'] & \pi_2^{-1}(V) \arrow[d, "\pi_2^V"] & \pi_1^{-1}(M\backslash a) \arrow[r, "h''"] \arrow[d, "\pi_1^{M\backslash a}"'] & \pi_2^{-1}(N\backslash b) \arrow[d, "\pi_2^{N\backslash b}"] \\
U \arrow[r, "h_U"] & V & M\backslash a \arrow[r, "h_{M\backslash a}"] & N\backslash b
\end{tikzcd}$$

where the biholomorphism h'' is defined by the remaining arrows (which are biholomorphisms), the biholomorphisms h' and h'' give rise to a biholomorphism $\bar{h} = h' \cup h'' : \bar{M} \to \bar{N}$. In fact, it suffices to find a holomorphic mapping $\bar{\alpha} : p^{-1}(\Omega) \to p^{-1}(\Delta)$ such that $p^\Delta \circ \bar{\alpha} = \alpha \circ p^\Omega$ (i.e. the commutativity of the inner rectangle) and a similar holomorphic mapping $\bar{\beta} : p^{-1}(\Delta) \to p^{-1}(\Omega)$ for α^{-1}, since then we obtain the commutative triangle

$$\begin{tikzcd} p^{-1}(\Omega) \arrow[rr, "\bar\beta\circ\bar\alpha"] \arrow[dr, "p^\Omega"'] & & p^{-1}(\Omega) \arrow[dl, "p^\Delta"] \\ & \Omega & \end{tikzcd}$$

which implies $\bar\beta \circ \bar\alpha = \mathrm{id}_{p^{-1}(\Omega)}$ (because we have the equality on the dense set $p^{-1}(\Omega)\backslash S_0$), and similarly $\bar\alpha \circ \bar\beta = \mathrm{id}_{p^{-1}(\Delta)}$. Obviously it suffices to find $\bar\alpha$ (because the construction of $\bar\beta$ is analogous) for sufficiently small Ω and Δ.

According to the Hadamard Lemma (since $\alpha(0) = 0$) one can choose neighbourhoods Ω, Δ such that $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i(z) = \sum_{j=1}^{n} a_{ij}(z) z_j$ and $\det a_{ij}(z) \neq 0$ in Ω. Define $a(z,w) = (\sum_{j=1}^{n} a_{1j}(z)w_j, ..., \sum_{j=1}^{n} a_{nj}(z)w_j)$ in $\Omega \times \mathbb{C}^n$; then $a(z,z) = \alpha(z)$ and $a(z,w) \neq 0$ for $w \neq 0$. Hence we may define a holomorphic mapping $\bar a : \Omega \times \mathbb{P} \ni (z, \mathbb{C}w) \to (\alpha(z), \mathbb{C}a(z,w)) \in \Delta \times \mathbb{P}$. Since $\bar a(z, \mathbb{C}z) = (\alpha(z), \mathbb{C}\alpha(z))$ for $z \in \Omega\backslash 0$ and $\bar a(0 \times \mathbb{P}) \subset 0 \times \mathbb{P}$, then we have the holomorphic restriction $\bar\alpha = \bar a_{\Pi_\Omega} : \Pi_\Omega \to \Pi_\Delta$, and hence $p^\Delta(\bar\alpha(z, \mathbb{C}z)) = \alpha(z) = \alpha(p^\Omega(z, \mathbb{C}z))$ for $z \in \Omega\backslash 0$, that is $p^\Delta \circ \bar\alpha = \alpha \circ p^\Omega$ by density of $\Pi_{\Omega\backslash 0}$ in Π_Ω. □

4. The proper inverse image

Let $\pi : \bar M \to M$ be a blowing-up at a point $a \in M$. The proper inverse image (by π) of a set $V \subset M$ closed in a neighbourhood of a (i.e. $V \cap U$ is a closed set in U for some neighbourhood U of a) is defined by

$$\bar V = \text{the closure of the set } \pi^{-1}(V\backslash a) = \pi^{-1}(V)\backslash S \text{ in } \pi^{-1}(V).$$

(It is obtained from the set $\pi^{-1}(V)\backslash S$ by adding to it its accumulation points belonging to S). If V is analytic in a neighbourhood of a then $\bar V$ is analytic in a neighbourhood of the exceptional set S (since $\pi^{-1}(V)$ and S are analytic in a neighbourhood of S). Obviously

$$\pi^{-1}(V) = \bar V \cup S.$$

If U is an open neighbourhood of a, then the proper inverse image of the set $V \cap U$ is $\bar V \cap \pi^{-1}(U)$. If $W \subset V$ then $\bar W \subset \bar V$, and if $V = \bigcup_{i=1}^{k} Z_i$, then $\bar V = \bigcup_{i=1}^{k} \bar Z_i$, (provided W, Z_i are closed in a neighbourhood of a). If $D \supset V$ is an open neighbourhood of a then $\bar V$ is the proper inverse image of V if and only if it is the same by the blowing-up π^D.

In Proposition 1 the biholomorphism $\bar{h}$ sends the exceptional submanifold $\pi_1^{-1}(a)$ onto the exceptional submanifold $\pi_2^{-1}(b)$, and the proper inverse image of V onto the proper inverse image of $h(V)$.

The proper inverse image of a linear subspace $L \subset \mathbb{C}^n$ of dimension k by the canonical blowing-up is $\bar{L} = \{(z, \lambda) \in L \times \mathbb{P}(L) : z \in \lambda\}$; it is a submanifold of dimension k and $p_{\bar{L}} : \bar{L} \to L$ is a blowing-up at 0. (For taking an isomorphism $\chi : L \to \mathbb{C}^k$ we have the commutative diagram

$$\begin{tikzcd} \bar{L} \arrow[r, "\psi_{\bar{L}}"] \arrow[d, "p_{\bar{L}}"'] & \Pi_k \arrow[d, "p_k"] \\ L \arrow[r, "\chi"] & \mathbb{C}^k \end{tikzcd}$$

where $\psi = \chi \times \chi' : L \times \mathbb{P}(L) \to \mathbb{C}^k \times \mathbb{P}_k$, $\chi' : \mathbb{P}(L) \ni \lambda \to \chi'(\lambda) \in \mathbb{P}_k$ are biholomorphisms and $\psi(\overline{L}) = \Pi_k$).

5. The transversality

Proposition 2. *If M is a linear space of dimension n then linear subspaces $L_1, ..., L_r \subset M$ intersect transversally (in M) if and only if in some linear coordinate system in M it is*

$$L_i = \{z_\upsilon = 0, \upsilon \in I_i\}, \quad \text{where } I_1, ..., I_r \subset \{1, ..., n\} \text{ are disjoint.}$$

Dowód. The sufficiency is obvious because $\operatorname{codim} L_i = \# I_i$. Conversely, if L_i intersect transversally, then the sum $\sum L_i^\perp = (\bigcap L_i)^\perp$ is direct because $\dim \sum L_i^\perp = \operatorname{codim} \bigcap L_i = \sum \operatorname{codim} L_i = \sum \dim L_i^\perp$. Hence there exists a basis $\phi_1, ..., \phi_n$ of the dual space M^* such that $\{\phi_\upsilon : \upsilon \in I_i\}$ generate $L_i^\perp$ where $I_i \subset \{1, ..., n\}$ are disjoint. Then $L_i = \{\phi_\upsilon = 0, \upsilon \in I_i\}$, that is $L_i = \{z_\upsilon = 0, \upsilon \in I_i\}$ in the coordinate system $\phi = (\phi_1, ..., \phi_n)$ (because $\phi^{-1}(\{z_\upsilon = 0, \upsilon \in I_i\}) = L_i$). □

Corollary 1. *If L_i, $i \in I$, intersect transversally and $J \subset I$, then also L_i, $i \in J$, intersect transversally. If $I \cap J = \emptyset$ and L_i, $i \in I \cup J$, intersect transversally then so do $\bigcap_I L_i$ and $\bigcap_J L_i$. If $L_1, ..., L_r, T$ intersect transversally then so do $L_1 \cap T, ..., L_r \cap T$ in T.*

Proposition 3. *If M is a manifold of dimension n, then submanifolds $N_1, ..., N_r$ intersect transversally at a point $a \in \bigcap N_i$ if and only if there exists a chart (a coordinate system at a) $\phi : U \to \Omega$, $\phi(a) = 0$, such that $\phi(N_i \cap U) = T_i \cap \Omega$, where*

$T_i \subset \mathbb{C}^n$ are subspaces that intersect transversally, so it may be

$$T_i = \{u_i = 0\}, \text{ where } z = (u_1, ..., u_r, v) \in \mathbb{C}^n = \mathbb{C}^{I_1} \times ... \times \mathbb{C}^{I_r} \times \mathbb{C}^J.$$

Dowód. The sufficiency is clear. For the necessity we may assume $M = \mathbb{C}^n$, $a = 0$ and $T_0 N_i = T_i$ as above. Then there exists an open neighbourhood $U = \Omega_1 \times ... \times \Omega_r \times \Delta$ of the origin in $\mathbb{C}^n$ and functions $\varepsilon_i(u_{(i)}, v)$ with values in $\mathbb{C}^{I_i}$, holomorphic in $U_i = \Omega_1 \times ..._{(i)}... \times \Omega_r \times \Delta$, such that $d_0\varepsilon_i = 0$ and $N_i \cap U = \{u_i = \varepsilon_i(u_{(i)}, v), (u_{(i)}, v) \in U_i\}$. After shrinking U the mapping $\phi : U \ni z \to (u_1 - \varepsilon_1(u_{(1)}, v), ..., u_r - \varepsilon_r(u_{(r)}, v), v) \in \Omega$ is a biholomorphism onto a neighbourhood Ω of the origin and hence $N_i \cap U = \phi^{-1}(T_i)$ which implies $\phi(N_i \cap U) = T_i \cap \Omega$. $\square$

Corollary 2. *If submanifolds N_i, $i \in I$, intersect transversally at a point a and $J \subset I$, then so do the submanifolds N_i, $i \in J$. If $I \cap J = \emptyset$ and submanifolds $N_i, i \in I \cup J$, intersect transversally at a then so do the submanifolds $\bigcap_I N_i$ and $\bigcap_J N_i$.*

Corollary 3. *If submanifolds N_i intersect transversally then $N = \bigcap N_i$ is a submanifold and* $\operatorname{codim} N = \sum \operatorname{codim} N_i$.

We say submanifolds N_i of a manifold M are <u>mutually transversal</u> in an open set $G \subset M$, if $N_i \cap G$ are closed and for each $a \in G$ submanifolds N_i containing a intersect transversally at a. Notice that if subspaces of a linear space intersect transversally then they are mutually transversal in this space (by Corollary 1 and from the fact that if subspaces intersect transversally, then they intersect transversally at each point of their intersection). Hence (by Proposition 3)

Corollary 4. *If submanifolds N_i intersect transversally at $a \in \bigcap N_i$, then they are mutually transversal in a neighbourhood of the point a.*

6. The effect of blowing-up

Let M be a manifold of dimension n and let $\pi : \bar{M} \to M$ be a blowing-up at point $a \in M$, and $S = \pi^{-1}(a) \subset \bar{M}$ – the exceptional set.

Proposition 4. *If $\Gamma \subset M$, $\Gamma \ni a$, is a submanifold of dimension s then its proper inverse image $\bar{\Gamma} \subset \bar{M}$ is a submanifold of dimension s which intersects S transversally and the submanifold $\bar{\Gamma} \cap S$ is biholomorphic to $\mathbb{P}_{s-1}$. Then $\pi_{\bar{\Gamma}} : \bar{\Gamma} \to \Gamma$ is a blowing-up at a with the exceptional set $\bar{\Gamma} \cap S$.*

Dowód. The set $\bar{\Gamma} \backslash S = \pi^{-1}(\Gamma \backslash a)$ is a submanifold of dimension s and $(\pi_{\bar{\Gamma}})^{\Gamma \backslash a} : \bar{\Gamma} \backslash S \to \Gamma \backslash a$ is a biholomorphism. Let us take a chart $\phi : U \to \Omega$, $\phi(a) = 0$, such that $\phi(\Gamma \cap U) = L \cap \Omega$, where $L = \{z_1 = ... = z_r = 0\}$ $(r = n - s)$. It suffices to show the proposition for π^U and $\Gamma \cap U$ because then the proper inverse image of $\Gamma \cap U$, that is $\bar{\Gamma} \cap \pi^{-1}(U)$, will be a submanifold (of dimension s) and $(\pi^U)_{\bar{\Gamma} \cap \pi^{-1}(U)} = (\pi_{\bar{\Gamma}})^{\Gamma \cap U}$ will be a blowing-up at a, whence $\bar{\Gamma}$ will be a submanifold and $\pi_{\bar{\Gamma}}$ a blowing-up at

a (see Remark 1). According to Proposition 1, it suffices to prove the proposition for p^Ω, $L\cap\Omega$ and 0. Since the proper inverse image of $L\cap\Omega$ is $\bar{L}\cap p^{-1}(\Omega)$, where $\bar{L}$ is the proper inverse image of L by p, and $(p^\Omega)_{\bar{L}\cap p^{-1}(\Omega)} = (p_{\bar{L}})^{L\cap\Omega}$, then it suffices to prove the proposition for p, L and 0. But $\bar{L}$ is a submanifold of dimension s, $p_{\bar{L}} : \bar{L} \to L$ is a blowing-up at 0 and $\bar{L}\cap S_0 = 0\times\mathbb{P}(L)$ (see Section 4). It remains to prove the transversality. We have (see (**) in Section 2)

$$\chi_k^{-1}(p^{-1}(L) = \begin{cases} \{z_k = 0\} \text{ if } k \leqslant r \\ \{z_k = 0\} \cup \{w_1 = ... = w_r = 0\} \text{ if } k > r, \end{cases}$$

so by $\chi_k^{-1}(S_0) = \{z_k = 0\}$ it is

$$\chi_k^{-1}(\bar{L}) = \begin{cases} \emptyset \text{ if } k \leqslant r \\ \{w_1 = ... = w_r = 0\} \text{ if } k > r, \end{cases}$$

whence (Proposition 2) the transversality of the intersection of $\bar{L}$ and S_0 follows. □

Proposition 5. *If submanifolds $\Gamma_1, ..., \Gamma_r \subset M$ intersect transversally at a and $\bar{\Gamma}_1, ..., \bar{\Gamma}_r$ are their proper inverse images then $\bar{\Gamma}_1, ..., \bar{\Gamma}_r, S$ are mutually transversal in a neighbourhood of S. If additionally Γ_i intersect transversally then the proper inverse image of $\Gamma = \bigcap \Gamma_i$ is $\bar{\Gamma} = \bigcap \bar{\Gamma}_i$.*

Dowód. If U is an open neighbourhood of a then the proper inverse image of $\Gamma_i\cap U$ $(\Gamma\cap U)$ is $\bar{\Gamma}_i\cap\pi^{-1}(U)$ $(\bar{\Gamma}\cap\pi^{-1}(U))$. By Propositions 3 and 1 it suffices to consider the canonical blowing-up p and $\Gamma_i = T_i = \{z_\upsilon = 0, \upsilon \in I_i\}$, I_i disjoint (by the fact $\bar{\Gamma}\backslash S = \bigcap(\bar{\Gamma}_i\backslash S)$). Let $\bar{T_i}$ denote the proper inverse image of T_i. We have (see (**) in Section 2)

$$\chi_k^{-1}(p^{-1}(T_i) = \begin{cases} \{z_k = 0\} \text{ if } k \in I_i \\ \{z_k = 0\} \cup \{w_\upsilon = 0, \upsilon \in I_i\} \text{ if } k \notin I_i, \end{cases}$$

so

$$\chi_k^{-1}(\bar{T}_i) = \begin{cases} \emptyset \text{ if } k \in I_i \\ \{w_\upsilon = 0, \upsilon \in I_i\} \text{ if } k \notin I_i, \end{cases}$$

which implies (Proposition 2) that $\bar{T}_i, ..., \bar{T}_r, S$ are mutually tranversal in Π. If $\bar{T}$ is the proper inverse image of $T = \bigcap T_i$ then $T = \{z_\upsilon = 0, \upsilon \in I\}$, where $I = \bigcup I_i$, and in the same way

$$\chi_k^{-1}(\bar{T}) = \begin{cases} \emptyset \text{ if } k \in I \\ \{w_\upsilon = 0, \upsilon \in I\} \text{ if } k \notin I, \end{cases}$$

so $\chi_k^{-1}(\bar{T}) = \bigcap \chi_k^{-1}(\bar{T}_i)$, whence $\bar{T} = \bigcap \bar{T}_i$. □

Let $\mathcal{C}(a) = \mathcal{C}(a, M)$ denote the set of curves $\Gamma \subset M$ (i.e. local analytic subsets of constant dimension 1) such that $a \in \Gamma$ and the germ Γ_a is irreducible. Then

$$\mathcal{C}(a) = \bigcup_{p=1}^{\infty} \mathcal{C}_p(a), \tag{6.1}$$

where $\mathcal{C}_p(a) = \mathcal{C}_p(a, M)$ denotes the set of curves Γ in $\mathcal{C}(a)$ having, in some coordinate system ϕ in a (i.e. ϕ is a chart such that $\phi(a) = 0$), the form (that is $\phi(\Gamma)$ is a set of the form)

$$\left\{ \begin{array}{l} z_1 = t^p \\ v = c(t)t^q \end{array} \right. \qquad |t| < \sigma, \tag{6.2}$$

where $v = (z_2, ...z_n)$, $q \geqslant p$, and c is a holomorphic function in $\{|t| < \sigma\}$ $(\sigma > 0)$. (For it is of the form $\{f(t) : |t| < \sigma\}$, where f is a holomorphic mapping, a homeomorphism onto its image, $f(0) = 0$; it is $f(t) = g(t)t^p$, $p \geqslant 1$, $g(0) \neq 0$, and after changing the system of coordinates one may have $g_1(0) \neq 0$; then $g_1 = \gamma^p$ with γ holomorphic in a neighbourhood of the origin, $\gamma(0) \neq 0$, and it suffices to change the parameter putting $\tau = \gamma(t)t$ in a neighbourhood of the origin). In particular, $\mathcal{C}_1(a)$ is the set of all curves $\Gamma \ni a$ smooth at a.

A set Γ_0 of the form (6.2) (without any restriction on q) is always a curve in $\mathbb{C}^n$ having its germ irreducible at 0. (For the mapping $\{|t| < \sigma\} \ni t \to (t^p, c(t)t^q) \in \{|z_1| < \sigma^p\} \subset \mathbb{C}^n$ is proper). Let us notice that replacing σ by $0 < \bar{\sigma} < \sigma$ we obtain an open neighbourhood of 0 in Γ_0 (precisely $\Gamma_0 \cap \{|z_1| < \bar{\sigma}^p\}$). If $0 < q < p$ and $c(0) \neq 0$ then $\Gamma_0 \in \mathcal{C}_q$. In fact, if for example $c_2(0) \neq 0$ then (changing the parameter to $\tau = t\gamma(t)$, where $\gamma^q = c_2$) for sufficiently small ε, a neighbourhood U_ε of the origin and holomorphic b_i, the sets $\Gamma_\varepsilon = \{z_1 = t^p, v = c(t)t^q, t \in U_\varepsilon\} = \{z_2 = \tau^q, z_i = b_i(\tau)\tau^q,\ i \neq 2,\ |\tau| < \varepsilon\}$ are neighbourhoods of 0 in Γ_0. But $\Gamma_{\varepsilon_0} \subset \Gamma_0 \cap \{|z_1| < \sigma_0\} \subset \Gamma_\varepsilon$ for some $\sigma_0, \varepsilon_0 > 0$, hence Γ_{ε_0} is an open set in Γ_ε and so in Γ_0.

It is

$$\mathcal{C}_p(a) = \mathcal{C}_1(a) \cup \bigcup \mathcal{C}_{p,q}(a), \tag{6.3}$$

where $\mathcal{C}_{p,q}(a), q > p$ is not divisible by p, is the set of all the curves in $\mathcal{C}(a)$ that have the form (6.2) in some coordinate system at a, where $c(0) \neq 0$. In fact, if in (6.2) we have $v = \sum c_{p\nu} t^{p\nu}$ then the curve (6.2) is smooth (it suffices to change the parameter to $\tau = t^p$). In the remaining cases $v = a_p t^p + ... + a_{kp} t^{kp} + c(t)t^q$, where $c(0) \neq 0$ and $pk < q < p(k+1)$, and it suffices to replace the coordinates to $z_1' = z_1, v' = v - a_p z_1 - ... - a_{kp} z_1^k$ (it is a biholomorphism of $\mathbb{C}^n$ onto $\mathbb{C}^n$).

Let us notice that if a curve $\Gamma \ni a$ is smooth at a, then its proper inverse image $\bar{\Gamma}$ intersects S at a unique point: $\bar{\Gamma} \cap S = \{\bar{a}\}$ and in a transversall way.

Proposition 6. *Let Γ be a curve in $\mathcal{C}_{p,q}$, $p > 1$. Then its proper inverse image $\bar{\Gamma}$ is a curve and $\bar{\Gamma} \cap S = \{\bar{a}\}$; if $q > 2p$ then $\bar{\Gamma} \in \mathcal{C}_{p,q-p}(\bar{a})$, and if $q < 2p$ then $\bar{\Gamma} \in \mathcal{C}_{q-p}(\bar{a})$.*

Dowód. We may restrict considerations to the canonical blowing-up ($a = 0$) and Γ of form (2), where $c(0) \neq 0$ and $|c(t)| \leqslant M$. Then (see $(**)$ in Section 2)

$$\begin{aligned}\chi_1^{-1}(p^{-1}(\Gamma)) &= \{z_1 = t^p, z_1 w_{(1)} = c(t)t^q, |t| < \sigma\} \\ &= \{z_1 = 0\} \cup \{z_1 = t^p, w_{(1)} = c(t)t^{q-p},\ |t| < \sigma\},\end{aligned}$$

and for $k > 1$

$$\begin{aligned}\chi_k^{-1}(p^{-1}(\Gamma)) &= \{z_k w_1 = t^p, ..., z_k = c_k(t)t^q, ...,\quad |t| < \sigma\} \\ &\subset \{z_k = 0\} \cup \{|z_k|^{q-p}|w_1|^q \geqslant M^{-p}\}.\end{aligned}$$

Hence

$$\chi_1^{-1}(\bar\Gamma) = \{z_1 = t^p, w_{(1)} = c(t)t^{q-p},\ |t| < \sigma\} \in \begin{cases} \mathcal{C}_{p,q-p}(0) \text{ if } q > 2p \\ \mathcal{C}_{q-p}(0) \text{ if } q < 2p \end{cases}$$

and $\chi_k^{-1}(\bar\Gamma) \cap \chi_k^{-1}(S) = \emptyset$ for $k > 1$. Then $\bar\Gamma \cap S = \{\bar a\}$, where $\bar a = \chi_1(0)$, and $\bar\Gamma \in \mathcal{C}_{p,q-p}(\bar a)$ if $q > 2p$, and $\bar\Gamma \in \mathcal{C}_{q-p}(\bar a)$ if $q < 2p$. $\square$

Smooth curves $\Gamma_1, \Gamma_2 \ni a$ are tangent of order p at a if in some (and then in each) coordinate system ϕ at a in which they are topographic: $\phi(\Gamma_i) = \{v = g_i(z_1), z_1 \in U_i\}$, the function $g_2 - g_1$ has a zero of order p at 0.

Proposition 7. *Let smooth curves $\Gamma_1, \Gamma_2 \ni a$ be tangent of order p at a, and let $\bar\Gamma_1, \bar\Gamma_2$ be their proper inverse images. If $p > 1$ then $\bar\Gamma_1 \cap S = \bar\Gamma_2 \cap S = \{\bar a\}$ and $\bar\Gamma_1, \bar\Gamma_2$ are tangent of order $p - 1$ at a; if $p = 1$ then $\bar\Gamma_1 \cap S \neq \bar\Gamma_2 \cap S$.*

Dowód. We may restrict considerations to the canonical blowing-up ($a = 0$) and $\Gamma_1 = \{v = 0, |z_1| < \sigma\}$, $\Gamma_2 = \{v = c(z_1)z_1^p, |z_1| < \sigma\}$, c is a holomorphic mapping, $c(0) \neq 0$, $|c(z_1)| \leqslant M$. Then (see $(**)$ in Section 2) $\chi_1^{-1}(p^{-1}(\Gamma_1)) = \{z_1 = 0\} \cup \{w_{(1)} = 0, |z_1| < \sigma\}$, $\chi_1^{-1}(p^{-1}(\Gamma_2)) = \{z_1 = 0\} \cup \{w_{(1)} = c(z_1)z_1^{p-1}, |z_1| < \sigma\}$ and $\chi_k^{-1}(p^{-1}(\Gamma_i)) \subset \{|z_k| \leqslant M|z_k w_1|^p\} \subset \{z_k = 0\} \cup \{|z_k|^p|w_1|^{p-1} \geqslant 1/M\}$ for $k > 1$. Hence $\chi_k^{-1}(\bar\Gamma_i) \cap \chi_k^{-1}(S) = \emptyset$ for $k > 1$ and $\chi_1^{-1}(\bar\Gamma_1) = \{w_{(1)} = 0, |z_1| < \sigma\}$ and $\chi_1^{-1}(\bar\Gamma_2) = \{w_{(1)} = c(z_1)z_1^{p-1}, |z_1| < \sigma\}$. So if $p > 1$ then $\bar\Gamma_1 \cap S = \bar\Gamma_2 \cap S = \{\bar a\}$, where $\bar a = \chi_1(0)$, and $\bar\Gamma_1, \bar\Gamma_2$ are tangent of order $p - 1$ at $\bar a$. If in turn $p = 1$ then $\bar\Gamma_1 \cap S = \{\chi_1(0)\}$ and $\bar\Gamma_2 \cap S = \{\chi_1(0, c(0))\}$. $\square$

A smooth curve $\Gamma \ni a$ is tangent of order p at a to a submanifold $N \ni a$ if it is tangent of order p at a to a smooth curve $\Gamma_0 = N \cap L$, where L is a submanifold of dimension $\operatorname{codim} N + 1$ transversal to N and containing a neighbourhood of a at Γ.

Proposition 8. *Let a smooth curve $\Gamma \ni a$ be tangent of order p at a to a submanifold $N \ni a$; let $\bar\Gamma, \bar N$ be their proper inverse images and let $\bar\Gamma \cap S = \{\bar a\}$. If $p > 1$ then $\bar a \in \bar N$ and $\bar\Gamma$ is tangent of order $p - 1$ at $\bar a$ to $\bar N$; if $p = 1$ then $\bar a \notin \bar N$.*

Dowód. One can assume that the submanifold L contains Γ, is transversal to N and the smooth curve $\Gamma_0 = N \cap L$ is tangent of order p at a to Γ. So, we have $\bar L \supset \bar\Gamma$, $\bar L$ is transversal to $\bar N$ and $\bar\Gamma_0 = \bar N \cap \bar L$ is a smooth curve (Proposition 5). According to Proposition 7: if $p > 1$ then $\bar{\Gamma}$ and $\bar{\Gamma}_0$ are tangent of order $p-1$ at $\bar{a}$, so $\bar{a} \in \bar{N}$ and $\bar{\Gamma}$ is tangent of order $p-1$ at $\bar{a}$ to $\bar{N}$; if $p = 1$ then $\bar{N} \cap \bar{L} \cap S = \bar{\Gamma}_0 \cap S = \{\bar{c}\}$, $\bar{c} \neq \bar{a}$, but $\bar{a} \in \bar{L}$, so $\bar{a} \notin \bar{N}$. □

7. Geometric desingularization of a curve in a manifold

Let M be a manifold. We say an analytic subset $V \subset M$ is a normal crossing subset if irreducible components of its germs V_a, $a \in V$, are germs of smooth hypersurfaces intersecting transversally at a. In particular such sets are:

Sets of type τ: they are unions of smooth compact hypersurfaces which are mutually transversal. By Propositions 4 and 5:

(1) The inverse-image of a set of type τ (with irreducible components $N_1, ..., N_r$ if $r > 0$) by a blowing-up is a set of type τ (with irreducible components $\bar{N}_1, ..., \bar{N}_r, S$ if $r > 0$, where S is the exceptional set).

A set of type τ' is one of type τ or one-point set. Obviously, the inverse image of a set of type τ' by a blowing-up is a set of type τ. Let $Z \subset M$ be of type τ'. We say a curve $\Gamma \subset M$ is crosswise to Z (at $c \in Z$) if it is closed, $\Gamma \cap Z = c$, Γ_c is irreducible and $\Gamma - c$ is smooth. In particular Γ is crosswise to c.

We say sets E_i are separated by a set F if $E_i \backslash F$ are disjoint. This property is preserved by the operation of taking inverse images.

(2) Let $\pi : \bar{M} \to M$ be a blowing-up at $a \in Z$, Z of type τ'. Then: Γ is crosswise to Z implies $\bar{\Gamma}$ is crosswise to $\pi^{-1}(Z)$, and $\pi^{-1}(\Gamma \cup Z) = \bar{\Gamma} \cup \pi^{-1}(Z)$ (by Propositions 6 and 4). If Γ is smooth, crosswise to Z and transversal to Z (in case Z is not one-point set) then $\bar{\Gamma}$ is smooth, crosswise and transversal to $\pi^{-1}(Z)$ (by Propositions 5 and 4). If Γ_i are crosswise to Z then: Γ_i are separated by Z implies $\bar{\Gamma}_i$ are separated by $\pi^{-1}(Z)$. If Γ_i are disjoint then $\bar{\Gamma}_i$ are disjoint.

A multiple blowing-up over $E \subset M$ is a composition of blowings-up $\pi = \pi_1 \circ ... \circ \pi_r : \bar{M} \to M$, where

$$\begin{array}{ccccccccc} & & E_{r-1} & & & & E_1 & & E_0 = E \\ & & \cap & & & & \cap & & \cap \\ \bar{M} = M_r & \xrightarrow{\pi_r} & M_{r-1} & \to & ... & \to & M_1 & \xrightarrow{\pi_1} & M_0 = M \end{array}$$

$\pi_i : M_i \to M_{i-1}$ is the blowing-up at a point of $E_{i-1}, i = 1, ..., r$, and $E_i = \pi_i^{-1}(E_{i-1}), i = 1, ..., r-1$. Then π is also a multiple blowing-up over $F \supset E$. If E is analytic and nowhere dense then π is a modification in E. Obviously:

(3) If $\pi : \bar{M} \to M$ is a multiple blowing-up over E and $\bar{\pi} : \overset{\circ}{M} \to \bar{M}$ – over $\pi^{-1}(E)$ then $\pi \circ \bar{\pi} : \overset{\circ}{M} \to M$ is a multiple blowing-up over E.

(4) If M is open in a manifold N and $\pi : \bar{M} \to M$ is a multiple blowing-up over $E \subset M$ then $\pi = \pi_1^M$, where $\pi_1 : \bar{N} \to N$ is a multiple blowing-up over E, $\bar{M}$ is open in $\bar{N}$ (by Proposition 1 and Remark 1).

(5) The inverse image of a set of type τ' by a multiple blowing-up is a set of type τ.

(6) Let $\pi : \bar{M} \to M$ be a multiple blowing-up over a set Z of type τ'. If Γ is crosswise to Z then consecutively using the operation of taking proper inverse images by $\pi_1, ..., \pi_r$ we obtain, according to (2), a curve $\bar{\Gamma} \subset \bar{M}$ which is crosswise to $\pi^{-1}(Z)$. It is called the proper inverse image of the curve Γ by the multiple blowing-up π, and then $\pi^{-1}(\Gamma \cup Z) = \bar{\Gamma} \cup \pi^{-1}(Z)$ (hence $\bar{\Gamma} = \overline{\pi^{-1}(\Gamma) \setminus \pi^{-1}(Z)}$). By (2):

(a) Γ smooth, crosswise to Z and transversal to Z (in case Z is not a one-point set) implies $\bar{\Gamma}$ is smooth, crosswise and transversal to $\pi^{-1}(Z)$. If Γ_i are crosswise to Z then:

(b) Γ_i separated by Z implies $\bar{\Gamma}_i$ separated by $\pi^{-1}(Z)$,

(c) Γ_i disjoint implies $\bar{\Gamma}_i$ disjoint. Moreover:

(d) If Γ is crosswise to Z and $\mathring{\Gamma}$ is the proper inverse image of $\bar{\Gamma}$ by a multiple blowing-up $\bar{\pi} : \mathring{M} \to \bar{M}$ over $\pi^{-1}(Z)$ then $\mathring{\Gamma}$ is the proper inverse image of Γ by $\pi \circ \bar{\pi}$.

(7) Let Γ be crosswise to a. By the first implication in (2) we recursively define a sequence of blowings-up $... \to M_i \xrightarrow{\pi_i} M_{i-1} \to ... \to M_1 \xrightarrow{\pi_1} M$ and a sequence of triplets $a_i \in \Gamma_i \subset M_i$, where Γ_i is crosswise to a_i, where $a_0 = a$, $\Gamma_0 = \Gamma$, $M_0 = M$, in such a way that: π_i is the blowing-up at a_{i-1}, Γ_i is the proper inverse image of Γ_{i-1} and $\{a_i\} = \Gamma_i \cap \pi_i^{-1}(a_{i-1})$. Then $\pi_{(k)} = \pi_1 \circ ... \circ \pi_k : M_k \to M$ is a multiple blowing-up over a by which Γ_k is the proper inverse image of Γ.

(A) If Γ is crosswise to a then there exists a multiple blowing-up over a such that the proper inverse image $\bar{\Gamma}$ is smooth.

In fact, let us take a sequence of blowings-up as in (7) for Γ. We will show that for some i the proper inverse image Γ_i of Γ by $\pi_{(i)}$ belongs to $\mathcal{C}_1(a_i)$, and so it is smooth. Namely $\Gamma = \Gamma_0$ belongs to some $\mathcal{C}_r(a_0)$ (see Section 6). By Proposition 6, if $\Gamma_\upsilon \in \mathcal{C}_{p,q}(a_\upsilon)$, $p > 1$, then $\Gamma_{\upsilon+1}$ belongs to $\mathcal{C}_{p,q-p}(a_{\upsilon+1})$ if $q > 2p$, and to $\mathcal{C}_{q-p}(a_{\upsilon+1})$ if $q < 2p$ (and then $q - p < p$). So, if $\Gamma_i \in \mathcal{C}_p(a_i)$, $p > 1$, then some Γ_j $(j > i)$ belongs to $\mathcal{C}_s(a_j)$, where $s < p$.

(B) If Γ, Γ' are smooth, crosswise to a and separated by a then there exists a multiple blowing-up over a such that proper inverse images $\bar{\Gamma}, \bar{\Gamma}'$ are disjoint.

In fact, let us consider constructions of sequences π_i, Γ_i, a_i for Γ and π_i', Γ_i', a_i' for Γ' described in (7). We may take the same first blowing-up $\pi_1 = \pi_1'$ at $a_0 = a_0' = a$, and (by the assumption) the curves $\Gamma_0, \Gamma_0' \ni a_0$ are separated by a_0; let p be their order of tangency. Let us consider the following condition:

(σ_k) for $i \leqslant k$ we can take the same blowings-up $\pi_i = \pi_i'$ at $a_{i-1} = a_{i-1}'$ and $\Gamma_{i-1}, \Gamma_{i-1}'$ are separated by a_{i-1} and tangent at a_{i-1} of order $p - i + 1$.

By the above (σ_1) holds. Suppose (σ_k) holds for $k < p$; then (σ_{k+1}) holds; in fact, $\Gamma_{k-1}, \Gamma_{k-1}'$ are tangent at a_{k-1} of order $p - k + 1$, so by Proposition 7 there is $a_k = a'_k$, and taking the same blowing-up $\pi_{k+1} = \pi'_{k+1}$ at a_k we have Γ_k, Γ'_k are tangent of order $p-k$ at a_k, crosswise to $\pi_k^{-1}(a_{k-1})$ and separated by $\pi_k^{-1}(a_{k-1})$ (see (2)), and so separated by a_k. In consequence (σ_p) holds, that is we may have $\pi_i = \pi'_i$ for $i \leqslant p$ and curves $\Gamma_{p-1}, \Gamma'_{p-1}$ are separated by a_{p-1} and tangent of order 1 at a_{p-1}. Hence by Proposition 7 the curves Γ_p, Γ'_p have different points a_p, a'_p in $\pi_p^{-1}(a_{p-1})$, but (see (2)) they are separated by $\pi_p^{-1}(a_{p-1})$ and so they are disjoint. Hence $\pi_{(p)}$ is a required multiple blowing-up over a.

(C) If Γ is smooth and crosswise at a to Z of type τ' then there exists a multiple blowing-up π over a such that the proper inverse image $\bar{\Gamma}$ of Γ intersect transversally $\pi^{-1}(Z)$.

In fact, let us take a sequence of blowings-up as in (7) for Γ (treated as crosswise to a). Then Γ_k are smooth and transversal to $\pi_k^{-1}(a_{k-1})$ (Proposition 4). The sets $Z_k = \pi_{(k)}^{-1}(Z)$ are of type τ. Since (see (6)) Γ_k is crosswise to $Z_k \ni a_k$ then $Z_k \cap \Gamma_k = \{a_k\}$. Let $N_1, ..., N_r$ be irreducible components of Z_k and consider the following condition

$$(\tau_p) \qquad N_i \ni a_k \Longrightarrow \Gamma_k \text{ is tangent of order} \leqslant p \text{ at } a_k \text{ to } N_i,$$

and notice that if $N_i \not\ni a_k$ then $\Gamma_k \cap N_i = \emptyset$. By (1) the irreducible components of Z_{k+1} are proper inverse images by π_{k+1} : $\bar{N}_1, ..., \bar{N}_r$ and $\pi_{k+1}^{-1}(a_k)$ (the latter is transversal to Γ_{k+1} at a_{k+1}). Hence, by Proposition 8, if Γ_k is tangent at a_k of order q to $N_i \ni a_k$ then Γ_{k+1} is tangent at a_{k+1} of order $q-1$ to $\bar{N}_i \ni a_{k+1}$ when $q > 1$, and $\bar{N}_i \not\ni a_{k+1}$ when $q = 1$. So, if (τ_p), $p > 1$, holds for k, then (τ_{p-1}) holds for $k+1$. Hence for some k the condition (τ_1) holds, and then Γ_{k+1} is disjoint with $\bar{N}_1, ..., \bar{N}_r$ and transversal to $\pi_{k+1}^{-1}(a_k)$ i.e. intersect transversally Z_{k+1}. Then $\pi_{(k+1)}$ is a required multiple blowing-up over a.

(8) If Γ is crosswise at a to Z of type τ' then there exists a multiple blowing-up π over a such that proper inverse image $\bar{\Gamma}$ of Γ is smooth, crosswise and transversal to $\pi^{-1}(Z)$.

In fact, by (A) there exists a multiple blowing-up $\pi_1 : M_1 \to M$ over a such that the proper inverse image $\bar{\Gamma} \subset M_1$ is smooth; by (6) it is crosswise to $\pi^{-1}(Z)$ of type τ (see (5)) at $c \in \pi_1^{-1}(\Gamma) \cap \pi_1^{-1}(Z) = \pi_1^{-1}(a)$, so by (C) there exists a multiple blowing-up $\pi_2 : M_2 \to M_1$ over c such that the proper inverse image $\mathring{\Gamma} \subset M_2$ of the curve $\bar{\Gamma}$ is smooth, transversal and croosswise (by (6) and $c \in \pi_1^{-1}(Z)$) to $\pi_2^{-1}(\pi_1^{-1}(Z)) = \pi^{-1}(Z)$, where $\pi = \pi_1 \circ \pi_2 : M_2 \to M$ is a multiple blowing-up over a (by (3) and $c \in \pi^{-1}(a)$), which satisfies the assertion (by (6) (d)).

Proposition 9. *If $\Gamma_1, ..., \Gamma_r$ are crosswise to a and separated by a then there exist a multiple blowing-up π over a such that the proper inverse images $\bar{\Gamma}_1, ..., \bar{\Gamma}_r$ are smooth, disjoint, and crosswise and transversal to $\pi^{-1}(a)$.*

Dowód. For the case $r = 1$ it is precisely (8) taking $Z = \{a\}$. Assume the proposition is true for $r-1$, $(r > 1)$; so there exists a multiple blowing-up $\pi_1 : M_1 \to M$ over a such that, if $\bar{\Gamma}_i \subset M_1$ are proper inverse images of Γ_i then $\bar{\Gamma}_1, ..., \bar{\Gamma}_{r-1}$

are smooth, disjoint, and crosswise and transversal to $Z_1 = \pi_1^{-1}(a)$ of type τ (see (5)). Then (see (6)) $\bar\Gamma_r$ is crosswise to Z_1 and we have $\bar\Gamma_r \cap Z_1 = \{a_1\}$. By (8) there exists a multiple blowing-up $\pi_2 : M_2 \to M_1$ over a_1 such that if $\mathring\Gamma_i \subset M_2$ are proper inverse images of $\bar\Gamma_i$ then $\mathring\Gamma_r$ is smooth, crosswise and transversal to $Z_2 = \pi_2^{-1}(Z_1) = \pi_0^{-1}(a)$, where $\pi_0 = \pi_1 \circ \pi_2 : M_2 \to M$ is the multiple blowing-up over a (see (3)). Then $\mathring\Gamma_1, ..., \mathring\Gamma_{r-1}$ are smooth, disjoint, and crosswise and transversal to Z_2 (see (6) (a) and (c)); moreover (see (6) (d)) the curves $\mathring\Gamma_i$ are proper inverse images of Γ_i by π_0 and so they are separated by Z_2 (see (6) (b)). If they are disjoint, π_0 satisfies the condition of the proposition. In the remaining cases is for example $\mathring\Gamma_r \cap \mathring\Gamma_1 = \{a_2\}$, $a_2 \in Z_2$, and then $\mathring\Gamma_r$ is disjoint with $\mathring\Gamma_2, ..., \mathring\Gamma_{r-1}$, that is $\mathring\Gamma_2, ..., \mathring\Gamma_r$ are disjoint. By (B) there exists a multiple blowing-up $\pi_3 : M_3 \to M_2$ over a_2 such that if $\Gamma'_i \subset M_3$ are proper inverse images of $\mathring\Gamma_i$ then Γ'_r and Γ'_1 are disjoint. But (see (6) (a)) Γ'_i are smooth, crosswise and transversal to $\pi_3^{-1}(Z_2) = \pi^{-1}(a)$, where $\pi = \pi_0 \circ \pi_3 : M_3 \to M$ is a multiple blowing-up over a (see (3)), under which Γ'_i are proper inverse images of Γ_i (see (6) (d)); moreover (see (6) (c)) $\Gamma'_1, ..., \Gamma'_{r-1}$ and $\Gamma'_2, ..., \Gamma'_r$ are disjoint and so Γ'_i are disjoint. Then π satisfies the condition of the proposition. □

Proposition 10. *If $\Gamma \subset M$ is a closed curve and the set of its singular points Γ^* is finite then there exists a multiple blowing-up π over Γ^* such that $\pi^{-1}(\Gamma) = \Lambda \cup Z$, where $Z = \pi^{-1}(\Gamma^*)$ is of type τ, and Λ a smooth, closed curve which intersects transversally Z. In other words: $\pi^{-1}(\Gamma) = N_1 \cup ... \cup N_r \cup \Lambda$, where N_i are smooth, compact hypersurfaces, Λ a smooth, closed curve, $N_1, ..., N_r, \Lambda$ are mutually tranversal and $\pi^{-1}(\Gamma^*) = N_1 \cup ... \cup N_r$.*

Dowód. Let $\Gamma^* = \{a_1, ..., a_k\}$ and assume the proposition is true for $k-1$ provided $k > 1$. There exists an open neighbourhood U of the point a_k such that $a_1, ..., a_{k-1} \notin U$ and $\Gamma \cap U = \Gamma_1 \cup ... \cup \Gamma_r$, where Γ_i are closed curves in U, crosswise to a_k and separated by a_k. By Proposition 9 there exists a multiple blowing-up $\pi_1 : M_1 \to M$ over a_k such that the proper inverse images $\bar\Gamma_i$ of the curves Γ_i by the multiple blowing-up π_1^U are closed in $U_1 = \pi_1^{-1}(U)$, smooth, disjoint and transversal to $Z_1 = \pi_1^{-1}(a_k)$ and (by (6)) $\pi^{-1}(\Gamma_i) = \bar\Gamma_i \cup Z_1$. Then $\bar\Gamma_0 = \bigcup \bar\Gamma_i$ is a closed curve in U_1, smooth and intersect transversally Z_1, and $\pi_1^{-1}(\Gamma \cap U) = \bar\Gamma_0 \cup Z_1$. The curve $\pi_1^{-1}(\Gamma)\backslash Z_1$ is closed in $M_1\backslash Z_1$ and its all singular points are $b_i = \pi_1^{-1}(a_i)$, $i = 1, ..., k-1$. Since $\bar\Gamma_0 \cap (U_1\backslash Z_1) = \pi_1^{-1}(\Gamma \cap U)\backslash Z_1 = (\pi_1^{-1}(\Gamma)\backslash Z_1) \cap (U_1\backslash Z_1)$ then $\bar\Gamma = (\pi_1^{-1}(\Gamma)\backslash Z_1) \cup \bar\Gamma_0$ is closed in M_1 which intersects transversally Z_1 and $\bar\Gamma^* = \{b_1, ..., b_{k-1}\}$. It is $\pi_1^{-1}(\Gamma) = \bar\Gamma \cup Z_1$ (since $\pi_1^{-1}(\Gamma) = \pi_1^{-1}(\Gamma \cap U) \cup \pi_1^{-1}(\Gamma\backslash a_k)$). If $k = 1$ then π_1 satisfies the conditions of the proposition. So, let us assume $k > 1$. Then (by the induction hypothesis) there exists a multiple blowing-up $\pi_2 : M_2 \to M_1$ over $\bar\Gamma^*$ such that $\pi_2^{-1}(\bar\Gamma) = \Lambda \cup Z_2$, where $\Lambda \subset M_2$ is a closed, smooth, intersect transversally $Z_2 = \pi_2^{-1}(\bar\Gamma^*)$ of type τ. Then $\pi = \pi_1 \circ \pi_2 : M_2 \to M$ is a multiple blowing-up over Γ^* (see (3)) and $\pi^{-1}(\Gamma) = \pi_2^{-1}(\bar\Gamma) \cup \pi_2^{-1}(Z_1) = \Lambda \cup Z$, where $Z = Z_2 \cup \pi_2^{-1}(Z_1) = \pi^{-1}(\Gamma^*)$. Since $Z_1 \subset U_1$ is disjoint with Γ^* then $\pi_2^{-1}(Z_1) \subset \pi_2^{-1}(U_1)$ is disjoint with Z_2 and $\pi_2^{U_1}$ is a biholomorphism. Then $\pi_2^{-1}(\bar{\Gamma}\cap U_1) = \Lambda\cap\pi_2^{-1}(U_1)$ intersect transversally $\pi_2^{-1}(Z_1)$ and so Λ intersect transversally $\pi_2^{-1}(Z_1)$. Then Λ intersect transversally Z and π satisfies the conditions of the proposition. $\square$

8. Blowing-up of submanifolds

Let M be a n-dimensional manifold and $n = p + q$. Let $f_1, ..., f_q \in \mathcal{O}(M)$ and assume $f = (f_1, ..., f_q) : M \to \mathbb{C}^q$ is a submersion. Then $L = f^{-1}(0)$ is a submanifold of dimension p. The subset

$$M_f = \{(z, \lambda) : f(z) \in \lambda\} \subset M \times \mathbb{P}_{q-1},$$

that is $M_f = \phi^{-1}(\Pi_q)$, where $\phi = f \times e : M \times \mathbb{P}_{q-1} \to \mathbb{C}^q \times \mathbb{P}_{q-1}$, $e = \mathrm{id}\,\mathbb{P}_{q-1}$, is also a submersion, is a closed submanifold of dimension n. The canonical projection

$$\pi_f : M_f \to M$$

is called an elementary blowing-up by functions $f_1, ..., f_q$. It is a modification in the set L called the centre of blowing-up. It is so because π_f is a proper mapping, $(M_f)_{M\setminus L} = M_f\setminus\pi_f^{-1}(L)$ is the graph of the holomorphic mapping $M\setminus L \ni z \to \mathbb{C}f(z) \in \mathbb{P}_{q-1}$, hence $\pi_f^{M\setminus L} : M_f\setminus\pi_f^{-1}(L) \to M\setminus L$ is a biholomorphism, and $\pi_f^{-1}(L) = L\times\mathbb{P}_{q-1}$ is a closed, smooth hypersurface called the exceptional set of the blowing-up. Of course $\pi_f^G = \pi_{f_G}$ is an elementary blowing-up by $(f_i)_G$ with centre $L \cap G$.

Proposition 11. *If additionally $g = (g_1, ..., g_q) : M \to \mathbb{C}^q$ is a submersion and $\sum \mathcal{O}(M)f_i = \sum \mathcal{O}(M)g_i$ (i.e. f_i and g_j generate the same ideal in $\mathcal{O}(M)$; then $g^{-1}(0) = f^{-1}(0) = L$), then the blowings-up π_f and π_g are isomorphic: the diagram*

$$\begin{tikzcd} M_f \arrow[rr, "\iota"] \arrow[dr, "\pi_f"'] & & M_g \arrow[dl, "\pi_g"] \\ & M & \end{tikzcd}$$

is commutative, where ι is a biholomorphism.

Corollary 5. *If $\pi_i : M_i \to M$ are elementary blowings-up with the centre L then arbitrary point $a \in L$ has an open neighbourhood U in M such that $\pi_1^U \approx \pi_2^U$.*

In particular we have the elementary blowing-up of $\mathbb{C}^n$ by $v = (z_{p+1}, ..., z_n)$:

$$\mathbb{C}^n_v = \{(z, \lambda) \in \mathbb{C}^n \times P_{q-1} : v \in \lambda\} = \mathbb{C}^p \times \Pi_q$$

and

$$\pi_v = (\mathrm{id}\,\mathbb{C}^p) \times \pi_q : \mathbb{C}^p \times \Pi_q \to \mathbb{C}^p \times \mathbb{C}^q.$$

The blowings-up π_v^Ω, where Ω is an open neighbourhood of 0 in $\mathbb{C}^n$ is called standard.

(#) Let $L \subset M$ be a p-dimensional submanifold. If $\phi : U \to \Omega$ is a chart at a ($\phi(a) = 0$) such that $\phi(L \cap U) = \{v = 0\} \cap \Omega$ then $\psi = (\phi_{p+1}, ..., \phi_n)$ is a submersion and the blowing-up π_ψ is isomorphic to the elementary blowing-up π_v^Ω

$$\begin{tikzcd}
U_\psi \arrow[r, "(\phi \times e)_{U_\psi}"] \arrow[d, "\pi_\psi"'] & (\mathbb{C}^p \times \pi_{q-1})_\Omega \arrow[d, "\pi_v^\Omega"] \\
U \arrow[r, "\phi"] & \Omega
\end{tikzcd}$$

Notice that if $f : \bar{M} \to M$ is a modification in an analytic set $Z \subset M$ and $G \subset M$ is an open set then $f^G : f^{-1}(G) \to G$ is a modification in $Z \cap G$.

Proposition 12. *If $f_i : M_i \to M$ are modifications ($i = 1, 2$) and $M = \bigcup G_\iota$ is an open cover then $f_1 \approx f_2$ if and only if $f_1^{G_\iota} \approx f_2^{G_\iota}$ for every ι.*

Corollary 6. *Elementary blowings-up of a manifold with the same centre are isomorphic.*

Proposition 13. *(on gluing modifications). If $M = \bigcup M_\iota$ is an open cover and $f_\iota : \bar{M}_\iota \to M$ are modifications such that $f_\iota^{M_\iota \cap M_k} \approx f_k^{M_\iota \cap M_k}$, then there exists a unique (up to isomorphism) modification $f : \bar{M} \to M$ such that $f^{M_\iota} \approx f_\iota$.*

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Let $L \subset M$ be a p-dimensional closed submanifold.

There exists a unique (up to an isomorphism) modification $\pi : \bar{M} \to M$ in L such that each point $a \in L$ has an open neighbourhood U_a such that π^{U_a} is isomorphic to an elementary blowing-up of U_a with the centre $L \cap U_a$. We will call it the blowing-up of the manifold M in the submanifold L (the latter is called the centre of blowing-up).

In fact, the uniqueness follows from Proposition 12 (applied to the cover: $M \backslash L$ and U_a for $a \in L$). For the existence: for every $a \in L$ we take an elementary blowing-up $\pi_a : M_a \to U_a$ of an open neighbourhood U_a of the point a with the centre $L \cap U_a$. By Proposition 12 and Corollary 6 we have $\pi_a^{U_a \cap U_b} \approx \pi_b^{U_a \cap U_b}$ (as blowings-up with the common centre $L \cap U_a \cap U_b$); we take also $e = \mathrm{id}\, M \backslash L$; then obviously $\pi_a^{U_a \backslash L} \approx e^{U_a \backslash L}$. By Proposition 13, there exists a modification $\pi : \bar{M} \to M$ such that $\pi^{U_a} \approx \pi_a$.

The subset $\pi^{-1}(L)$ is a closed and smooth hypersurface. (There is: $\pi^{-1}(V)$ is isomorphic to $V \times \mathbb{P}_q$, where V are sufficently small open neighbourhoods of points in L; moreover $\pi^L : \pi^{-1}(L) \to L$ is a locally trivial fibration with the fiber $\mathbb{P}_{q-1}$). It is called the exceptional set of the blowing-up π.

Proposition 14. *If $\pi : \bar{M} \to M$ is a blowing-up in L and $N \subset M$ is a submanifold of dimension s which intersect transversally L, then $\pi^{-1}(N)$ is a submanifold of dimension s which intersect transversally $\pi^{-1}(L)$.*

Dowód. Let $a \in N \cap L$. By Proposition 3 we take a chart $\phi : U \to \Omega$ at a such that $\phi(L \cap U) = \{v = 0\} \cap \Omega$ and $\phi(N \cap U) = \{t = 0\} \cap \Omega$, where $t = (z_1, ..., z_r)$, $r = n - s \leqslant p$; then $L \cap U = \psi^{-1}(0)$, where $\psi = (\phi_{p+1}, ..., \phi_n)$. Shrinking U we may assume that π^U is isomorphic to an elementary blowing-up π_f of U, isomorphic in turn to π_ψ (Proposition 11) which is isomorphic to π_v^Ω (by (#)), that is π^U is isomorphic to π_v^Ω over ϕ

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\text{bihol.}"] \arrow[d, "\pi^U"'] & \pi_v^{-1}(\Omega) \arrow[d, "\pi_v^\Omega"] \\ U \arrow[r, "\phi"] & \Omega \end{tikzcd}$$

Then $\pi^{-1}(L \cap U)$, $\pi^{-1}(N \cap U)$ correspond to $\pi_v^{-1}(\{v = 0\} \cap \Omega)$, $\pi_v^{-1}(\{t = 0\} \cap \Omega)$ by the biholomorphism $\pi^{-1}(U) \to \pi_v^{-1}(\Omega)$. But $\pi_v^{-1}(\{v = 0\}) = \mathbb{C}^p \times (0 \times \mathbb{P}_{q-1})$ and $\pi_v^{-1}(\{t = 0\}) = \{u \in \mathbb{C}^r : t = 0\} \times \Pi_q$ (a submanifold of dimension s), where $u = (z_1, ..., z_p)$, intersect transversally in $\mathbb{C}^p \times \Pi_q$, so the inverse images $\pi^{-1}(L)$, $\pi^{-1}(N)$ in $\pi^{-1}(U)$ (the second is a submanifold of dimension s) intersect transversally which implies that $\pi^{-1}(N)$ is a submanifold of dimension s and intersect transversally $\pi^{-1}(L)$ (because the sets of the form $\pi^{-1}(U)$ cover $\pi^{-1}(L) \cap \pi^{-1}(N)$). $\square$

Theorem 1. *If $\Gamma \subset M$ is a closed curve with Γ^* finite then there exists a modification $\pi : \bar{M} \to M$ in Γ such that $\pi^{-1}(\Gamma)$ is a finite union of smooth, closed and mutually transversal hypersurfaces in M.*

Dowód. Let us take a multiple blowing-up $\pi_1 : M_1 \to M$ as in Proposition 10 and the blowing-up $\pi_2 : M_2 \to M_1$ of the curve Λ. Then $\pi = \pi_1 \circ \pi_2 : M_2 \to M$ is a modification in Γ. Submanifolds $N_1, ..., N_r \subset M_1$ are mutually transversal in M_1 and pairs N_i, N_j $(i \neq j)$ intersect outside Λ. Hence $\pi_2^{-1}(N_i) \subset M_2$ are smooth hypersurfaces (Proposition 14), compact, mutually transversal in $M_2 \backslash \pi_2^{-1}(\Lambda)$ and pairs $\pi_2^{-1}(N_i)$, $\pi_2^{-1}(N_j)$, $i \neq j$, intersect only outside $\pi_2^{-1}(\Lambda)$; moreover by Proposition 14 each $\pi_2^{-1}(N_i)$ intersect transversally $\pi^{-1}(\Lambda)$. Then smooth, closed hypersurfaces $\pi_2^{-1}(N_1), ..., \pi_2^{-1}(N_r), \pi_2^{-1}(\Lambda)$ with the union equal to $\pi^{-1}(\Gamma)$ are mutually transversal in M_2. $\square$

9. Desingularization of a coherent sheaf of ideals on a 2-dimensional manifold

Let M be a 2-dimensional manifold.

A parameter at a point $a \in M$ is a germ $\phi \in \mathfrak{m}_a$ such that $d_a\phi \neq 0$. We say ϕ correspond to a germ of smooth curve A if $V(\phi) = A$; then it is a generator of $I(A)$ unique up to an invertible factor. We say parameters ϕ, ψ at a are transversal, if $V(\phi), V(\psi)$ are transversal, which means $d_a\phi, d_a\psi$ are linearly independent, or equivalently $(\bar{\phi}, \bar{\psi})$ is a chart (a system of coordinates at a) for some representatives $\bar{\phi}, \bar{\psi}$.

We say a germ $f \in \mathcal{O}_a$ is of type (NC) if $f \sim \phi^\alpha \psi^\beta$, where ϕ, ψ are transversal parameters at a. (It means that in some chart it has the form $az_1^\alpha z_2^\beta$, $a(0) \neq 0$). It holds if and only if $V(f) = A \cup B$ or $= A$ or $= \emptyset$, where A, B are germs of transversal smooth curves. Then, respectively to the above cases, $f \sim \phi^\alpha \psi^\beta$ or $f \sim \phi^\alpha$ or $f \sim 1$, where ϕ, ψ are parameters corresponding to A, B.

We say a function $f \in \mathcal{O}_M$ is of type (NC) if its all germs f_z, $z \in M$ are of type (NC). Then by Proposition 10 we have

Proposition 15. *If $f \in \mathcal{O}_M$ and $V(f)^*$ is finite, then there exists a blowing-up $\pi : \bar{M} \to M$ over $V(f)^*$ such that $f \circ \pi$ is of type (NC).*

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By a coherent sheaf of ideals on M we mean a family $\mathcal{T}$ of ideals $\mathcal{T}_z \subset \mathcal{O}_z$, $z \in M$, such that each point in M has an open neighbourhood U in which $\mathcal{T}$ has a finite set of generators i.e. there exist $\phi_1, ..., \phi_r \in \mathcal{O}_U$ such that $(\phi_1)_z, ..., (\phi_r)_z$ generate $\mathcal{T}_z$ for every $z \in U$ ($\mathcal{T}$ corresponds to a sheaf according to the standard definition - obtained by the presheaf: $\{f \in \mathcal{O}_G : f_z \in \mathcal{T}_z \text{ for } z \in G\}_{G \text{ open in } M}$). The set of its zeros is defined by $V(\mathcal{T}) = \{z \in M : \mathcal{T}_z \neq \mathcal{O}_z\}$; since $V(\mathcal{T}) \cap U = \{\phi_1 = ... = \phi_r = 0\}$ if $\phi_1, \ldots, \phi_r$ generate $\mathcal{T}$ in U, then it is an analytic subset of the manifold M.

If $f : N \to M$ is a holomorphic mapping between manifolds we define the coherent sheaf $f^*\mathcal{T}$ on N (called the inverse image of the sheaf $\mathcal{T}$) by: $(f^*\mathcal{T})_\xi \subset \mathcal{O}_\xi$ is the ideal generated by $\mathcal{T}_{f(\xi)} \circ f_\xi$ that is by $\phi_1 \circ f_\xi, ..., \phi_r \circ f_\xi$, provided $\phi_1, ..., \phi_r$ generate $\mathcal{T}_{f(\xi)}$ (so, if ψ_i generate $\mathcal{T}$ in U then $\psi_i \circ f$ generate $f^*\mathcal{T}$ in $f^{-1}(U)$). It is obviously $V(f^*\mathcal{T}) = f^{-1}(V(\mathcal{T}))$. If $g : L \to N$ is a holomorphic mapping between manifolds then

$$(f \circ g)^*\mathcal{T} = g^*(f^*\mathcal{T}).$$

We say a sequence of germs $\phi_1, ..., \phi_r \in \mathcal{O}_a$ is of type (NC) if $\phi_i \sim \phi^{\alpha_i}\psi^{\beta_i}$, where ϕ, ψ are transversal parameters at a. We say a sequence of functions $f_1, ..., f_r \in \mathcal{O}_M$ is of type (NC) if each sequence of germs $(f_1)_z, ..., (f_r)_z$, $z \in M$, is of type (NC). Notice that if $f_1, ..., f_r \in \mathcal{O}_M$ then if the sequence $(f_1)_a, ..., (f_r)_a$ is of type (NC) then for an open neighbourhood U of the point a the sequence $(f_1)_U, ..., (f_r)_U$ is of type (NC).

We say an ideal of the ring $\mathcal{O}_a$ is of type (NC*), respectively (NC), if there exists a sequence of generators of the ideal of type (NC), respectively one generator of type (NC). We say a sheaf $\mathcal{T}$ is of type (NC*), respectively (NC), at a point $z \in M$ if $\mathcal{T}_z$ is of type (NC*), respectively (NC). At last we say a sheaf $\mathcal{T}$ is of type (NC*), respectively (NC) if it is of type (NC*), respectively (NC), in each point $z \in M$.

By $\sigma\mathcal{T}$ we will denote the set of points in which $\mathcal{T}$ is not of type (NC). Obviously $\sigma\mathcal{T} \subset V(\mathcal{T})$ (in general the inclusion $\sigma\mathcal{T} \subset V(\mathcal{T})^*$ does not hold, for example the point $0 \in \mathbb{C}^2$ and the sheaf generated in $\mathbb{C}^2$ by z_1^2 and $z_1 z_2$).

Lemma 1. *If $\phi_1, ..., \phi_r$ are holomorphic in an open neighbourhood U of a point a and $(\phi_i)_a \neq 0$, then after shrinking U there is $\phi_i = \psi_1^{\alpha_{i1}} ... \psi_s^{\alpha_{is}}$ for some $\psi_v \in O(U)$ such that $V(\psi_v)$ are (in U) crosswise to a, separated by a and $d_z\psi_v \neq 0$ for $z \in V(\psi_v) \backslash a$.*

Dowód. In fact, it suffices to take as ψ_v representatives, in a sufficiently small neighbourhood U, of all non-associated, irreducible divisors of the germs $(\phi_i)_a$. □

Hence

(1) The set $\sigma\mathcal{T}$ is isolated.

It suffices to take generators ϕ_i in U and ψ_v as above and let $z \in U \backslash a$. If $z \in V(\mathcal{T})$ then z belongs to a unique $V(\psi_s)$ and then $\mathcal{T}_z = O_z(\psi_s)_z^\alpha$, where $\alpha = \min(\alpha_{1s}, ..., \alpha_{rs})$.

Proposition 16. *If $\mathcal{T}$ is a coherent sheaf of ideals in M with $\sigma\mathcal{T}$ finite then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over $\sigma\mathcal{T}$ such that $\pi^*\mathcal{T}$ is of type (NC*).*

Dowód. Let $a_1, ..., a_k$ be all the points in which $\mathcal{T}$ is not of type (NC*) (their number is finite because they belong to $\sigma\mathcal{T}$). Using induction with respect to k, by (3) in Section 7, it suffices to show that there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a_k such that $\pi^{-1}(a_1), ..., \pi^{-1}(a_{k-1})$ are unique points of $\bar{M}$ in which $\pi^*\mathcal{T}$ is not of type (NC*). Really, let us take generators $\phi_1, ..., \phi_r$ of the sheaf $\mathcal{T}$ in an open neighbourhood U of the point a_k, and $\psi_1, ..., \psi_s \in \mathcal{O}(U)$ as in Lemma 1 (after shrinking U). By Proposition 9 applied to $V(\psi_v)$ (and by (4) and (6) in Section 7), there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a_k and curves $L_1, ..., L_q \subset U$ smooth, closed and mutually tranversal in U, such that each $V(\psi_v \circ \pi) = \pi^{-1}(V(\psi_v))$ is the union of some of them. Let $c \in \pi^{-1}(U)$. It suffices to show that the sequence $(\psi_1 \circ \pi)_c, ..., (\psi_s \circ \pi)_c$ is of type (NC). If $c \notin \cup L_i$ then $V(\psi_v \circ \pi) = \emptyset$, so $(\psi_v \circ \pi)_c \sim 1$. If c belongs to a unique L_i then $V(\psi_v \circ \pi) = \emptyset$ or $= (L_i)_c$, so $(\psi_v \circ \pi)_c \sim \phi^{\alpha_v}$, where ϕ is a parameter corresponding to L_c. If at last $c \in L_i \cap L_j$, $i \neq j$, then $V(\psi_v \circ \pi) = \emptyset$ or $= (L_i)_c$ or $= (L_i)_c \cup (L_j)_c$, so $(\psi_v \circ \pi)_c \sim \phi^{\alpha_v}\psi^{\beta_v}$, where ψ, ϕ are parameters corresponding to $(L_i)_c, (L_j)_c$. □

Let $\pi : \bar{M} \to M$ be a blowing-up at a. Let σ_ξ be a parameter corresponding to S_ξ for $\xi \in S$.

Let ϕ be a parameter at a. The inverse image $\bar{\Gamma}$ of a representative of $V(\phi)$ intersects S precisely in one point a_ϕ, and the parameter $\bar{\phi}$ at a_ϕ corresponding to $\bar{\Gamma}_{a_\phi}$ is tranversal to σ_{a_ϕ} (see Proposition 4); notice that if parameters ϕ, ψ at a are transversal then $a_\phi \neq a_\psi$. It is

$$\phi \circ \pi_\xi \sim \begin{cases} \sigma_\xi \text{ for } \xi \in S \backslash a_\phi, \\ \bar{\phi}\sigma_\xi \text{ for } \xi = a_\phi. \end{cases}$$

In fact, it suffices to consider the canonical blowing-up p and $\phi = (z_1)_0$. Then $\phi \circ (p \circ \chi_1)_u = (z_1)_u$ for $u \in \{z_1 = 0\}$ and $\phi \circ (p \circ \chi_2)_v = (z_2 w_1)_v$ for $v \in \{z_2 = 0\}$, and $a_\phi = \chi_2(0)$ and $\bar{\phi} \circ \chi_2 = (w_1)_0$ (because $\chi_2^{-1}(p^{-1}(V(z_1))) = \{z_2 = 0\} \cup \{w_1 = 0\}$). It implies that if $f \sim \phi^\alpha \psi^\beta$, where ϕ, ψ are transversal parameters at a, then, putting $c = a_\phi, d = a_\psi$,

$$(\#) \qquad f \circ \pi_\xi \sim \begin{cases} \sigma_\xi^{\alpha+\beta} & \text{if } \xi \in S \backslash (a_\phi, a_\psi), \\ \bar{\phi}^\alpha \sigma_\xi^{\alpha+\beta} & \text{if } \xi = a_\phi, \\ \bar{\psi}^\beta \sigma_\xi^{\alpha+\beta} & \text{if } \xi = a_\psi. \end{cases}$$

A pair $f, g \in \mathcal{O}_a$ of type (NC): $f \sim \phi^\alpha \psi^\beta$, $g \sim \phi^{\alpha'} \psi^{\beta'}$, ϕ, ψ transversal parameters at a, is called unessential, if $(\alpha' - \alpha)(\beta' - \beta) \geqslant 0$; then f is a divisor of g or g is a divisor of f. If $(\alpha' - \alpha)(\beta' - \beta) < 0$ then we call the pair f, g essential of type (p, q), where $p = \min(|\alpha' - \alpha|, |\beta' - \beta|)$, $q = \max(|\alpha' - \alpha|, |\beta' - \beta|)$.

(2) Let $f, g \in O_a$ be a pair of type (NC). Then

(a) All the pairs $P_\xi = (f \circ \pi_\xi, g \circ \pi_\xi)$, $\xi \in S$, are of type (NC).

(b) If the pair f, g is unessential then all the pairs P_ξ, $\xi \in S$, are unessential.

(c) If the pair f, g is essential of type (p, p), then all the pairs $P_\xi, \xi \in S$, are unessential.

(d) If the pair f, g is essential of type (p, q), $p < q$, then there exists $c \in S$ such that all the pairs $P_\xi, \xi \in S \backslash c$, are unessential, and the pair P_c is essential of type $(p, q - p)$ or $(q - p, p)$ depending on whether $q \geqslant 2p$ or $q \leqslant 2p$.

Dowód. (a) and (b) are obvious by (#). The case (c) follows (by (#)) from the fact that then $\alpha + \beta = \alpha' + \beta'$. Let us pass to the proof of (d). We may assume (changing f and g if necessary) that $\alpha + \beta < \alpha' + \beta'$. If $\alpha < \alpha'$ then $\beta > \beta'$ and P_{a_ϕ} is unessential; then $p = \beta - \beta'$, $q = \alpha' - \alpha$, $q - p = (\alpha' + \beta') - (\alpha + \beta)$ and the pair P_{a_ψ} is essential of type – as in (d). If $\alpha > \alpha'$ then $\beta < \beta'$, so P_{a_ψ} is unessential: then $p = \alpha - \alpha'$, $q = \beta' - \beta$, $q - p = (\alpha' + \beta') - (\alpha + \beta)$ and the pair P_{a_ϕ} is essential of type – as in (d). □

Let $f, g \in \mathcal{O}_M$ be a pair of type (NC). We say it is unessential at a point $z \in M$, respectively, essential of type (p, q), if the pair of germs f_z, g_z is such a pair. Let us notice that each point has a neighbourhood U such that the pair f, g is unessential at each point $z \in U \backslash a$. From (2) it follows:

(3) Let $f, g \in \mathcal{O}_M$ be a pair of type (NC). Then the pair $f \circ \pi$, $g \circ \pi$ is also of type (NC). If the pair f, g is unessential in M then the pair $f \circ \pi$, $g \circ \pi$ is unessential in $\bar{M}$. Assume that the pair f, g is unessential at all the points of $M \backslash a$ and essential of type (p, q) at a. If $p = q$ then $f \circ \pi$, $g \circ \pi$ is unessential at all the points of $\bar{M}$; if $p < q$ then there exists $c \in \bar{M}$ such that $f \circ \pi$, $g \circ \pi$ is unessential at all the points $\bar{M} \backslash c$ and essential at c, of type $(p, q - p)$ or $(q - p, p)$ depending on whether $q \geqslant 2p$ or $q \leqslant 2p$.

(4) If the pair $f, g \in \mathcal{O}_M$ of type (NC) is unessential at all the points of $M \backslash a$ then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a such that the pair $f \circ \pi$, $g \circ \pi$ (also of type (NC) by (3)) is unessential at all the points of $\bar{M}$.

In fact, if the pair f, g is essential at a, we may define (by (3)) a sequence $\bar{M} = M_r \overset{\pi_r}{\to} \dots \overset{\pi_1}{\to} M_0 = M$, where $\pi_i : M_i \to M_{i-1}$ is the blowing-up at a_{i-1} $(i = 1, \dots, r)$, $a_0 = a$, and $a_i \in \pi_i^{-1}(a_{i-1})$ is the unique point of M_i in which the pair $f \circ \pi_1 \circ \dots \circ \pi_i$, $g \circ \pi_1 \circ \dots \circ \pi_i$ is essential $(i = 1, \dots, r - 1)$, and in particular of type (p, p) if $i = r - 1$ (because if $0 < p \leqslant q$ and the sequence $(p_i, q_i) \in \mathbb{N}^2$ is defined by $(p_0, q_0) = (p, q)$ and

$$(p_i, q_i) = \begin{cases} (p_{i-1}, q_{i-1} - p_{i-1}), & \text{if } q_{i-1} \geqslant 2p_{i-1}, \\ (q_{i-1} - p_{i-1}, p_{i-1}), & \text{if } q_{i-1} \leqslant 2p_{i-1}, \end{cases}$$

then there must be $p_{r-1} = q_{r-1}$ for some r).

Let $\pi : \bar{M} \to M$ be a multiple blowing-up. From (#) it follows:

(5) If $\xi \in \bar{M}$ and the sequence $f_1, \dots, f_r \in O_{\pi(\xi)}$ is of type (NC) then also the sequence $f_1 \circ \pi_\xi, \dots, f_r \circ \pi_\xi \in O_\xi$.

For it suffices to check it for a blowing-up. Hence (taking $r = 1$):

(6) If $\mathcal{T}$ is a coherent sheaf of ideals then $\sigma(\pi^* \mathcal{T}) \subset \pi^{-1}(\sigma \mathcal{T})$. Hence (by (1)), if $\sigma \mathcal{T}$ is finite then also the set $\sigma(\pi^* \mathcal{T})$ is finite.

(For if ϕ is a generator of type (NC) of the ideal $\mathcal{T}_{\pi(\xi)}$ then $\phi \circ \pi_\xi$ is a generator of type (NC) of the ideal $(\pi^* \mathcal{T})_\xi$).

Theorem 2 (Hironaka Theorem on 2-dimensional manifold)**.** *If $\mathcal{T}$ is a coherent sheaf of ideals on M for which $\sigma \mathcal{T}$ is finite, then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over $\sigma \mathcal{T}$ such that $\pi^* \mathcal{T}$ is of type (NC).*

Dowód. By Proposition 12 (and by (3) in Section 7 and (6)) we may assume that $\mathcal{T}$ is of type (NC*).

Let us introduce the following definitions: An ideal $I \subset \mathcal{O}_z$ is of type (n), where $n \geqslant 1$, if I has a sequence at most n generators of type (NC). A sheaf $\mathcal{T}$ on M is of type (n) if $\sigma \mathcal{T}$ is finite and each $\mathcal{T}_z$, $z \in M$, is of type (n). Then (by (5) and (6)) for every multiple blowing-up $\pi : \bar{M} \to M$ the sheaf $\pi^* \mathcal{T}$ is also of type (n). A sheaf $\mathcal{T}$ of type (n) is of type (n, r), where $r \geqslant 0$, if, with exception of r points, each $\mathcal{T}_z$ is of type $(n - 1)$. Each sheaf $\mathcal{T}$ of type (n) is (because $\sigma \mathcal{T}$ is finite) of type (n, r) for some $r \geqslant 0$. A sheaf of type $(n, 0)$ is of type $(n - 1)$ and a sheaf of type (1) is of type (NC). Since $\mathcal{T}$ is of some type (n) (because $\sigma\mathcal{T}$ is finite) then it suffices (by (3) in Section 7 and (6)) to prove that if $\mathcal{T}$ is of type (n, r), $n \geqslant 2$, $r \geqslant 1$, then there exists a multiple blowing-up $\pi : \bar{M} \to M$ over $\sigma\mathcal{T}$ such that $\pi^*\mathcal{T}$ is of type $(n, r-1)$.

So, let $\mathcal{T}$ be of type (n, r), $n \geqslant 2$, $r \geqslant 1$. Then there exist points $a_1, ..., a_r \in M$ such that $\mathcal{T}_{a_i}$ are of type (n), and for $z \neq a_1, ..., a_r$ the ideals $\mathcal{T}_z$ are of type $(n-1)$. There exists a sequence $f_1, ..., f_n \in \mathcal{O}_U$ of type (NC) of generators of $\mathcal{T}$ in an open neighbourhood U of the point a_r, and (shrinking U) we may additionally assume that the pair f_1, f_2 is unessential at each points of the set $U \setminus a_r$. By (5) (and by (4) in Section 7) there exists a multiple blowing-up $\pi : \bar{M} \to M$ over a_r such that the pair $f_1 \circ \pi$, $f_2 \circ \pi \in \mathcal{O}_{\pi^{-1}(U)}$ is unessential at all the points of the set $\pi^{-1}(U)$. Then, if $\xi \in \pi^{-1}(U)$ then in the sequence $(f_i \circ \pi)_\xi$ of generators of the ideal $(\pi^*\mathcal{T})_\xi$ we may omit one of the generators $(f_1 \circ \pi)_\xi$, $(f_2 \circ \pi)_\xi$, that is $(\pi^*\mathcal{T})_\xi$, $\xi \in \pi^{-1}(U)$, are of type $(n-1)$. Since for $\xi \in \bar{M} \setminus \pi^{-1}(a_r)$ different of $\pi^{-1}(a_1), ..., \pi^{-1}(a_{r-1})$, the ideals $(\pi^*\mathcal{T})_\xi$ are obviously of type $(n-1)$ then $\pi^*\mathcal{T}$ is of type $(n, r-1)$. $\square$

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Analytic and Algebraic Geometry
Łódź University Press 2013, 33 – 40

NECESSARY CONDITIONS FOR IRREDUCIBILITY OF ALGEBROID PLANE CURVES

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Abstract. Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and let $f \in \mathbb{K}[[X]][Y]$ be monic. Using the properties of approximate roots given in [J. Algebra 343 (2011), pp. 143–159] we propose some necessary conditions for irreducibility of f in $\mathbb{K}[[X]][Y]$. The result is expressed only in terms of intersection multiplicities of f with its approximate roots.

1. Introduction

We recall that for a monic polynomial $f \in R[Y]$ of degree k, where R is a commutative ring with unity, and for a positive integer $l|k$ satisfying $\gcd(l, \operatorname{char} R) = 1$, there exists a unique monic polynomial $g \in R[Y]$ with the property

$$\deg_Y(f - g^l) < k - \frac{k}{l}.$$

The polynomial g is called an *approximate l-th root of f* and is denoted by $\sqrt[l]{f}$ (cf. [Abh77, Definition (4.3)]).

Now, let $\mathbb{K}$ be an algebraically closed field of *characteristic* 0, $\mathbb{K}[[X]]$ – the ring of power series in one variable X with coefficients in $\mathbb{K}$ and $\mathbb{K}((X))$ – its field of fractions. Let $f \in \mathbb{K}((X))[Y]$ be a *monic* and *irreducible* polynomial. In [Brz11] we proved an extension of the results of Abhyankar and Moh concerning approximate roots of f (see [AM73]) to the case of so-called 'non-characteristic' approximate roots of f. The necessary excerpt from [Brz11, Theorem 5] is given in Theorem 1. In the present work, we use this theorem and the properties of characteristic sequences to give some necessary conditions for the irreducibility of $f \in \mathbb{K}[[X]][Y]$ when $\operatorname{char} \mathbb{K} = 0$ (one can think $\mathbb{K} = \mathbb{C}$). These conditions are effective in the case

2010 *Mathematics Subject Classification.* Primary 12E05, Secondary 12E10, 14C17.
Key words and phrases. Approximate root, irreducibility condition, characteristic sequence, intersection multiplicity.

of $f \in \mathbb{K}[X,Y]$. Namely, Theorem 4 below can be easily turned into a test algorithm for reducibility, main point of which is the process of division with remainder (it serves to compute the intersection multiplicity (cf. [GP13]) and approximate roots (cf. [Brz11, Remark 1])).

Let us remark that the problem of testing irreducibility has been fully solved by Abhyankar in [Abh89], but his criterion is more technical than our numeric conditions as it involves analyzing the form of G-adic expansions of polynomials. From this criterion one can easily deduce necessary conditions for irreducibility ([Abh90, p. 183], presented in Theorem 2 below) similar in nature to ours (Theorem 4). We show by example (Example 2) that in general our necessary conditions are stronger than those in Theorem 2.

For an interesting combinatorial criterion of irreducibility see the recent work [GG10].

2. Characteristic Sequences (cf. [Abh77, § 6])

Let $\mathbb{K}$ be an algebraically closed field (for simplicity – of characteristic 0) and let $f \in \mathbb{K}((X))[Y]$ be a monic and irreducible polynomial. By Newton Theorem ([Abh77, Theorem (5.19)]), f can be written in the form

$$f(t^k, Y) = \prod_{\varepsilon \in U_k(\mathbb{K})} (Y - y(\varepsilon t)), \tag{2.1}$$

where $U_k(\mathbb{K}) := \{\varepsilon \in \mathbb{K} : \varepsilon^k = 1\}$ and $y(t) = \sum_{j \in \mathbb{Z}} y_j t^j \in \mathbb{K}((t))$. We recall that the *support* $\operatorname{Supp}_t y(t)$ of $y(t)$ is the set of those exponents of the powers of t that occur with a non-zero coefficient in the Laurent expansion of $y(t)$. Note also that from the irreducibility of f it follows that $\gcd(\{k\} \cup \operatorname{Supp}_t y(t)) = 1$.

The basic characteristic sequences of f. To begin with, we put $m_0 := k$, $d_1 := k$ and $m_1 := \operatorname{ord}_t y(t)$. If, now, $y(t) = 0$ then putting $h := 0$ we end the construction. In the opposite case, let $d_2 := \gcd(m_0, m_1)$. Inductively, if $m_0, \ldots, m_i$ and $d_1, \ldots, d_{i+1}$ are already defined for some $i \geqslant 1$, put

$$m_{i+1} := \inf\{j \in \operatorname{Supp}_t y(t) : j \not\equiv 0 (\operatorname{mod} d_{i+1})\}.$$

If, now, $m_{i+1} < +\infty$, we also define

$$d_{i+2} := \gcd(m_0, \ldots, m_{i+1}),$$

whereas in the case $m_{i+1} = +\infty$ we put $h := i$ and finish the inductive definition.

Since in the above construction there is always $0 < d_{j+1} < d_j$ for $j \geqslant 2$, the process ends after finitely many steps. Thus we end up with two sequences:

$$m := (m_0, m_1, \ldots, m_{h+1})$$

and

$$d := (d_1, \ldots, d_{h+1}).$$

We call them, respectively: *the characteristic of* f and *the sequence of characteristic divisors of* f.

Using the sequences m and d we also define the following **derived characteristic sequence of** f:

$$r = (r_0, \dots, r_{h+1}),$$

where $r_0 := m_0$, $r_i := \frac{1}{d_i}(m_1 d_1 + \sum_{2 \leqslant j \leqslant i} (m_j - m_{j-1}) d_j)$ for $1 \leqslant i \leqslant h$, and $r_{h+1} := +\infty$.

Note that the characteristic sequences defined above do not depend on the choice of a particular $y(t)$ satisfying (2.1).

Immediately from the definitions we get:

Property 1. *The sequences m, d, r are integer-valued (or $+\infty$). What is more,*
1. $h \geqslant 1$ *unless* $f = Y$,
2. $m_1 < m_2 < \dots < m_{h+1} = +\infty$,
3. $d_{i+1} = \gcd(m_0, \dots, m_i) = \gcd(d_i, m_i) = \gcd(d_i, r_i) = \gcd(r_0, \dots, r_i)$ *for* $1 \leqslant i \leqslant h$,
4. $1 = d_{h+1} | d_h | \dots | d_1 = k$ *and* $d_{h+1} < d_h < \dots < d_2 \leqslant d_1$,
5. *if* $M \in \mathbb{Z} \cup \{+\infty\}$ *and* $m_{i-1} < M \leqslant m_i$ *for some* $i \in \{2, \dots, h+1\}$ *(or only* $M \leqslant m_i$ *if* $i = 1$*), then*

$$\gcd(\{k\} \cup (\operatorname{Supp}_t y(t) \cap (-\infty, M))) = \gcd(m_0, \dots, m_{i-1}) = d_i,$$

6. $r_i d_i = r_{i-1} d_{i-1} + (m_i - m_{i-1}) d_i$ *for* $2 \leqslant i \leqslant h$,
7. $r_1 d_1 < r_2 d_2 < \dots < r_{h+1} d_{h+1} = +\infty$.

3. The Preliminary Result

We start with the following (here m, d, r are the characteristic sequences of f with $h + 1$ equal to the length of the divisor sequence d).

Theorem 1. *Let* $\mathbb{K}$ *be an algebraically closed field,* $\operatorname{char} \mathbb{K} = 0$, *let* $f \in \mathbb{K}((X))[Y]$ *be of the form (2.1) and let* l *be a positive divisor of* k. *Define* $i := \max\{1 \leqslant j \leqslant h+1 : l | d_j\}$. *Then*

$$\operatorname{ord}_t(\sqrt[l]{f}(t^k, y(t))) = r_i \frac{d_i}{l}. \tag{3.1}$$

Proof. The case $l \neq d_i$ is the non-characteristic case stated in [Brz11, Theorem 5, item 5]; if $l = d_i$ and $l \neq k$ then $2 \leqslant i$, and this is the characteristic case proved in [Abh77, Theorem (8.2)].

It remains to prove the case of $l = k$. Now, if $k = 1$ then $\sqrt[l]{f} = f$, $i = h + 1$ and $r_{h+1} = \infty$, so (3.1) is valid by the very definitions (cf. Section 2). Hence, in the following we may assume that $k \geqslant 2$. Property 1 implies that in this case

$$i \in \{1, 2\} \text{ and } d_1, \dots, d_i = k; \tag{3.2}$$

also $h \geqslant i$ since $k \geqslant 2$. Let $f(t^k, Y) = Y^k + v(t^k)Y^{k-1} + \dots$. From Viète's formulas it follows that

$$v(t^k) = -\sum_{\varepsilon \in U_k(\mathbb{K})} y(\varepsilon t) = -\sum_{\varepsilon \in U_k(\mathbb{K})} \Big(\sum_{j<m_i} (y_j \varepsilon^j t^j) + y_{m_i}\varepsilon^{m_i} t^{m_i}\Big) + \text{ terms of order} > m_i.$$

By the definitions of i and the characteristic sequences of f, we have $d_{i+1} = \gcd(d_i, m_i) < d_i = k$ and also $\gcd(\{k\} \cup (\operatorname{Supp}_t y(t) \cap (-\infty, m_i))) = d_i = k$ (by Property 1). Consequently, for a k-th primitive root of unity $\varepsilon_0 \in U_k(\mathbb{K})$,

$$\begin{cases} \varepsilon_0^j = 1, & \text{if } j < m_i \\ \varepsilon_0^j \neq 1, & \text{if } j = m_i \end{cases}$$

and so

$$\sum_{\varepsilon \in U_k(\mathbb{K})} \varepsilon^j = \begin{cases} k, & \text{if } j < m_i \\ 0, & \text{if } j = m_i \end{cases} .$$

It follows that

$$v(t^k) = -k \cdot \sum_{j<m_i} y_j\, t^j + \text{ terms of order} > m_i.$$

Now, by the definition of an approximate root, one sees easily that $\sqrt[l]{f} = \sqrt[k]{f} = Y + \frac{v(t)}{k}$. Thus we have

$$\sqrt[l]{f}(t^k, y(t)) = y(t) + \frac{v(t^k)}{k} = y_{m_i}\, t^{m_i} + \text{ terms of order} > m_i,$$

and since $y_{m_i} \neq 0$,

$$\operatorname{ord}_t(\sqrt[l]{f}(t^k, y(t))) = m_i.$$

It remains to see that (according to (3.2))

$$m_i = \left\{ \begin{array}{ll} r_1, & \text{if } i = 1 \\ \frac{m_1 d_1 + (m_2 - m_1) d_2}{d_2} = r_2, & \text{if } i = 2 \end{array} \right\} = r_i \frac{d_i}{l}.$$

□

4. Necessary Conditions for Irreducibility

Throughout this section $\mathbb{K}$ denotes an algebraically closed field of characteristic 0.

Notation 1. *For monic polynomials $f, g \in \mathbb{K}[[X]][Y]$ we write $\mathcal{I}(f, g)$ to denote the* intersection multiplicity *of f and g at $0 = (0, 0)$, which is, by definition, equal to the dimension of the $\mathbb{K}$-vector space $\mathbb{K}[[X, Y]]/(f, g)$ (see e.g.* [Pło13, Section 3]*).*

We recall that a monic $f \in \mathbb{K}[[X]][Y]$ with $f(0) = 0$ is called *Y-distinguished* if $f = Y^k + a_1(X)\, Y^{k-1} + \ldots + a_k(X)$ and $a_1(0) = \ldots = a_k(0) = 0$.

The simplest test for reducibility is the following well-known

Property 2. *If a monic $f \in \mathbb{K}[[X]][Y]$, $f(0) = 0$, is not distinguished, then f is reducible in $\mathbb{K}[[X]][Y]$.*

Proof. This can be deduced from Hensel's Lemma. An alternative proof is the following. Suppose that f is irreducible. It is clear that f is also irreducible in $\mathbb{K}((X))[Y]$. By Newton Theorem we can assume that f is of the form (2.1). Since $f \in \mathbb{K}[[X]][Y]$ we have $y(t) \in \mathbb{K}[[t]]$ and since: $f(0) = 0$, $f(t^k, 0) = \pm \prod_{\varepsilon \in U_k(\mathbb{K})} y(\varepsilon t)$ — we have $y(0) = 0$. This means that f is distinguished. □

The above property implies that the only interesting case to deal with is that of a distinguished polynomial. Hence in the following we will consider only such polynomials. The starting point for our further considerations is:

Theorem 2 (**Abhyankar's Necessary Conditions for Irreducibility** [Abh90, p. 183]). *Let $f \in \mathbb{K}[[X]][Y]$ be Y-distinguished of degree $k \geqslant 2$. Put $r'_0 := d'_1 := k$, $r'_1 := \mathcal{I}(f, Y)$, $d'_2 := \gcd(d'_1, r'_1)$ and then $r'_e := \mathcal{I}(f, \sqrt[d_e]{f})$, $d'_{e+1} := \gcd(d'_e, r'_e)$, for $e = 2, \ldots, h' + 1$, where the number $h' \geqslant 1$ is defined in such a way that $d'_{h'} > d'_{h'+1} = d'_{h'+2}$ and where (by convention) every integer divides ∞. If either*

(*A*1) $\quad d'_{h'+1} \neq 1$

or

(*A*2) $\quad$ *the sequence $(r'_1\, d'_1, \ldots, r'_{h'+1}\, d'_{h'+1})$ is not strictly increasing,*

then the polynomial f is reducible in $\mathbb{K}[[X]][Y]$.

Proof. If f is irreducible, one can use the Abhyankar-Moh result on characteristic approximate roots (cf. Theorem 1) and Property 1 item 3 to see that in such a case none of the above conditions hold. Indeed, it is enough to note that the sequences $(d'_1, \ldots, d'_{h'+1})$, $(r'_0, \ldots, r'_{h'+1})$ are in fact the characteristic sequences d, r (respectively) defined in section 2. □

Theorem 1 of section 3 can be restated as follows.

Theorem 3. *Let $f \in \mathbb{K}[[X]][Y]$ be Y-distinguished of degree k. Let $(l_1, \ldots, l_a)$ be the strictly decreasing sequence of all the positive divisors of the number k. Define $\Delta := \{\delta_j : j = 0, \ldots, a\}$ where*

$$\delta_j := \mathcal{I}(f, \sqrt[l_j]{f}) \cdot l_j \ (j = 1, \ldots, a)$$

and

$$\delta_0 := \mathcal{I}(f, Y) \cdot k.$$

If f is irreducible in $\mathbb{K}[[X]][Y]$ and (m, d, r) denote the characteristic sequences of f with $h + 1$ equal to the length of the divisor sequence d, then

$$\Delta = \{r_e \cdot d_e : e = 1, \ldots, h + 1\}.$$

Proof. By the same argument as in the proof of Property 2, we can assume that f is of the form (2.1), where $y(t) \in \mathbb{K}[[t]]$ and $y(0) = 0$. Hence $(t^k, y(t))$ is a normalization of the algebroid curve $f = 0$. By the well-known property of intersection multiplicity, for any $g \in \mathbb{K}[[X]][Y]$ we have (cf. [Cam80, Chapter 2.3] or [Pło13])

$$\mathcal{I}(f,g) = \operatorname{ord}_t g(t^k, y(t)).$$

Thus, $\delta_0 = \mathcal{I}(f,Y)\cdot k = \operatorname{ord}_t y(t)\cdot k = m_1 k = r_1 d_1$. Moreover, from Theorem 1 and the definition of the derived sequence r it follows that

$$\delta_j = \mathcal{I}(f, \sqrt[l_j]{f})\cdot l_j = \operatorname{ord}_t \sqrt[l_j]{f}(t^k, y(t))\cdot l_j \in \{r_e\cdot d_e : e = 1,\dots,h+1\},$$

for $j = 1,\dots,a$. In particular, if $l_j = d_e < k$ we have $\delta_j = r_e d_e$, for $e = 2,\dots,h+1$; if $l_j = d_2 = k$, we still have $\delta_j = r_2 d_2$. Consequently, $\Delta = \{r_e\cdot d_e : e = 1,\dots,h+1\}$. □

Now we can strengthen Abhyankar's criterion.

Theorem 4. *Let $f \in \mathbb{K}[[X]][Y]$ be Y-distinguished of degree $k \geqslant 2$. Define the sequences d', r' as in Theorem 2 and the set Δ as in Theorem 3. If any of the conditions $(A1)$, $(A2)$,*

$(B1)$ $\quad \Delta \neq \{r'_e\, d'_e : 1 \leqslant e \leqslant h'+1\}$

or

$(B2)$ *there exists $j \in \{1,\dots,a\}$ such that for $i := \max\{1 \leqslant e \leqslant h'+1 : l_j | d'_e\}$ it is*

$$\delta_j \neq r'_i\, d'_i$$

holds, then f is reducible in $\mathbb{K}[[X]][Y]$.

Proof. As in the proof of Theorem 2, if f is irreducible then the sequences $(d'_1,\dots, d'_{h'+1})$, $(r'_0,\dots,r'_{h'+1})$ are in fact the characteristic sequences d and r of f. Hence the condition $(B1)$ is fulfilled by Theorem 3. As for condition $(B2)$, putting $i(l_j) := \max\{1 \leqslant e \leqslant h'+1 : l_j | d'_e\}$ for $j = 1,\dots,a$, thanks to Theorem 1 we get

$$\delta_j = \mathcal{I}(f, \sqrt[l_j]{f})\cdot l_j = r'_{i(l_j)} \frac{d'_{i(l_j)}}{l_j}\cdot l_j = r'_{i(l_j)} d'_{i(l_j)}, \text{ for } j = 1,\dots,a.$$

This finishes the proof. □

We illustrate Theorem 4 with some examples.

Example 1. *Take Kuo's example considered in* [Abh89]*:*

$$f := (Y^2 - X^3)^2 - X^7.$$

We easily compute $\sqrt[4]{f} = Y$, $\sqrt[2]{f} = (Y^2 - X^3)$ and, naturally, $\sqrt[1]{f} = f$. Hence $(r'_1 d'_1,\dots, r'_{h'+1} d'_{h'+1}) = (6\cdot 4, 14\cdot 2)$. By the condition $(A1)$ of Theorem 2 we deduce that f is reducible. *Now we change f a little:*

$$f := (Y^2 - X^3)^2 - 4\,X^5 Y - X^7.$$

The approximate roots are as before but now $(r'_1 d'_1,\dots, r'_{h'+1} d'_{h'+1}) = (6\cdot 4, 13\cdot 2, \infty\cdot 1)$. Moreover, $(\delta_j)_{j=0}^3 = (24, 24, 26, \infty)$. This easily implies that none of the *conditions* $(A1)$–$(B2)$ *of Theorem 4 is fulfilled and we may suspect (which is indeed the case) that* f *is* irreducible.

The next example shows that the conditions $(B1)$–$(B2)$ of Theorem 4 are sometimes stronger than Abhyankar's conditions $(A1)$–$(A2)$.

Example 2. *Consider* $f \in \mathbb{C}[[X]][Y]$ *of the form*

$$\begin{aligned} f :=&(Y^2 - X)^6 - 2\,X^3Y(Y^2 - X)^3 - 24\,X^4Y(Y^2 - X)^2 \\ &+ (-32\,X^5Y + X^6)(Y^2 - X) + 64\,X^8Y. \end{aligned}$$

One easily checks that

$$\begin{aligned} \sqrt[2]{f} &= (Y^2 - X)^3 - X^3Y \\ \sqrt[3]{f} &= (Y^2 - X)^2 \\ \sqrt[4]{f} &= Y^3 - \frac{3}{2}XY \\ \sqrt[6]{f} &= Y^2 - X \\ \sqrt[12]{f} &= Y \end{aligned}$$

and then $\delta_0 = \delta_1 = \mathcal{I}(f, Y) \cdot 12 = 6 \cdot 12 = 72$, $\delta_2 = \mathcal{I}(f, \sqrt[6]{f}) \cdot 6 = 17 \cdot 6 = 102$, $\delta_3 = \mathcal{I}(f, \sqrt[4]{f}) \cdot 4 = 18 \cdot 4 = 72$, $\delta_4 = \mathcal{I}(f, \sqrt[3]{f}) \cdot 3 = 34 \cdot 3 = 102$, $\delta_5 = \mathcal{I}(f, \sqrt[2]{f}) \cdot 2 = 40 \cdot 2 = 80$, $\delta_6 = \mathcal{I}(f, \sqrt[1]{f}) = \infty$. *Hence* $\Delta = \{72, 80, 102, \infty\}$.

On the other hand, performing the test of Theorem 2, we have $(r'_e\, d'_e)_{e=1,\ldots,h'+1} = (6 \cdot 12, 17 \cdot 6, \infty \cdot 1) = (72, 102, \infty)$ *which easily shows that the conditions* $(A1)$–$(A2)$ *are not fulfilled. Hence in this case one cannot decide reducibility of* f *using the criterion of Theorem 2. But since* $\Delta \supsetneqq \{r'_e\, d'_e : e = 1, \ldots, h' + 1\}$, *the condition* $(B1)$ *of Theorem 4 is fulfilled and we may conclude that* f *is reducible.*

Remark. *Abhyankar's criterion (Theorem 2) is valid over any algebraically closed field* $\mathbb{K}$ *of characteristic* $\operatorname{char} \mathbb{K} =: p$ *as long as* $k \not\equiv 0 \pmod{p}$. *Theorem 4, however, requires even more assumptions in such generality. Namely, in the notations of Theorem 4, for every positive divisor* l *of the number* k *one has to assume that* $\binom{\frac{d'_{i+1}}{l} - 1}{u} \cdot \mathbf{1} \neq 0$ *in* $\mathbb{K}$, *where* $i := \max\{1 \leqslant e \leqslant h' + 1 : l | d'_e\}$ *and* $u := \max\{0 \leqslant e \leqslant \frac{d'_{i+1}}{l} : \binom{\frac{d'_{i+1}}{l}}{e} \cdot \mathbf{1} \neq 0$ *in* $\mathbb{K}\}$. *This follows from* Theorem 11 *in* [Brz08] *which generalizes* Theorem 5 *of* [Brz11], *the main ingredient for the results of the present paper.*

Acknowledgement. *We wish to thank the anonymous referee whose remarks have led to a significant simplification of the exposition.*

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Analytic and Algebraic Geometry
Łódź University Press 2013, 41 – 49

EUCLIDEAN ALGORITHM AND POLYNOMIAL EQUATIONS AFTER LABATIE

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Abstract. We recall Labatie's effective method of solving polynomial equations with two unknowns by using the Euclidean algorithm.

Introduction

The French mathematician Labatie[1] published in 1835 a booklet on a method of solving polynomial systems of equations in two unknowns (see [Fin1]). He used the polynomial division to replace the given system of equations by the collection of triangular systems. Labatie's theorem can be found in some old Algebra books: by Finck [Fin2], Serret [Se] and Netto [Ne], but as far as we know, not in any Algebra text book written in the twentieth century.

In this paper we recall Labatie's method following Serret [Se] (pp. 196-206). Then we give, in a modern setting, an improvement of Labatie's result due to Bonnet [Bo].

Let $\mathbf{K}$ be a field of arbitrary characteristic. We shall consider polynomials with coefficients in $\mathbf{K}$. If $W = W(x, y) \in \mathbf{K}[x, y]$ then we denote by $\deg_y W$ the degree of W with respect to y. We say that a non-zero polynomial W is *y-primitive* if it is a primitive polynomial in the ring $\mathbf{K}[x][y]$, that is, if 1 is the greatest common divisor of all the non-zero coefficients that are dependent on x. If V, $W \in \mathbf{K}[x, y]$ satisfy the condition $0 < \deg_y V \leq \deg_y W$ then there are polynomials Q (quotient), R (remainder) in $\mathbf{K}[x, y]$ and a non-zero polynomial $u = u(x) \in \mathbf{K}[x]$ such that $uW = QV + R$, where $\deg_y R < \deg_y V$ or $R = 0$.

2010 *Mathematics Subject Classification.* Primary 12xxx; Secondary 14H20.
Key words and phrases. Polynomial equations, Euclidean algorithm, intersection multiplicity.
The first-named author was partially supported by the Spanish Project PNMTM 2007-64007.

[1]Anthilde-Gabriel Labatie (1786-1866), graduated from l'École Polytechnique.

The greatest common divisor of polynomials V, W may be computed using the Euclidean algorithm, see [Bô] chapter XVI. Recently Hilmar and Smyth [H-S] gave a very simple proof of Bézout's theorem for plane projective curves using as a main tool the Euclidean division.

1. Euclidean algorithm

Let V_1, $V_2 \in \mathbf{K}[x,y]$ be coprime and y-primitive polynomials such that $0 < \deg_y V_2 \leq \deg_y V_1$.

Using the polynomial division we get a sequence of y-primitive polynomials $V_3, \ldots,$ V_{n+1} of decreasing y-degrees $0 < \deg_y V_{n+1} < \cdots < \deg_y V_3 < deg_y V_2$ such that

$$\begin{array}{rcl} u_1V_1 & = & Q_1V_2 + v_1V_3, \\ u_2V_2 & = & Q_2V_3 + v_2V_4, \\ & \vdots & \\ u_{n-1}V_{n-1} & = & Q_{n-1}V_n + v_{n-1}V_{n+1}, \\ u_nV_n & = & Q_nV_{n+1} + v_n, \end{array}$$

where $u_1, \ldots, u_n$, $v_1, \ldots, v_n$ are non-zero polynomials of the ring $\mathbf{K}[x]$. Let be $V_{n+2} = 1$ and write the above equalities in the form

$$(1)_i \qquad u_iV_i = Q_iV_{i+1} + v_iV_{i+2} \quad \text{for } i = 1, \ldots, n.$$

In what follows we call n the number of steps performed by the Euclidean algorithm on input (V_1, V_2). We will keep the above notation in all this note.

2. Labatie's elimination

Let us define two sequences $d_1, \ldots, d_n$ and $w_1, \ldots, w_n$ of polynomials in x determined by the sequences $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ in a recurrent way. We let $d_1 = \gcd(u_1, v_1)$, $w_1 = \frac{u_1}{d_1}$ and $d_i = \gcd(w_{i-1}u_i, v_i)$, $w_i = \frac{w_{i-1}u_i}{d_i}$ for $i \in \{2, \ldots, n\}$. It is easy to see that $w_i = \frac{u_1 \cdots u_i}{d_1 \cdots d_i}$ in $\mathbf{K}[x]$ for all $i \in \{1, \ldots, n\}$.

For any V, $W \in \mathbf{K}[x,y]$ we let $\{V = 0, W = 0\} = \{\, P \in \mathbf{K}^2 : V(P) = W(P) = 0 \,\}$.

Theorem 2.1 (Labatie 1835)**.** *With notations and assumptions given above we have*

$$\{V_1 = 0, V_2 = 0\} = \bigcup_{i=1}^{n} \left\{ V_{i+1} = 0, \frac{v_i}{d_i} = 0 \right\}.$$

We present the proof of the above theorem in Section 4.

Labatie's theorem shows that the system of equations $V_1(x,y) = 0$, $V_2(x,y) = 0$ is equivalent to the collection of triangular systems

$$V_{i+1}(x,y) = 0,\ \frac{v_i}{d_i}(x) = 0 \qquad (i = 1, \ldots, n).$$

Labatie's theorem fell into oblivion for a longtime. At the beginning of the 1990's Lazard in [La] proved that every system of polynomial equations in many unknowns with a finite number of solutions in the algebraic closure of $\mathbf{K}$ is equivalent to the union of triangular systems, which can be obtained from Gröbner bases. Kalkbrener in [Kalk1] and [Kalk2] developed the theory of elimination sequences based on the Euclidean algorithm. His method of computing solutions of systems of polynomials equations turned out to be very efficient if applied to systems of two or three unknowns (see [Kalk2] and the references given therein for the comparison with Gröbner basis methods). Neither Lazard nor Kalkbrener mentioned Labatie's work. Only Glashof in [Glas] recalled Labatie's method after Netto [Ne] and compared it with Kalkbrener's approach to polynomials equations. In what follows we need the notion of multiplicity of a solution of a system of two equations in two unknowns. The definition we are going to present is quite sophisticated. The reader not acquainted with it may assume the five properties of multiplicity given below as axiomatic definition of this notion.

Let $P \in \mathbf{K}^2$. We define the local ring of rational functions regular at P to be

$$\mathbf{K}[x,y]_P = \left\{ \frac{R}{S}\ :\ R, S \in \mathbf{K}[x,y], S(P) \neq 0 \right\}.$$

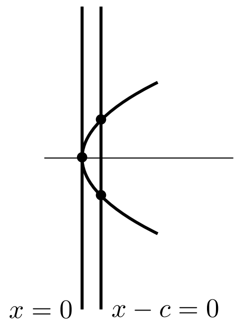
The ring $\mathbf{K}[x,y]_P$ is a unique factorization domain. The units of $\mathbf{K}[x,y]_P$ are rational functions $\frac{R}{S}$ such that $R(P)S(P) \neq 0$.

Let $(V,W)_P$ be the ideal generated by polynomials V and W in $\mathbf{K}[x,y]_P$. Following [Ful], we define the *intersection multiplicity* $i_P(V,W)$ to be the dimension of the $\mathbf{K}$-vector space $\mathbf{K}[x,y]_P/(V,W)_P$. We call also $i_P(V,W)$ the *multiplicity of the solution* P of the system $V = 0$, $W = 0$.

Let us recall the basic properties of the intersection multiplicity which hold for any field $\mathbf{K}$ (not necessarily algebraically closed):

(1) $i_P(V,W) < +\infty$ if and only if $P \notin \{\gcd(V,W) = 0\}$,
(2) $i_P(V,W) > 0$ if and only if $P \in \{V = W = 0\}$,
(3) $i_P(V,WW') = i_P(V,W) + i_P(V,W')$,
(4) $i_P(V,W)$ depends only on the ideal $(V,W)_P$.
Intuitively: $i_P(V,W)$ does not change when we replace the system $V = 0$, $W = 0$ by another one equivalent to it near P.
Moreover, it is easy to check that
(5) if $P = (a,b)$ is a solution of the triangular system $W(x,y) = 0$, $w(x) = 0$ then $i_P(W,w) = (\mathrm{ord}_a w)(\mathrm{ord}_b W(a,y))$, where $\mathrm{ord}_c p$ denotes the multiplicity of the root c in the polynomial $p = p(x) \in \mathbf{K}[x]$. By convention $\mathrm{ord}_c p = 0$ if $p(c) \neq 0$.

The following example may be helpful to acquire an intuition of intersection multiplicity. Let us consider the parabola $y^2 - x = 0$ over the field of real numbers. Applying Property 5 to the triangular system $y^2 - x = 0$, $x - c = 0$ we check that the axis $x = 0$ intersects the parabola in $(0, 0)$ with multiplicity 2 but the line $x - c = 0$, where $c > 0$ intersects it in two points $(c, \sqrt{c})$ and $(c, -\sqrt{c})$, each with multiplicity 1. If $c \to 0^+$ then the two points coincide.



Note also that the system of equations $y^2 - x = 0$, $x - c = 0$ has for $c \neq 0$ two complex solutions, which are arbitrary close to the origin for small enough complex c. This observation leads to the *dynamic definition* of intersection multiplicity for algebraic complex curves (see [Te], Section 6).

The following theorem due to Bonnet [Bo] is an improvement of Labatie's result:

Theorem 2.2 (Bonnet 1847). *For any $P \in \mathbf{K}^2$ we have*

$$i_P(V_1, V_2) = \sum_{i=1}^{n} i_P\Big(V_{i+1}, \frac{v_i}{d_i}\Big).$$

Bonnet, like Labatie, considered polynomials with complex coefficients and used the definition of the intersection multiplicity in terms of Puiseux series. In Section 5 we present a short proof of Theorem 2.2 based on Labatie's calculations (Section 3) and the properties of the intersection multiplicity listed above.

Example 2.3. *Let $V_1 = y^5 - x^3$, $V_2 = y^3 - x^4$. Using the Euclidean algorithm we get $y^5 - x^3 = y^2(y^3 - x^4) + x^3(xy^2 - 1)$, $x(y^3 - x^4) = y(xy^2 - 1) + y - x^5$ and $xy^2 - 1 = (xy + x^6)(y - x^5) + x^{11} - 1$. Hence we have $(u_1, u_2, u_3) = (1, x, 1)$, $(v_1, v_2, v_3) = (x^3, 1, x^{11} - 1)$ and $(d_1, d_2, d_3) = (1, 1, 1)$. By Labatie's theorem, we get*

$$\{y^5 - x^3 = 0, y^3 - x^4 = 0\} =$$
$$\{y^3 - x^4 = 0, x^3 = 0\} \cup \{xy^2 - 1 = 0, 1 = 0\} \cup \{y - x^5 = 0, x^{11} - 1 = 0\}.$$

Therefore the systems $V_1 = 0$, $V_2 = 0$ has two solutions $(0, 0)$ and $(1, 1)$ in $\mathbf{K}$ and ten solutions in the algebraic closure of $\mathbf{K}$. To compute the multiplicities of the *solutions we use Bonnet's theorem:*

$i_0(y^5-x^3, y^3-x^4) = i_0(y^3-x^4, x^3)+i_0(xy^2-1, 1)+i_0(y-x^5, x^{11}-1) = 3{\cdot}3+0+0 = 9.$

The remaining multiplicities are equal to one. Thus the system $V_1 = 0$, $V_2 = 0$ *has* $9 + 11 = 20$ *solutions counted with multiplicities.*

3. Auxiliary lemmas

Recall that the polynomials w_i and $\frac{v_i}{d_i}$ are coprime.

Lemma 3.1. *There exist two sequences of polymomials* $G_0,\dots,$ G_n *and* $H_0,\dots,$ H_n *in the ring* $\mathbf{K}[x, y]$ *such that*

$$w_{i-1}V_1 = G_{i-1}V_i + G_{i-2}V_{i+1}\frac{v_{i-1}}{d_{i-1}}, \qquad (2)_i$$

$$w_{i-1}V_2 = H_{i-1}V_i + H_{i-2}V_{i+1}\frac{v_{i-1}}{d_{i-1}} \qquad (3)_i$$

for $i \in \{2, \dots, n + 1\}$.

Proof. We proceed by induction on i. Let's check the first identity. From the equality $u_1V_1 = Q_1V_2 + v_1V_3$ it follows that $d_1 = \gcd(u_1, v_1)$ divides the product Q_1V_2 and consequently the polynomial Q_1 since V_2 is y-primitive. Letting $G_0 = 1$, $G_1 = \frac{Q_1}{d_1}$ we get $w_1V_1 = G_1V_2+G_0V_3\frac{v_1}{d_1}$ that is $(2)_2$. Suppose now that $2 \le i < n+1$ and that for some polynomials G_{i-1} and G_{i-2} the identity $(2)_i$ holds. Multiplying the identity $(2)_i$ by the polynomial u_i we get

$$w_{i-1}u_iV_1 = u_iG_{i-1}V_i + u_iG_{i-2}V_{i+1}\frac{v_{i-1}}{d_{i-1}}.$$

Let us insert to the identity above $u_iV_i = Q_iV_{i+1} + v_iV_{i+2}$. After simple computations we get:

$$w_{i-1}u_iV_1 = \Big(G_{i-1}Q_i + u_iG_{i-2}\frac{v_{i-1}}{d_{i-1}}\Big)V_{i+1} + G_{i-1}v_iV_{i+2}.$$

Since $d_i = \gcd(w_{i-1}u_i, v_i)$ and the polynomial V_{i+1} is y-primitive we get that $G_i := \frac{G_{i-1}Q_i}{d_i} + G_{i-2}\frac{u_iv_{i-1}}{d_id_{i-1}}$ is a polynomial and we have

$$w_iV_1 = G_iV_{i+1} + G_{i-1}V_{i+2}\frac{v_i}{d_i},$$

which is the identity $(2)_{i+1}$. This proves the first part of the lemma.

To prove the identity $(3)_i$ note that

$$w_1V_2 = H_1V_2 + H_0V_3\frac{v_1}{d_1}$$

if we let $H_0 = 0$ and $H_1 = \frac{u_1}{d_1}$. This proves $(3)_2$. To check $(3)_i$ we proceed analogously to the proof of $(2)_i$: it suffices to replace G_i by H_i. ∎

Remark 3.2. *The polynomials G_i are defined by $G_0 = 1$, $G_1 = \frac{Q_1}{d_1}$, $G_i = \frac{G_{i-1}Q_i}{d_i} + \frac{G_{i-2}u_i v_{i-1}}{d_{i-1}d_i}$ and the polynomials H_i by $H_0 = 0$, $H_1 = \frac{u_1}{d_1}$ and $H_i = \frac{H_{i-1}Q_i}{d_i} + \frac{H_{i-2}u_i v_{i-1}}{d_{i-1}d_i}$.*

Lemma 3.3. *With the notations of Lemma 3.1 we have the identities*

$$(4)_i \qquad (-1)^i \frac{v_1 \cdots v_{i-1}}{d_1 \cdots d_{i-1}} V_{i+1} = H_{i-1}V_1 - G_{i-1}V_2 \qquad \textit{for } i \in \{2, \dots, n+1\}.$$

Proof. Let $D_i = G_i H_{i-1} - G_{i-1}H_i$ for $i \in \{2, \dots, n\}$. Consider the system of equations $(2)_i$, $(3)_i$ as a linear system with unknowns V_i, $V_{i+1}\frac{v_{i-1}}{d_{i-1}}$ with determinant equals D_{i-1}. Using Cramer's rule we get

$$\begin{aligned} D_{i-1}V_i &= w_{i-1}\left(H_{i-2}V_1 - G_{i-2}V_2\right), \\ D_{i-1}V_{i+1}\frac{v_{i-1}}{d_{i-1}} &= -w_{i-1}(H_{i-1}V_1 - G_{i-1}V_2). \end{aligned}$$

Replacing in the first equality i by $i+1$ we obtain

$$(1) \qquad D_i V_{i+1} = w_i(H_{i-1}V_1 - G_{i-1}V_2).$$

Multiplying the second equality by $\frac{u_i}{d_i}$ we get

$$(2) \qquad D_{i-1}V_{i+1}\frac{v_{i-1}}{d_{i-1}}\frac{u_i}{d_i} = -w_i(H_{i-1}V_1 - G_{i-1}V_2).$$

Comparing the left sides of (1) and (2) and cancelling V_{i+1} we have $D_i = -\frac{v_{i-1}u_i}{d_{i-1}d_i}D_{i-1}$. Moreover $D_1 = G_1H_0 - G_0H_1 = -\frac{u_1}{d_1}$ and by induction we have

$$D_i = (-1)^i w_i \frac{v_1 \cdots v_{i-1}}{d_1 \cdots d_{i-1}}$$

which inserted into formula (1) gives the identity $(4)_i$. ■

4. Proof of Labatie's theorem

We can now give the proof of Theorem 2.1: fix a point $P \in \mathbf{K}^2$. If $V_i(P) = \frac{v_{i-1}}{d_{i-1}}(P) = 0$ for a value $i \in \{2, \dots, n+1\}$ then from Lemma 3.1 it follows that $V_1(P) = V_2(P) = 0$ given that $w_{i-1}(P) \neq 0$ since w_{i-1}, $\frac{v_{i-1}}{d_{i-1}}$ are coprime.

Suppose now that $V_1(P) = V_2(P) = 0$. From the identity $(4)_{n+1}$ of Lemma 3.3 we get $\frac{v_1 \cdots v_n}{d_1 \cdots d_n}(P) = 0$. Therefore at least one of polynomials $\frac{v_1}{d_1}, \dots, \frac{v_n}{d_n}$ vanishes at P. If $\frac{v_1}{d_1}(P) = 0$ then $P \in \{V_2 = \frac{v_1}{d_1} = 0\}$.

If the smallest index i for which $\frac{v_i}{d_i}(P) = 0$ is strictly greater than 1 then we get, by the identity $(4)_i$, that $V_{i+1}(P) = 0$ because $\frac{v_1 \cdots v_{i-1}}{d_1 \cdots d_{i-1}}(P) \neq 0$ by the definition of i. This proves the theorem.

5. Proof of Bonnet's theorem

Fix a point $P \in \mathbf{K}^2$. If $\frac{v_1 \cdots v_n}{d_1 \cdots d_n}(P) \neq 0$ then by $(4)_{n+1}$ we get

$$1 \in (V_1, V_2)_P \tag{3}$$

which implies $i_P(V_1, V_2) = 0$.

On the other hand we have $i_P\left(V_{i+1}, \frac{v_i}{d_i}\right) = 0$ since $\frac{v_i}{d_i}(P) \neq 0$ for $i \in \{1, \dots, n\}$ and the theorem holds in the case under consideration.

Suppose now that $\frac{v_1 \cdots v_n}{d_1 \cdots d_n}(P) = 0$ and let i_0 be the smallest index $i \in \{1, \dots, n\}$ such that $\frac{v_{i_0}}{d_{i_0}}(P) = 0$. Therefore we have $w_{i_0}(P) \neq 0$ since $\frac{v_{i_0}}{d_{i_0}}$ and w_{i_0} are coprime. Let us check that

$$(V_1, V_2)_P = \left(V_{i_0+1}, V_{i_0+2}\frac{v_{i_0}}{d_{i_0}}\right)_P. \tag{4}$$

From $(2)_{i_0+1}$ and $(3)_{i_0+1}$ we get

$$V_1, V_2 \in \left(V_{i_0+1}, V_{i_0+2}\frac{v_{i_0}}{d_{i_0}}\right)_P. \tag{5}$$

On the other hand, from $(4)_{i_0}$ (if $i_0 > 1$, the case $i_0 = 1$ being obvious), we obtain

$$V_{i_0+1} \in (V_1, V_2)_P \tag{6}$$

and from $(4)_{i_0+1}$, we have

$$\frac{v_{i_0}}{d_{i_0}} V_{i_0+2} \in (V_1, V_2)_P. \tag{7}$$

Combining (5), (6) and (7) we get (4). Equality (4) and the additive property of intersection multiplicity give

$$i_P(V_1, V_2) = i_P\left(V_{i_0+1}, \frac{v_{i_0}}{d_{i_0}}\right) + i_P(V_{i_0+1}, V_{i_0+2}). \tag{8}$$

If $i_0 = n$ then (8) reduces to

$$i_P(V_1, V_2) = i_P\left(V_{n+1}, \frac{v_n}{d_n}\right) \tag{9}$$

since $V_{n+2} = 1$.

To prove Theorem 2.2 we shall proceed by induction on the number n of steps performed by the Euclidean algorithm. For $n = 1$ the theorem follows from (9) since $n = 1$ implies $i_0 = 1$. Let $n > 1$ and suppose that the theorem holds for all pairs of polynomials for which the number of steps performed by the Euclidean algorithm is strictly less than n.

We assume that $i_0 < n$ since for $i_0 = n$ the theorem is true by (9).

Let us put $\overline{V}_j = V_{i_0+j}$, where $j \in \{1, 2, \ldots, n - i_0 + 2\}$. The number of steps performed by the Euclidean algorithm on input $(\overline{V}_1, \overline{V}_2)$ is equal to $\overline{n} = n - i_0 < n$. We have $\overline{u}_j = u_{i_0+j}$ and $\overline{v}_j = v_{i_0+j}$ for $j \in \{1, \ldots, \overline{n}\}$. To relate $\overline{d}_j$ and d_{i_0+j} we introduce some notation. We will write $u \sim \tilde{u}$ for polynomials $u, \tilde{u}$ associated in the local ring $\mathbf{K}[x, y]_P$. If $u, \tilde{u} \in \mathbf{K}[x]$ then $u \sim \tilde{u}$ if and only if there exist polynomials $r, s \in \mathbf{K}[x]$ such that $su = r\tilde{u}$ and $r(P)s(P) \neq 0$. Note that $\gcd(u, v) \sim \gcd(\tilde{u}, v)$ if $u \sim \tilde{u}$. We claim that

$$\overline{d}_j \sim d_{i_0+j}, \quad \overline{w}_j \sim w_{i_0+j} \quad \text{for} \quad j \in \{1, \ldots, \overline{n}\}. \tag{10}$$

Let us check (10) by induction on j.

If $j = 1$ then $\overline{d}_1 = \gcd(\overline{u}_1, \overline{v}_1) = \gcd(u_{i_0+1}, v_{i_0+1}) \sim \gcd(w_{i_0} u_{i_0+1}, v_{i_0+1}) = d_{i_0+1}$ since $w_{i_0} \sim 1$. Hence we get $\overline{w}_1 = \frac{\overline{u}_1}{\overline{d}_1} = \frac{u_{i_0+1}}{\overline{d}_1} \sim \frac{w_{i_0} u_{i_0+1}}{d_{i_0+1}}$, which proves (10) for $j = 1$.

Suppose that (10) holds for a $j < \overline{n}$. Then we get

$$\overline{d}_{j+1} = \gcd(\overline{w}_j \overline{u}_{j+1}, \overline{v}_{j+1}) \sim \gcd(w_{i_0+j} u_{i_0+j+1}, v_{i_0+j+1}) = d_{i_0+j+1}$$

since $\overline{w}_j \sim w_{i_0+j}$ by the induction assumption, and

$$\overline{w}_{j+1} = \frac{\overline{w}_j \overline{u}_{j+1}}{\overline{d}_{j+1}} \sim \frac{w_{i_0+j} u_{i_0+j+1}}{d_{i_0+j+1}} = w_{i_0+j+1}.$$

This finishes the proof of (10).

Now we can pass to the proof of the theorem. By the inductive assumption applied to the pair $\overline{V}_1$, $\overline{V}_2$ we get

$$\begin{aligned} i_P(V_{i_0+1}, V_{i_0+2}) &= i_P(\overline{V}_1, \overline{V}_2) = \sum_{j=1}^{\overline{n}} i_P\left(\overline{V}_{j+1}, \frac{\overline{v}_j}{\overline{d}_j}\right) \\ &= \sum_{j=1}^{\overline{n}} i_P\left(V_{i_0+j+1}, \frac{v_{i_0+j}}{d_{i_0+j}}\right) = \sum_{i=i_0+1}^{n} i_P\left(V_{i+1}, \frac{v_i}{d_i}\right) \end{aligned}$$

since $\overline{d}_j \sim d_{i_0+j}$ by (10) which together with (8) proves the inductive step and so the theorem.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 51 – 55

ON SMOOTH HYPERSURFACES CONTAINING A GIVEN SUBVARIETY

ZBIGNIEW JELONEK

Abstract. We reprove some results about affine complete intersections.

1. Introduction.

Let k be an algebraically closed field. Let X^n be a smooth affine variety (of dimension n). Let us recall that a variety $H \subset X$ is a hypersurface if the ideal $I(H) \subset k[X]$ is generated by a single polynomial. Let $Y^r \subset X^n$ be a smooth subvariety. It was proved in [2] (see also [3]), that if $n \geq 2r+1$ then there is a smooth complete intersection $Z^{2r} \subset X^n$ such that $Y^r \subset Z^{2r}$. In general this result can not be improved- see Example 2.2. We also show how to use results from [6] to improve the result above in some special cases. In particular we show:

Theorem 1.1. *(Greco, Valabrega) Let X^n be a smooth variety and let Y^r be a smooth subvariety of X. Assume that the r^{th} Chow group $CH^r(Y^r)$ vanishes. If $n \geq 2r$, then there is a smooth complete intersection $Z^{2r-1} \subset X$ such that $Y^r \subset Z^{2r-1}$.*

and

Theorem 1.2. *(Murthy) Let $Y^r \subset \mathbb{A}^n$ be a smooth subvariety. If $n \geq 2r$ then there is a smooth hypersurface $H \subset \mathbb{A}^n$ such that $Y \subset H$.*

In particular a smooth surface $S \subset \mathbb{A}^4$ is contained in a smooth hypersurface $H \subset \mathbb{A}^4$. Let us note that this is not true in the projective case: it is well known that a smooth surface $S \subset \mathbb{P}^4$ is not contained in any smooth hypersurface $H \subset \mathbb{P}^4$,

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2010 *Mathematics Subject Classification.* 14R10, 14R99.
Key words and phrases. Algebraic vector bundle, complete intersection, unimodular row.
The author was partially supported by the grant of Polish Ministry of Science 2010-2013.

unless it is a complete intersection. Our approach is slightly different than the original ones.

2. Main Result.

We start with:

Theorem 2.1. *Let $Y \subset X$ be smooth affine varieties. Then there is a smooth hypersurface $V(f) \subset X$ which contains Y if and only if the normal bundle of Y contains a one dimensional trivial summand i.e.,*

$$\mathbf{N}_{X/Y} = \mathbf{T} \oplus \mathbf{E}^1,$$

where $\mathbf{E}^1$ denotes a trivial line bundle.

Proof. Assume that there is a smooth hypersurface $H = V(f) \subset X$ which contains Y. We have

$$TY \subset TH \subset TX,$$

in particular

$$\mathbf{N}_{X/Y} = \mathbf{N}_{H/Z} \oplus \mathbf{N}_{X/H}|_Y.$$

However, the normal bundle of the smooth hypersurface $H = V(f)$ is trivial (in fact the class of f is a generator of the conormal bundle of H).

Conversely, assume that

$$\mathbf{N}_{X/Y} = \mathbf{T} \oplus \mathbf{E}^1.$$

Hence also

$$\mathbf{N}^*_{X/Y} = \mathbf{T}^* \oplus \mathbf{E}^1.$$

This means that the conormal bundle $\mathbf{N}^*_{X/Y}$ has a nowhere vanishing section $\mathbf{s} \in \Gamma(Y, \mathbf{N}^*_{X/Y})$. But $\Gamma(Y, \mathbf{N}^*_{X/Y}) = I(Y)/I(Y)^2$, where $I(Y) \subset k[X]$ denotes the ideal of the subvariety Y. Hence $\mathbf{s}$ determines a polynomial $s \in I(X)$ such that the class of s is $\mathbf{s}$. Take a point $a \in Y$ and local coordinates $(u_1, ..., u_n)$ at a such that Y is described by local equations $u_1, ..., u_t$ ($t = \text{codim} Y$) near a. Since $\mathbf{u}_1, ..., \mathbf{u}_t$ freely generate the bundle $\mathbf{N}^*_{X/Y}$ near the point a, we have

$$\mathbf{s} = \sum_{i=1}^{t} \alpha_i \mathbf{u}_i,$$

where $\alpha_i \in k[U_a]$ (U_a denotes some open neighborhood of a in Y). Since the section $\mathbf{s}$ nowhere vanishes, there exists at least one i_0 such that $\alpha_{i_0} \neq 0$. Let us compute the derivative $d_y s$ of the polynomial s at the point $y \in Y$. We have

$$s = \sum_{i=1}^{t} \alpha_i u_i \mod I(Y)^2,$$

hence there are polynomials $f_j, h_j \in I(Y),\ j = 1, ..., m$, such that

$$s = \sum_{i=1}^{t} \alpha_i u_i + \sum_{j=1}^{m} f_j h_j.$$

Now we easily see that

$$d_a s = \sum_{i=1}^{t} \alpha_i d_a u_i.$$

Since $d_a u_i, i = 1, ..., n$, are linearly independent and not all α_i vanish at y we have $d_y s \neq 0$. Hence the hypersurface $V(s)$ is smooth along Y. Let $I(Y) = (g_1, ..., g_r)$. Consider the linear system on X given by the polynomials $(s, g_1^2, ..., g_r^2)$. The base locus of this system is exactly the subvariety Y. We can extend the set $\{g_1^2, ..., g_r^2\}$ adding some polynomials $\{g_j^2\alpha_i,\ j = 1, ..., s, i = 0, 1, ..., k\}$ in such a way that a new system $(s, g_1^2, ..., g_r^2, g_j^2\alpha_i)$ is unramified outside Y. Indeed, let $x \in X \setminus Y$. There is a polynomial $g_x \in I(Y)$, such that $g_x(x) \neq 0$. Let $\alpha_1, ..., \alpha_{2k+1}$ $(k = \dim X)$ be polynomials which gives an embedding of X into k^{2n+1}. In some neighbourhood U_x of X we still have $g_x \neq 0$. Since $X \setminus Y$ is quasi-compact we can cover $X \setminus Y$ by a finite set $U_{x_i}, i \in I$ of such neighbourhoods. Associate with every such U_x the set $S_x := \{g_x^2, g_x^2\alpha_1, ..., g_x^2\alpha_{2k+1}\}$. It is easy to see, that the system given by polynomials $\{s, g_1^2, ..., g_r^2\} \cup \bigcup_{i\in I} S_{x_i}$ is unramified on $X \setminus Y$.

Hence by the Bertini Theorem (see [4], Corollary 12 and [5], Theorem 3.1) the hypersurface $V(s + \sum_{i=1}^{r} \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s)$ for generic $\beta_i,\ \beta_{j,s}$ is smooth outside Y. But for $y \in Y$,

$$d_y(s + \sum_{i=1}^{r} \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s) = d_y s.$$

This implies that the hypersurface $V(s + \sum_{i=1}^{r} \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s)$ is also smooth along Y. Hence we can take $f = s + \sum_{i=1}^{r} \beta_i g_i^2 + \sum \beta_{j,s} g_j^2 \alpha_s$. $\square$

Let X^{2n} be a smooth variety and Y^n be a smooth subvariety of X^{2n}. We show that in general does not exist a smooth hypersurface $H \subset X^{2n}$, such that $Y^n \subset H$. Indeed we have:

Example 2.2. Let $H_d \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree $d > n + 2$. Let $Y \subset H$ be an affine open subset. By [7] we have $CH^n(Y) \neq 0$. Take a non-zero $z \in CH^n(Y)$. By Riemann-Roch without denominators and Serre Splitting Theorem (Theorem 2.3 below), there exists an algebraic vector bundle $\mathbf{F}$ on Y of rank n such that $c_n(\mathbf{F}) = (n-1)!z$. Since $CH^n(Y)$ has no $(n-1)!$ torsion (see e.g. [6]) we have $c_n(\mathbf{F}) \neq 0$. Now let X denote the total space of this vector bundle. Then $Y \subset X$ (as the zero-section) and $\mathbf{N}_{X/Y} \cong \mathbf{F}$. Since the top Chern class of $\mathbf{F}$ does not vanish, the bundle $\mathbf{F}$ does not have a one dimensional trivial summand. In particular Y is not contained in any smooth hypersurface in X (see Theorem 2.1).

In the sequel we need the following (see [1], p.177, Th. 7.1.8 and [5], Corollary 3.4):

Theorem 2.3. *(Serre Splitting Theorem) Let X be a smooth affine variety and let $\mathbf{F}$ be an algebraic vector bundle on X. If rank $\mathbf{F} > \dim \mathbf{X}$, then $\mathbf{F}$ has a one dimensional trivial summand i.e.,*

$$\mathbf{F} = \mathbf{T} \oplus \mathbf{E}^1.$$

Now we are in a position to prove:

Theorem 2.4. *Let X^n be a smooth variety and let Y^r be a smooth subvariety of X. If $n \geq 2r+1$ then there is a smooth complete intersection $Z^{2r} \subset X^n$ such that $Y^r \subset Z^{2r}$. Assume additionally that the r^{th} Chow group $CH^r(Y^r)$ vanishes. If $n \geq 2r$, then there is a smooth complete intersection $Z^{2r-1} \subset X$ such that $Y^r \subset Z^{2r-1}$.*

Proof. Assume first that $s = n - 2r > 0$. Since $\dim Y^r <$ rank $\mathbf{N}_{X/Y}$, Theorem 2.3 shows that $\mathbf{N}_{X/Y} = \mathbf{T} \oplus \mathbf{E}^1$, where $\mathbf{E}^1$ denotes a trivial line bundle. By Theorem 2.1 there exists a smooth hypersurface $H = V(f)$ (where f is a reduced polynomial) such that $Y \subset H$. Now we can apply the mathematical induction. This completes the proof of the first part of Theorem 2.4.

For the proof of the second part let us note that the bundle $\mathbf{F} = \mathbf{N}^*_{Z^{2r}/Y^r}$ has a one dimensional trivial summand as $c_r(\mathbf{F}) = 0$, by the Theorem of Murthy (see [6], Th. 3.8). Now we can finish by applying Theorem 2.1. □

Theorem 2.5. *Let X^n, Y^r be as above. If $n \geq 2r+1$ then there is a smooth hypersurface $H = V(f)$ such that $Y^r \subset H$. If the r^{th} Chow group $CH^r(X^n)$ vanishes, then it is enough to assume $n \geq 2r$.*

Proof. It is enough to consider only the last statement. Moreover, we can assume that $n = 2r$. Let $Y^r = \bigcup_{i=1}^s Y_i$ be the decomposition of Y into irreducible components. Of course $Y_i \cap Y_j = \emptyset$ for $i \neq j$. We show that the bundle $\mathbf{F} = \mathbf{N}_{X/Y}$ has a one dimensional trivial summand over every Y_i. Indeed, if dim $Y_i < r$ then it follows from the Serre Splitting Theorem. Assume that $\dim Y_i = r$. Let $\iota : Y_i \to X$ be the inclusion. By the self-intersection formula we have the following expression for the top Chern class of the normal bundle of Y_i:

$$c_r(\mathbf{F}|_{Y_i}) = \iota^* \circ \iota_*[Y_i],$$

where $[Y_i] \in CH^0[Y_i] = \mathbb{Z}$ is a generator. By our assumption we have $c_r(\mathbf{F}|_{Y_i}) = 0$. Now by the Theorem of Murthy, invoked above, the normal bundle $\mathbf{N}_{X/Y}$ splits over Y_i in a suitable way. Finally we can use Theorem 2.1. □

The last statement of Theorem 2.5 can be applied to $X = \mathbb{A}^n$, or more generally to $X =$ *open affine subset of* $\mathbb{A}^n$. In particular we have:

Corollary 2.6. *Let $Y^r \subset \mathbb{A}^n$ be a smooth subvariety. If $n \geq 2r$ then there is a smooth hypersurface $H \subset \mathbb{A}^n$ such that $Y \subset H$.*

Theorems above suggest that if all (positive) Chow groups of X and Y vanish, then it is easier to find a smooth hypersurface which contains a given smooth subvariety $Y \subset X$. However, we show that also in that case there are examples of smooth subvarieties $Y \subset X$ which are not contained in any smooth hypersurface of X. In our example X will be an open affine subset of $\mathbb{A}^9$ and Y be an affine open subset of $\mathbb{A}^7$. In particular Y and X have all positive Chow groups trivial.

Example 2.7. Consider the variety $\Gamma = \{(x, y) \in k^3 \times k^3 : \sum_{i=1}^3 x_i y_i = 1\}$. By the Raynaud Theorem (see [8] and [9]) the algebraic vector bundle given by the unimodular row (x_1, x_2, x_3) is not free. Let $\Lambda = \{(x, y) \in k^3 \times k^3 : \sum_{i=1}^3 x_i y_i = 0\}$ be an affine cone and let $Y' = \mathbb{A}^6 \setminus \Lambda$. Hence Y' is an affine open subset of $\mathbb{A}^6$. Moreover, the algebraic vector bundle $\mathbf{F}$ given by the unimodular row (x_1, x_2, x_3) is not trivial, because it is not trivial after restriction to Γ. Since every stably trivial line bundle is trivial and rank $\mathbf{F} = 2$, we see that the vector bundle $\mathbf{F}$ does not split.

Take $Y'' = Y' \times k,\ X = Y' \times k^3$ and consider the embedding

$$\phi : Y'' \ni ((x, y), t) \mapsto ((x, y), x_1 t, x_2 t, x_3 t) \in X.$$

Take $Y = \phi(Y'')$. By direct computations we see that the normal bundle $\mathbf{N}_{X/Y}$ restricted to the subvariety $Y' \times \{0\}$ is equal to

$$\mathbf{E}^3 / < (x_1, x_2, x_3) > \cong \mathbf{F}$$

(where $\mathbf{E}^s$ denotes the trivial bundle of rank s). Since the bundle $\mathbf{F}$ does not split, neither does $\mathbf{N}_{X/Y}$. In particular Y is not contained in any smooth hypersurface in X. Moreover, X is an open subset of $\mathbb{A}^9$ and Y is isomorphic to an open subset of $\mathbb{A}^7$.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 57 – 79

RINGS OF CONSTANTS OF POLYNOMIAL DERIVATIONS AND p-BASES

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Abstract. We present a survey of results concerning p-bases of rings of constants with respect to polynomial derivations in characteristic $p > 0$. We discuss characterizations of rings of constants, properties of their generators and a general characterization of their p-bases. We also focus on some special cases: one-element p-bases, eigenvector p-bases and when a ring of constants is a polynomial graded subalgebra.

Introduction

In Section 1 we introduce the notation and definitions concerning derivations, rings of constants and p-bases. Then we discuss characterizations of rings of constants in Section 2 and we present some basic information on the number of generators for rings of constants of polynomial derivations in Section 3. For a wider panorama of contemporary differential algebra we refer to the book of Nowicki ([41]), and for problems connected with locally nilpotent derivations we refer to the book of Freudenburg ([10]).

Next two sections contain a general characterization of p-bases of rings of constants with respect to polynomial derivations, based on the author's paper [26]. In Section 4 we present generalizations of Freudenburg's lemma (Theorems 4.7 and 4.8). The main theorem (Theorem 5.4) and its motivations are presented in Section 5. In Section 6 (based on the results of [23] and [18]) we discuss analogies and differences between single generators of rings of constants in zero and positive characteristic, and we focus on some special cases. Section 7, based on [24], is devoted to specific properties of eigenvector p-bases (Theorem 7.2). Finally, in

2010 *Mathematics Subject Classification.* Primary 13N15, Secondary 13F20.
Key words and phrases. Polynomial, derivation, ring of constants, p-basis.

Section 8 (based on the paper [28], joint with Nowicki) we describe rings of constants of homogeneous polynomial derivations in positive characteristic, which are polynomial algebras.

1. Basic definitions and notation

Throughout this article, by a ring we mean a commutative ring with unity, and by a domain we mean a commutative ring with unity, without zero divisors. If K is a ring, then by $K[x_1, \ldots, x_n]$ we denote a polynomial K-algebra. If R is a domain, then by R_0 we denote its field of fractions.

Let A be a domain. By A^* we denote the set of all invertible elements of A. We call two elements $a, b \in A$ associated and denote it by $a \sim b$, if $a = bc$ for some $c \in A^*$. An element $a \in A$ is called square-free if $b^2 \nmid a$ for every $b \in A \setminus A^*$.

Let A be a domain of characteristic $p > 0$. Then

$$A^p = \{a^p;\ a \in A\}$$

is a subring of A. Let B a subring of A, containing A^p. An element $a \in A$ is called B-free if $b \nmid a$ for every $b \in B \setminus A^*$. If $A = k[x_1, \ldots, x_n]$ is a polynomial algebra over a field k of characteristic $p > 0$, then $k[x_1^p, \ldots, x_n^p]$-free elements are called shortly p-free.

If A is a domain of characteristic $p > 0$ and B is a subring of A, containing A^p, then for elements $f_1, \ldots, f_m \in A$ we define the following subring of A:

$$C_B(f_1, \ldots, f_m) = B_0(f_1, \ldots, f_m) \cap A = B_0[f_1, \ldots, f_m] \cap A.$$

Note that the equality $B_0(f_1, \ldots, f_m) = B_0[f_1, \ldots, f_m]$ can easily be proved directly, but it also follows from the fact that the field extension $B_0 \subset B_0(f_1, \ldots, f_m)$ is algebraic.

Let A be a ring. An additive map $d\colon A \to A$ satisfying the Leibniz rule

$$d(fg) = d(f)g + fd(g)$$

for $f, g \in A$, is called a derivation of A. The set

$$A^d = \{f \in A :\ d(f) = 0\}$$

is called the ring of constants of d; it is a subring of A. Moreover, if A is a field, then A^d is a subfield of A.

If A is a K-algebra, where K is a ring, then a K-linear derivation $d\colon A \to A$ is called a K-derivation. In this case A^d is a K-subalgebra of A. When K is a subring of A, d is a K-derivation if and only if $K \subset A^d$.

If d is a K-derivation of a polynomial algebra $K[x_1, \ldots, x_n]$, where K is a ring, then

$$d(f) = \frac{\partial f}{\partial x_1} d(x_1) + \ldots + \frac{\partial f}{\partial x_n} d(x_n)$$

for every $f \in K[x_1, \dots, x_n]$.

On the other hand, for arbitrary polynomials $g_1, \dots, g_n \in K[x_1, \dots, x_n]$ there exists exactly one K-derivation d of $K[x_1, \dots, x_n]$ such that

$$\begin{cases} d(x_1) &= g_1 \\ &\vdots \\ d(x_n) &= g_n \end{cases}$$

and this derivation is of the form

$$d = g_1 \frac{\partial}{\partial x_1} + \dots + g_n \frac{\partial}{\partial x_n}.$$

Let A be a domain. Then every derivation $d\colon A \to A$ can be uniquely extended to a derivation $\delta\colon A_0 \to A_0$, which is defined by the formula

$$\delta\left(\frac{f}{g}\right) = \frac{d(f)g - fd(g)}{g^2}$$

for $f, g \in A$, $g \neq 0$. If A is a K-domain (that is, a K-algebra and a domain), where K is a domain, and d is a K-derivation, then δ is a K_0-derivation.

If A is a domain of characteristic $p > 0$ and $d\colon A \to A$ is a derivation, then $d(a^p) = 0$ for every $a \in A$, so $A^p \subset A^d$. If A is also a K-algebra, where K is a domain of characteristic $p > 0$, and d is a K-derivation, then $KA^p \subset A^d$, so d is a KA^p-derivation. For example, if A is a polynomial K-algebra: $A = K[x_1, \dots, x_n]$, where $\operatorname{char} K = p > 0$, then $A^p = K^p[x_1^p, \dots, x_n^p]$ and $KA^p = K[x_1^p, \dots, x_n^p]$.

Lemma 1.1. *Let K be a domain of characteristic $p > 0$, consider a polynomial $f \in K[x_1, \dots, x_n]$. Then $f \in K[x_1^p, \dots, x_n^p]$ if and only if $\frac{\partial f}{\partial x_i} = 0$ for $i = 1, \dots, n$.*

Recall the definition of a p-basis. We restrict our interests to finite p-bases, see [35], 38.A, p. 269, for a definition of a p-basis of arbitrary cardinality.

Definition 1.2. *Let R be a domain of characteristic $p > 0$ and B a subring of R, containing R^p. Let $f_1, \dots, f_m \in R$.*

a) *The elements $f_1, \dots, f_m$ are called p-independent over B if the elements of the form $f_1^{\alpha_1} \dots f_m^{\alpha_m}$, where $\alpha_1, \dots, \alpha_m \in \{0, \dots, p-1\}$, are linearly independent over B.*

b) *We say that the elements $f_1, \dots, f_m$ form a p-basis of R over B if R is a free B-module with a basis of the form*

$$f_1^{\alpha_1} \dots f_m^{\alpha_m},$$

where $\alpha_1, \dots, \alpha_m \in \{0, \dots, p-1\}$.

Note that the elements $f_1, \ldots, f_m$ form a p-basis of R over B if and only if they are p-independent over B and generate R as a B-algebra. If the elements $f_1, \ldots, f_m$ form a p-basis of R over B, then every element $f \in R$ can be presented in the form

$$f = \sum_{0 \leqslant \alpha_1, \ldots, \alpha_m < p} a_\alpha f_1^{\alpha_1} \ldots f_m^{\alpha_m},$$

where $a_\alpha \in B$, and this presentation is unique.

The notion of a p-basis is a specific positive characteristic analog of a transcendental basis. It fits into the same abstract notion of dependency, see [52], II.12, p. 97 and II.17, p. 129.

Example 1.3. *The elements $x_1, \ldots, x_n$ form:*

a) *a p-basis of $K[x_1, \ldots, x_n]$ over $K[x_1^p, \ldots, x_n^p]$,*

b) *a p-basis of $k(x_1, \ldots, x_n)$ over $k(x_1^p, \ldots, x_n^p)$,*

c) *a p-basis of $K[[x_1, \ldots, x_n]]$ over $K[[x_1^p, \ldots, x_n^p]]$,*

where K is a domain, k is a field, $\operatorname{char} K = \operatorname{char} k = p > 0$.

Theorem 1.4. ([15], p. 180)
If M is a subfield of a field L of characteristic $p > 0$, such that $L^p \subset M$, then there exists a p-basis (possibly infinite) of L over M.

Various conditions for existence of p-bases of ring extensions have been studied for a long time (see, for example, [46] and its references).

Given polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, where K is a ring, and $j_1, \ldots, j_m \in \{1, \ldots, n\}$, by $\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m}$ we denote the Jacobian determinant of $f_1, \ldots, f_m$ with respect to $x_{j_1}, \ldots, x_{j_m}$. If $m = n$, then the Jacobian determinant of $f_1, \ldots, f_n$ with respect to $x_1, \ldots, x_n$ we denote by $\operatorname{jac}(f_1, \ldots, f_n)$.

It is convenient to introduce the following notion of a differential gcd of polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, where K is a UFD:

$$\operatorname{dgcd}(f_1, \ldots, f_m) = \gcd\big(\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m},\ j_1, \ldots, j_m \in \{1, \ldots, n\}\big).$$

We put $\operatorname{dgcd}(f_1, \ldots, f_m) = 0$ if $\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m} = 0$ for every $j_1, \ldots, j_m$.

Note that $\operatorname{dgcd}(f_1, \ldots, f_m)$ is defined up to a factor from K^*. We have

$$\operatorname{dgcd}(f) \sim \gcd\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)$$

for a single polynomial $f \in K[x_1, \ldots, x_n]$ and

$$\operatorname{dgcd}(f_1, \ldots, f_n) \sim \operatorname{jac}(f_1, \ldots, f_n)$$

for n polynomials $f_1, \ldots, f_n \in K[x_1, \ldots, x_n]$.

From a generalized Laplace expansion we obtain the following ([26], Lemma 3.2).

Lemma 1.5. *Consider arbitrary pairwise different numbers $i_1, \ldots, i_r$ belonging to $\{1, \ldots, m\}$, where $1 \leqslant r \leqslant m$.*

a) *If* $\operatorname{dgcd}(f_{i_1}, \ldots, f_{i_r}) \neq 0$, *then* $\operatorname{dgcd}(f_{i_1}, \ldots, f_{i_r}) \mid \operatorname{dgcd}(f_1, \ldots, f_m)$.

b) *If* $\operatorname{dgcd}(f_{i_1}, \ldots, f_{i_r}) = 0$, *then* $\operatorname{dgcd}(f_1, \ldots, f_m) = 0$.

Recall the following known positive characteristic analog of the well known criterion of algebraic dependence in characteristic zero.

Lemma 1.6. *Let K be a domain of characteristic $p > 0$. Polynomials $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ are p-dependent over $K[x_1^p, \ldots, x_n^p]$ if and only if* $\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m} = 0$ *for every $j_1, \ldots, j_m \in \{1, \ldots, n\}$.*

2. Characterizations of rings of constants

Recall some characterizations of fields of constants with respect to derivations of fields. The case of characteristic zero was considered by Suzuki in [49] (Theorem 1) under the assumption of finite transcendence degree and genralized by Nowicki in [42], Theorem 4.2 (see also [41], Theorem 3.3.2).

Theorem 2.1. (Suzuki, Nowicki)
Let $K \subset L$ be an extension of fields of characteristic 0. A subfield $M \subset L$ such that $K \subset M$, is a field of constants of some K-derivation of L if and only if M is algebraically closed in L.

Similarly, in the positive characteristic case, Baer considered extensions of finite degree (see [15], IV.7, p. 185). Gerstenhaber proved the theorem in the general case in [12] (Remark at the end of Section 1) and, explicitly, in [13], Lemma 2.

Theorem 2.2. (Baer, Gerstenhaber)
Let $K \subset L$ be an extension of fields of characteristic $p > 0$ satisfying the condition $L^p \subset K$. Then every subfield $M \subset L$ such that $K \subset M$, is a field of constants of some K-derivation of L.

A characterization of rings of constants with respect to derivations of domains was obtained by Nowicki in [42], Theorem 5.4 (see also [41], Theorem 4.1.4).

Theorem 2.3. (Nowicki)
Let A be a finitely generated k-domain, where k is a field of characteristic zero. Let R be a k-subalgebra of A. The following conditions are equivalent:

(1) *R is the ring of constants of some k-derivation of A,*

(2) *R is integrally closed in A and $R_0 \cap A = R$.*

The author observed in [16] and, more generally, in [19], that analogous characterization (without the condition that R is integrally closed) holds in the positive characteristic case.

Theorem 2.4. ([16], Theorem 1.1, [19], Theorem 2.5)
Let A be a finitely generated K-domain, where K is a domain of characteristic $p > 0$. Let R be a subring of A. The following conditions are equivalent:

(1) *R is the ring of constants of some K-derivation of A,*

(2) *$KA^p \subset R$ and $R_0 \cap A = R$.*

The implications (1) $\Rightarrow$ (2) in Theorems 2.3 and 2.4 hold without the assumption A is finitely generated, and there are counter-examples to the reverse implications ([17], see Example 2.7 below).

Daigle noted ([5], 1.4) that the two conditions in (2) in Theorem 2.3 can be replaced by one condition of algebraic closedness (in the ring sense). The author observed in [22] that we can apply this condition to the positive characteristic case if we modify it to separable algebraic closedness. We call R separably algebraically closed in A, if each element of A, separably algebraic over R, belongs to R ([22], Definition 2.1).

Theorem 2.5. ([22], Theorem 3.1)
Let A be a finitely generated K-domain, where K is a domain (of arbitrary characteristic). Let R be a K-subalgebra of A. If $\operatorname{char} K = p > 0$, we assume additionally that $A^p \subset R$ and we put $B = KA^p$. The following conditions are equivalent:

(1) *R is the ring of constants of some K-derivation of A,*

(2) *R is separably algebraically closed in A,*

(3) *R is a maximal element in one of the following families of rings:*

$$\begin{cases} \Phi_m = \{R :\ K \subset R \subset A,\ \operatorname{tr}\deg_K R \leqslant m\} & \text{if } \operatorname{char} A = 0, \\ \Psi_m = \{R :\ B \subset R \subset A,\ (R_0 : B_0) \leqslant p^m\} & \text{if } \operatorname{char} A = p > 0, \end{cases}$$

where $m = 0, 1, 2, \ldots$

Now, let A be a domain of characteristic $p > 0$ and let B be a subring of A, containing A^p. Consider arbitrary elements $f_1, \ldots, f_m \in A$. Recall a notation

$$C_B(f_1, \ldots, f_m) = B_0(f_1, \ldots, f_m) \cap A = B_0[f_1, \ldots, f_m] \cap A.$$

If A is finitely generated as a B-algebra, then $C_B(f_1, \ldots, f_m)$ is the smallest (with respect to inclusion) ring of constants of a B-derivation containing the elements $f_1, \ldots, f_m$. Under this assumption, the elements $f_1, \ldots, f_m$ form a p-basis (over B) of the ring of constants of some B-derivation if and only if $f_1, \ldots, f_m$ are p-independent over B and $C_B(f_1, \ldots, f_m) = B[f_1, \ldots, f_m]$.

Remark that the notion of the ring $C_k(f)$, for a polynomial f over a field k of characteristic 0, was introduced by Nowicki in [40].

Let k be a field of characteristic $p > 0$. Note that, if $f \notin k[x^p, y^p]$, then f is a one-element p-basis of $k[x^p, y^p, f]$.

Example 2.6. *Let d be a k-derivation of $k[x, y]$ such that*

$$\begin{cases} d(x) &= x \\ d(y) &= -y. \end{cases}$$

Then the polynomial xy is a (one-element) p-basis of $k[x, y]^d$:

$$k[x, y]^d = C_B(xy) = k[x^p, y^p, xy],$$

where $B = k[x^p, y^p]$.

The following example from [24] (Example 4.3), motivated by Examples 6, 7 from [17], shows that in Theorem 2.4 the assumption that A is finitely generated is necessary.

Example 2.7. *Let k be a field of characteristic $p > 0$, let $A = k[x_0, x_1, x_2, \dots]$ be a polynomial k-algebra, put $B = k[x_0^p, x_1^p, x_2^p, \dots]$. For $i = 1, 2, \dots$ put $f_i = x_i^{r_i} - x_0$, where $r_i > 1$ and $p \nmid r_i$. Consider the ring*

$$C_B(f_1, f_2, f_3, \dots) = B_0(f_1, f_2, f_3, \dots) \cap A.$$

Then:

a) *the polynomials $f_1, f_2, f_3, \dots$ form a p-basis of $C_B(f_1, f_2, f_3, \dots)$ over B,*

b) *$C_B(f_1, f_2, f_3, \dots)$ is not a ring of constants of any B-derivation of A.*

3. Generators of rings of constants

The case of characteristic zero. Let k be a field of characteristic 0.

Recall the following theorem of Zariski ([51]).

Theorem 3.1. (Zariski)
Let L be a subfield of $k(x_1, \dots, x_n)$ containing k. If $\operatorname{tr}\deg_k L \leqslant 2$, then the ring

$$L \cap k[x_1, \dots, x_n]$$

is finitely generated over k.

Nowicki and Nagata in [43] (Theorem 2.6) applied Zariski's theorem to rings of constants of derivations.

Theorem 3.2. (Nowicki, Nagata)
Let d be a k-derivation of $k[x_1, \dots, x_n]$. If $n \leqslant 3$, then $k[x_1, \dots, x_n]^d$ is finitely generated over k.

The following example was obtained by Kuroda in [30] and [31] (see [10], 7.6, p. 175). This example is very important in the context of Hilbert's Fourteenth Problem. It solved the Problem for ordinary derivations, while for locally nilpotent derivations the case of $n = 4$ remains open (we refer to [10] for details).

Example 3.3. (Kuroda)
Let d be a k-derivation of $k[x, y, z, t]$ such that

$$\begin{cases} d(x) &= x(4x^4 - y^4 - z^4) \\ d(y) &= y(4y^4 - x^4 - z^4) \\ d(z) &= z(4z^4 - x^4 - y^4) \\ d(t) &= -20x^3y^3z^3. \end{cases}$$

Then $k[x, y, z, t]^d$ is not a finitely generated k-algebra.

Nowicki and Strelcyn in [44] constructed examples of k-derivations with arbitrary finite (minimal) number of generators of rings of constants.

Example 3.4. (Nowicki, Strelcyn)
Let $n \geqslant 3$ and $r \geqslant 0$. Then r is the minimal number of generators of $k[x_1, \dots, x_n]^d$ as a k-algebra, for the following k-derivation d.

a) *Let $r < n$. Consider a k-derivation d such that $d(x_i) = 0$ if $i \leqslant r$ and $d(x_i) = x_i$ if $i > r$. Then*

$$k[x_1, \dots, x_n]^d = k[x_1, \dots, x_r].$$

b) *Let $r \geqslant n$. Consider a k-derivation d such that*

$$\begin{cases} d(x_1) &= x_1 \\ d(x_2) &= x_2 \\ d(x_3) &= (n - r - 2)x_3 \\ d(x_i) &= 0 \ \text{ for } \ i > 3. \end{cases}$$

Then

$$k[x_1, \dots, x_n]^d = k[f_0, f_1, \dots, f_{r-n+2}, x_4, \dots, x_n],$$

where $f_j = x_1^j x_2^{r-n+2-j} x_3$ for $j = 0, \dots, r - n + 2$.

Now, recall the following theorem of Zaks ([50]).

Theorem 3.5. (Zaks)
If R is a Dedekind subring of $k[x_1, \dots, x_n]$ containing k, then $R = k[f]$ for some $f \in k[x_1, \dots, x_n]$.

Using Zaks' theorem, Nowicki and Nagata proved ([43], Theorem 2.8, [41], Theorem 7.1.4, Corollary 7.1.5) the following.

Theorem 3.6. (Nowicki, Nagata)
If d is a k-derivation of $k[x_1, \dots, x_n]$, such that $\operatorname{tr}\deg_k k[x_1, \dots, x_n]^d \leqslant 1$, then $k[x_1, \dots, x_n]^d = k[f]$ for some $f \in k[x_1, \dots, x_n]$.

Corollary 3.7. *If d is a nonzero k-derivation of $k[x, y]$, then $k[x, y]^d = k[f]$ for some $f \in k[x, y]$.*

Note also in this context Miyanishi's theorem ([36], see [10], Theorem 5.1, p. 108).

Theorem 3.8. (Miyanishi)
If d is a nonzero locally nilpotent k-derivation of $k[x, y, z]$, then $k[x, y, z]^d = k[f, g]$ for some algebraically independent $f, g \in k[x, y, z]$.

The case of positive characteristic. Now, let k be a field of characteristic $p > 0$.

Recall the results of Nowicki and Nagata ([43], Proposition 4.1, Proposition 4.2).

Theorem 3.9. (Nowicki, Nagata)
If d is a k-derivation of $k[x_1, \ldots, x_n]$, then $k[x_1, \ldots, x_n]^d$ is finitely generated as a $k[x_1^p, \ldots, x_n^p]$-algebra.

Theorem 3.10. (Nowicki, Nagata)
If $\operatorname{char} k = 2$ *and d is a nonzero k-derivation of $k[x, y]$, then there exists a polynomial $f \in k[x, y]$ such that $k[x, y]^d = k[x^2, y^2, f]$.*

Nowicki and Nagata proved that, if $p > 2$, the ring of constants of the Euler's derivation in $k[x, y]$ is not of the form $k[x^p, y^p, f]$ for any polynomial $f \in k[x, y]$ ([43], Example 4.3). Li in [34] proved that in this case $p - 1$ is the minimal number of generators of $k[x, y]^d$ as a $k[x^p, y^p]$-algebra.

Example 3.11. *Let d be a k-derivation of $k[x, y]$ such that*

$$\begin{cases} d(x) &= x \\ d(y) &= y. \end{cases}$$

Then, for $B = k[x^p, y^p]$ we have:

$$k[x, y]^d = C_B(x^{p-1}y) = k[x^p, x^{p-1}y, \ldots, xy^{p-1}, y^p].$$

Li in [33] (Theorem) obtained the following generalization of Theorem 3.10 for arbitrary characteristic $p > 0$.

Theorem 3.12. (Li)
Let d be a nonzero k-derivation of $k[x, y]$. Then:

a) *$k[x, y]^d$ is a free $k[x^p, y^p]$-module of rank p or 1,*

b) *there exist $g_1, \ldots, g_{p-1} \in k[x, y]$ such that $k[x, y]^d = k[x^p, y^p, g_1, \ldots, g_{p-1}]$.*

Note also that Nowicki and Nagata gave an example of a derivation, which ring of constants is not a free module ([43], Example 4.6).

Example 3.13. *Let $n \geqslant 3$ and let d be a k-derivation of $k[x_1, \ldots, x_n]$ such that $d(x_i) = x_i^p$ for $i = 1, \ldots, n$. Then $k[x_1, \ldots, x_n]^d$ is not a free $k[x_1^p, \ldots, x_n^p]$-module.*

4. Freudenburg's lemma

The key preparatory fact for the main characterization of p-bases of rings of constants with respect to polynomial derivations (Theorem 5.4) is a positive characteristic generalization of the following lemma, obtained by Freudenburg in [9].

Lemma 4.1. (Freudenburg)
Given a polynomial $f \in \mathbb{C}[x, y]$, suppose $g \in \mathbb{C}[x, y]$ is an irreducible non-constant divisor of both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Then there exists $c \in \mathbb{C}$ such that g divides $f + c$.

This lemma was generalized by van den Essen, Nowicki and Tyc in [8], Proposition 2.1.

Proposition 4.2. (van den Essen, Nowicki, Tyc)
Let k be an algebraically closed field of characteristic zero. Let Q be a prime ideal in $k[x_1, \ldots, x_n]$ and $f \in k[x_1, \ldots, x_n]$. If for each i the partial derivative $\frac{\partial f}{\partial x_i}$ belongs to Q, then there exists $c \in k$ such that $f - c \in Q$.

The following example from [8], Remark 2.4, shows that the condition that k is algebraically closed can not be dropped in the above theorem. We can, however, make a positive conclusion, as in point b).

Example 4.3. *Consider polynomials $f = x^3 + 3x$, $g = x^2 + 1 \in \mathbb{R}[x]$. Then g is irreducible, $g \mid f'$ and:*

a) *$g \nmid f - c$ for any $c \in \mathbb{R}$,*

b) *$g \mid f^2 + 4$, where $w(x) = x^2 + 4$ is irreducible.*

Note the following generalization of the Freudenburg's lemma for a UFD of arbitrary characteristic.

Proposition 4.4. ([21], Theorem 3.1)
Let K be a UFD, let Q be a prime ideal of $K[x_1, \ldots, x_n]$. Consider a polynomial $f \in K[x_1, \ldots, x_n]$ such that $\frac{\partial f}{\partial x_i} \in Q$ for $i = 1, \ldots, n$.

a) *If $\operatorname{char} K = 0$, then there exists an irreducible polynomial $w(x) \in K[x]$ such that $w(f) \in Q$.*

b) *If $\operatorname{char} K = p > 0$, then there exist $b, c \in K[x_1^p, \ldots, x_n^p]$ such that $\gcd(b, c) \sim 1$, $b \notin Q$ and $bf + c \in Q$.*

Now, let K be a UFD of characteristic $p > 0$.

Lemma 4.5. *Let $f \in K[x_1, \ldots, x_n]$ and let $g \in K[x_1, \ldots, x_n]$ be an irreducible polynomial. If $g \mid f$ and $g \mid \frac{\partial f}{\partial x_i}$ for every i, then $g^2 \mid f$ or $g \in K[x_1^p, \ldots, x_n^p]$.*

In the case of a principal ideal in positive characteristic we obtain from Proposition 4.4 the following equivalence.

Corollary 4.6. *Consider a polynomial $f \in K[x_1, \ldots, x_n]$ and an irreducible polynomial $g \in K[x_1, \ldots, x_n]$. The following conditions are equivalent:*

(1) $g \mid \frac{\partial f}{\partial x_i}$ *for* $i = 1, \ldots, n$,

(2) *there exist* $b, c \in K[x_1^p, \ldots, x_n^p]$ *such that* $g \nmid b$, $\gcd(b, c) \sim 1$ *and*

$$\begin{cases} g^2 \mid bf + c & \text{if } g \notin K[x_1^p, \ldots, x_n^p], \\ g \mid bf + c & \text{if } g \in K[x_1^p, \ldots, x_n^p]. \end{cases}$$

Now we are going to present generalizations of Freudenburg's lemma for an arbitrary number of polynomials instead of one. Theorem 4.7 is a generalization of Proposition 4.4 b), and Theorem 4.8 is a generalization of Corollary 4.6.

Theorem 4.7. ([26], Proposition 3.5)
Let $A = K[x_1, \ldots, x_n]$ be a polynomial K-algebra, where K is a UFD of characteristic $p > 0$. Put $B = K[x_1^p, \ldots, x_n^p]$. Let $f_1, \ldots, f_m \in A$, $m \geqslant 1$, and let Q be a prime ideal of A. If $\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m} \in Q$ for every $j_1, \ldots, j_m \in \{1, \ldots, n\}$, then there exist $i \in \{1, \ldots, m\}$ and

$$b, c \in B[f_1, \ldots, \widehat{f_i}, \ldots, f_m],$$

$b \notin Q$, such that $bf_i + c \in Q$.

Proof. (Sketch.)
Consider the factor algebra $\overline{A} = A/Q$ and denote $\overline{f} = f + Q$ for an element $f \in A$, and by $\overline{T}$ the canonical homomorphic image in $\overline{A}$ of a subring $T \subset A$.

If $\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m} \in Q$ for every $j_1, \ldots, j_m \in \{1, \ldots, n\}$, then the rank of the matrix

$$\begin{bmatrix} \overline{\partial f_1/\partial x_1} & \overline{\partial f_1/\partial x_2} & \cdots & \overline{\partial f_1/\partial x_n} \\ \overline{\partial f_2/\partial x_1} & \overline{\partial f_2/\partial x_2} & \cdots & \overline{\partial f_2/\partial x_n} \\ \vdots & \vdots & & \vdots \\ \overline{\partial f_m/\partial x_1} & \overline{\partial f_m/\partial x_2} & \cdots & \overline{\partial f_m/\partial x_n} \end{bmatrix}$$

over the field $(\overline{A})_0$ is less than m. From the linear dependence of the rows of this matrix we infer that:

$(*)$ there exist $s_1, \ldots, s_m \in A$, where $s_i \notin Q$ for some $i \in \{1, \ldots, m\}$, such that $s_1 d(f_1) + \ldots + s_m d(f_m) \in Q$ for every K-derivation d of A.

Now, denote $R_i = B[f_1, \ldots, \widehat{f_i}, \ldots, f_m]$. For every $\overline{R_i}$-derivation δ of $\overline{A}$ there exists a K-derivation d of A such that $\delta(\overline{f}) = \overline{d(f)}$ for every $f \in A$ ([21], Lemma 3.2). Then, by $(*)$, $d(f_i) \in Q$, so $\delta(\overline{f_i}) = \overline{0}$. Hence, $\overline{f_i}$ belongs to $(\overline{R_i})_0 \cap \overline{A}$ – the smallest ring of constants of any $\overline{R_i}$-derivation of $\overline{A}$, so there exist $b, c \in R_i$ such that $\overline{b} \neq \overline{0}$ and $\overline{f_i} = -\frac{\overline{c}}{\overline{b}}$. $\square$

Theorem 4.8. ([26], Theorem 3.6)
Let K be a UFD of characteristic $p > 0$. Let $A = K[x_1, \ldots, x_n]$, put $B = K[x_1^p, \ldots, x_n^p]$. Consider arbitrary polynomials $f_1, \ldots, f_m \in A$, where $m \geqslant 1$, and denote

$$R_i = B[f_1, \ldots, \widehat{f_i}, \ldots, f_m]$$

for $i = 1, \ldots, m$, and, if $m > 1$,

$$R_{ij} = B[f_1, \ldots, \widehat{f_i}, \ldots, \widehat{f_j}, \ldots, f_m]$$

for $i, j = 1, \ldots, m$, such that $i \neq j$.

Then $\operatorname{dgcd}(f_1, \ldots, f_m)$ is divisible by an irreducible polynomial $g \in A$ if and only if at least one of the following conditions holds:

(i) $g \notin B$ and $g^2 \mid bf_i + c$ for some $i \in \{1, \ldots, m\}$ and $b, c \in R_i$ such that $g \nmid b$,

(ii) $g \in B$ and $g \mid bf_i + c$ for some $i \in \{1, \ldots, m\}$ and $b, c \in R_i$ such that $g \nmid b$,

(iii) $g \mid b_1 f_i + c_1$ and $g \mid b_2 f_j + c_2$ for some $i, j \in \{1, \ldots, m\}$, $i \neq j$, and $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $g \nmid b_1$ and $g \nmid b_2$.

Proof. (Sketch.)
($\Rightarrow$) If $\operatorname{dgcd}(f_1, \ldots, f_m)$ is divisible by an irreducible polynomial $g \in A$, then $\operatorname{jac}^{f_1,\ldots,f_m}_{j_1,\ldots,j_m} \in (g)$ for every $j_1, \ldots, j_m \in \{1, \ldots, n\}$. Hence, by Theorem 4.7, $bf_i + c = gh$ for some $i \in \{1, \ldots, m\}$, $b, c \in R_i$ such that $g \nmid b$, and $h \in A$.

The condition (i) holds if $g \notin B$ and $g \mid h$, and the condition (ii) holds if $g \in B$, so we assume that $g \notin B$ and $g \nmid h$. Applying, for arbitrary $j_1, \ldots, j_m \in \{1, \ldots, n\}$, the Jacobian derivation d_i defined by

$$d_i(f) = \operatorname{jac}^{f_1,\ldots,f_{i-1},f,f_{i+1},\ldots,f_m}_{j_1,\ldots,j_m},$$

we infer that $g \mid \operatorname{jac}^{f_1,\ldots,f_{i-1},g,f_{i+1},\ldots,f_m}_{j_1,\ldots,j_m}$. Then the condition $(*)$ from the proof of Theorem 4.7 holds for polynomials $f_1, \ldots, f_{i-1}, g, f_{i+1}, \ldots, f_m$, where (one can show that) $g \nmid s_j$ for some $j \neq i$, so since $\overline{g} = \overline{0}$, we obtain that $\overline{f_j} \in (\overline{R_{ij}})_0$. Recall that $\overline{f_i} \in (\overline{R_i})_0$, but $R_i = R_{ij}[f_j]$, so $\overline{f_i} \in (\overline{R_{ij}})_0$, and then (iii) holds.

($\Leftarrow$) If $bf_i + c = g^2 h$ for some irreducible polynomial $g \in A \setminus B$, some $h \in A$ and $b, c \in R_i$ such that $g \nmid b$, then we apply the derivation d_i defined above, and obtain that $g \mid \operatorname{jac}^{f_1,\ldots,f_m}_{j_1,\ldots,j_m}$ for arbitrary $j_1, \ldots, j_m \in \{1, \ldots, n\}$, so $g \mid \operatorname{dgcd}(f_1, \ldots, f_m)$. We proceed similarly, if (ii) holds.

If $g \mid b_1 f_i + c_1$ and $g \mid b_2 f_j + c_2$ for some irreducible polynomial g, $i \neq j$ and $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $g \nmid b_1$ and $g \nmid b_2$, then $g \mid \operatorname{dgcd}(b_1 f_i + c_1, b_2 f_j + c_2)$, so

$$g \mid \operatorname{dgcd}(f_1, \ldots, b_1 f_i + c_1, \ldots, b_2 f_j + c_2, \ldots, f_m)$$

by Lemma 1.5. Then we show that

$$\operatorname{dgcd}(f_1, \ldots, b_1 f_i + c_1, \ldots, b_2 f_j + c_2, \ldots, f_m)$$

$$= b_1 b_2 \operatorname{dgcd}(f_1, \dots, f_i, \dots, f_j, \dots, f_m)$$

and obtain the conclusion: $g \mid \operatorname{dgcd}(f_1, \dots, f_m)$. $\square$

Let us remark that the zero characteristic analog of Theorem 4.8 for $m = n$ ([25], Theorem 4.1) is connected with a characterization of Keller maps and an equivalent formulation of the Jacobian Conjecture.

5. A characterization of p-bases of rings of constants

A characterization of p-bases of the whole polynomial algebra $k[x_1, \dots, x_n]$ was obtained by Nousiainen in [39], see Niitsuma, [37] or [38].

Theorem 5.1. (Nousiainen)
Given polynomials $f_1, \dots, f_n \in k[x_1, \dots, x_n]$, where k is a field of characteristic $p > 0$, the following conditions are equivalent:

(1) *there exist k-derivations $d_1, \dots, d_n$ of $k[x_1, \dots, x_n]$ such that $d_i(f_j) = \delta_{ij}$ (the Kronecker delta) for $i, j = 1, \dots, n$,*

(2) *there exist k-derivations $d_1, \dots, d_n$ of $k[x_1, \dots, x_n]$ such that $\det(d_i(f_j)) \in k \setminus \{0\}$,*

(3) *the Jacobian matrix $\left[\dfrac{\partial f_i}{\partial x_j}\right]$ is invertible,*

(4) $k[x_1, \dots, x_n] = k[x_1^p, \dots, x_n^p, f_1, \dots, f_n]$,

(5) *the polynomials $f_1, \dots, f_n$ form a p-basis of $k[x_1, \dots, x_n]$ over $k[x_1^p, \dots, x_n^p]$.*

Note that Lang and Mandal obtained in [32], Theorem 2.2, some other equivalent conditions in terms of Jacobian derivations.

Nousiainen's theorem is connected with the positive characteristic version of the Jacobian Conjecture formulated by Adjamagbo ([1], see [7], 10.3.16, p. 261).

Conjecture 5.2. *Let $f_1, \dots, f_n \in \mathbb{F}_p[x_1, \dots, x_n]$. If $\operatorname{jac}(f_1, \dots, f_n) \in \mathbb{F}_p \setminus \{0\}$ and p does not divide the degree of the field extension $\mathbb{F}_p(f_1, \dots, f_n) \subset \mathbb{F}_p(x_1, \dots, x_n)$, then $\mathbb{F}_p[f_1, \dots, f_n] = \mathbb{F}_p[x_1, \dots, x_n]$.*

Theorem 5.3. (Adjamagbo, [1], see [7], 10.3.17, p. 261)
If the above conjecture is true for all $n \geqslant 1$ and all primes p, then the Jacobian Conjecture is true.

Now we present a general theorem about p-bases of rings of constants of polynomial derivations. In the case $m = n$ it extends the Nousiainen's theorem with the condition (3) below.

Theorem 5.4. ([26], Theorem 4.4)
Let K be a UFD of characteristic $p > 0$, let $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$, where $m \in \{1, \ldots, n\}$. Denote: $B = K[x_1^p, \ldots, x_n^p]$, $R_i = B[f_1, \ldots, \widehat{f_i}, \ldots, f_m]$ for $i = 1, \ldots, m$, and $R_{ij} = B[f_1, \ldots, \widehat{f_i}, \ldots, \widehat{f_j}, \ldots, f_m]$ for $i, j = 1, \ldots, m$, such that $i \neq j$.

The following conditions are equivalent:

(1) $\operatorname{dgcd}(f_1, \ldots, f_m) \sim 1$,

(2) *the polynomials $f_1, \ldots, f_m$ form a p-basis of the ring of constants of some K-derivation,*

(3) *the polynomial $bf_i + c$ is square-free and B-free for every $i \in \{1, \ldots, m\}$ and $b, c \in R_i$ such that* $\gcd(b, c) \sim 1$, *and, if $m > 1$, then* $\gcd(b_1 f_i + c_1, b_2 f_j + c_2) \sim 1$ *for every $i, j \in \{1, \ldots, m\}$, $i \neq j$, and $b_1, b_2, c_1, c_2 \in R_{ij}$ such that* $\gcd(b_1, c_1) \sim 1$ *and* $\gcd(b_2, c_2) \sim 1$.

Proof. (Sketch.)
(1) $\Rightarrow$ (2) Assume that $\operatorname{dgcd}(f_1, \ldots, f_m) \sim 1$. By Lemma 1.6, $f_1, \ldots, f_m$ are p-independent over B. We will show that for every $b \in B \setminus \{0\}$ and $a_\alpha \in B$, $0 \leqslant \alpha_1, \ldots, \alpha_m < p$, the following holds:

$(*)$ if $b \mid \sum_{0 \leqslant \alpha_1, \ldots, \alpha_m < p} a_\alpha f_1^{\alpha_1} \ldots f_m^{\alpha_m}$, then $b \mid a_\alpha$ for every $\alpha_1, \ldots, \alpha_m \in \{0, \ldots, p-1\}$.

Denote by s the maximal sum $\alpha_1 + \ldots + \alpha_m$ such that $a_\alpha \neq 0$. If $s = 0$, $(*)$ holds. Assume that $s > 0$ and $(*)$ holds for $s - 1$. Let $b \mid \sum_{0 \leqslant \alpha_1, \ldots, \alpha_m < p} a_\alpha f_1^{\alpha_1} \ldots f_m^{\alpha_m}$. Applying, for each i, the Jacobian derivation d_i defined by

$$d_i(f) = \operatorname{jac}^{f_1, \ldots, f_{i-1}, f, f_{i+1}, \ldots, f_m}_{j_1, \ldots, j_m},$$

we obtain that $b \mid \sum_{0 \leqslant \alpha_1, \ldots, \alpha_m < p} \alpha_i a_\alpha f_1^{\alpha_1} \ldots f_i^{\alpha_i - 1} \ldots f_m^{\alpha_m} \operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m}$. Then

$$b \mid \sum_{0 \leqslant \alpha_1, \ldots, \alpha_m < p} \alpha_i a_\alpha f_1^{\alpha_1} \ldots f_i^{\alpha_i - 1} \ldots f_m^{\alpha_m},$$

because $\gcd\left(\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m},\ j_1, \ldots, j_m \in \{1, \ldots, n\}\right) \sim 1$, and it is enough to use the induction hypotheses.

Now, observe that any element of the ring

$$C_B(f_1, \ldots, f_m) = B_0[f_1, \ldots, f_m] \cap A$$

is the form $\sum_{0 \leqslant \alpha_1, \ldots, \alpha_m < p} \frac{a_\alpha}{b} f_1^{\alpha_1} \ldots f_m^{\alpha_m}$, where $b \in B \setminus \{0\}$, $a_\alpha \in B$, so, by $(*)$, it belongs to $B[f_1, \ldots, f_m]$.

(2) $\Rightarrow$ (3) Assume that $f_1, \ldots, f_m$ form a p-basis of the ring $R = C_B(f_1, \ldots, f_m)$.

If $g^2 \mid bf_i + c$ for some $i \in \{1, \ldots, m\}$, $b, c \in R_i$ such that $\gcd(b, c) \sim 1$, and a noninvertible polynomial g, then one can show that the polynomial $\frac{1}{g^p} \cdot (bf + c)^{p-1}$ belongs to R and does not belong to $B[f_1, \ldots, f_m]$.

If $g \mid bf_i + c$ for some $i \in \{1, \ldots, m\}$, $b, c \in R_i$ such that $\gcd(b, c) \sim 1$, and a noninvertible polynomial $g \in B$, then $\frac{bf+c}{g} \in R \setminus B[f_1, \ldots, f_m]$.

If $g \mid b_1 f_i + c_1$ and $g \mid b_2 f_j + c_2$ for some $i, j \in \{1, \ldots, m\}$, $i \neq j$, $b_1, b_2, c_1, c_2 \in R_{ij}$ such that $\gcd(b_1, c_1) \sim 1$, $\gcd(b_2, c_2) \sim 1$ and a noninvertible polynomial g, then $\frac{1}{g^p} \cdot (b_1 f_i + c_1)^{p-1}(b_2 f_j + c_2) \in R \setminus B[f_1, \ldots, f_m]$.

$\neg(1) \Rightarrow \neg(3)$ If $g \mid \operatorname{dgcd}(f_1, \ldots, f_m)$ for irreducible polynomial g, then at least one of the conditions (i), (ii), (iii) of Theorem 4.8 holds. Now, if $bf_i + c$ is divisible by g or by g^2, it is enough to take h – a product of g and all (if any) irreducible factors of b, which do not divide c, and then $bf_i + c + h^p$ remains being divisible by g, resp. by g^2, but $\gcd(b, c + h^p) \sim 1$. $\square$

6. Closed polynomials and one-element p-bases

The properties of single generators of rings of constants were studied by many authors.

Theorem 6.1. (Nowicki, Nagata, Ayad, Arzhantsev, Petravchuk)
Let k be a field, let $f \in k[x_1, \ldots, x_n] \setminus k$. Denote by $\overline{k}$ the algebraic closure of k. Consider the following conditions:

(1) *$k[f]$ is the ring of constants of some k-derivation of $k[x_1, \ldots, x_n]$,*

(2) *$k[f]$ is integrally closed in $k[x_1, \ldots, x_n]$,*

(3) *$k[f]$ is a maximal element (with respect to inclusion) of the family $\{k[g];\ g \in k[x_1, \ldots, x_n]\}$,*

(4) *for some $c \in \overline{k}$ the polynomial $f + c$ is irreducible over $\overline{k}$,*

(5) *for all but finitely many $c \in \overline{k}$ the polynomial $f + c$ is irreducible over $\overline{k}$.*

a) *If* $\operatorname{char} k = 0$, *then the conditions* (1) – (5) *are equivalent.*

b) *If k is a perfect field, then the conditions* (2) – (5) *are equivalent.*

c) *For arbitrary field the conditions* (2) *and* (3) *are equivalent.*

Nowicki and Nagata proved the equivalence of the conditions (1), (2) and (3) in characteristic zero ([40], Theorem 2.1; [41], Proposition 5.2.1; [43], Lemma 3.1). Ayad added the condition (4) in $\operatorname{char} k = 0$ ([3], Théorème 8, Remarque), based on the theorem of Płoski ([47], see [48], 3.3, Corollary 1, p. 220), and observed that the equivalence (2) $\Leftrightarrow$ (3) holds also for $\operatorname{char} k = p > 0$. Arhzantsev and Petravchuk ([2], Theorem 1) considered the case of a perfect field and added the condition (5).

Note also that Nowicki and Nagata in [40] and [43] defined a closed polynomial in characteristic zero as a polynomial f satisfying the condition (3) above.

Now, let k be a field of characteristic $p > 0$.

Consider the following families of subrings of $k[x_1, \dots, x_n]$:

$$\begin{aligned}
\mathcal{A} &= \{k[g];\ g \in k[x_1, \dots, x_n]\},\\
\mathcal{B} &= \{k[x_1^p, \dots, x_n^p, g];\ g \in k[x_1, \dots, x_n]\},\\
\mathcal{C} &= \{R \subset k[x_1, \dots, x_n] :\ k[x_1^p, \dots, x_n^p] \subset R,\ (R_0 : k(x_1^p, \dots, x_n^p)) = p\},
\end{aligned}$$

where $(L : K)$ denotes the degree of a field extension $K \subset L$.

The family $\mathcal{A}$ plays its role in characteristic zero, the family $\mathcal{B}$ is a natural positive characteristic analog, since rings of constants are $k[x_1^p, \dots, x_n^p]$-algebras. The family $\mathcal{C}$, however, has the property that its maximal elements are rings of constants (see Theorem 2.5).

Note that we do not have any implication, in general, between the maximality of respective rings in $\mathcal{A}$ and in $\mathcal{B}$ ([23], Examples 2.1, 2.2), and even the maximality in $\mathcal{C}$ does not imply, in general, the maximality in $\mathcal{A}$. Moreover, the maximality in $\mathcal{B}$ does not imply, in general, the maximality in $\mathcal{C}$ ([23], Example 2.3). The only implication is that if an element of $\mathcal{B}$ is maximal in $\mathcal{C}$, then it is also maximal in $\mathcal{B}$.

Example 6.2. a) *Put $f_1 = x_1^p x_2$. Then the ring $k[f_1]$ is maximal in $\mathcal{A}$, and the ring $k[x_1^p, \dots, x_n^p, f_1]$ is not maximal in $\mathcal{B}$.*

b) *Put $f_2 = x_1 + x_1^p$. Then the ring $k[x_1^p, \dots, x_n^p, f_2]$ is maximal in $\mathcal{B}$ and in $\mathcal{C}$, and the ring $k[f_2]$ is not maximal in $\mathcal{A}$.*

c) *Put $f_3 = x_1^{p-1} x_2$. Then the ring $k[x_1^p, \dots, x_n^p, f_3]$ is maximal in $\mathcal{B}$, and is not maximal in $\mathcal{C}$.*

Now we are going to analyze a characterization of single generators of rings of constants. In order to understand better the condition (3) in Theorem 6.4 below, observe the following positive characteristic analog of a known property of polynomials. Recall that k denotes a field of characteristic $p > 0$.

Lemma 6.3. *Consider a polynomial $f \in k[x_1, \dots, x_n] \setminus k[x_1^p, \dots, x_n^p]$. Then*

$$\gcd\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \sim 1$$

if and only if f is square-free and p-free.

From Theorem 5.4 in the case of $m = 1$ we have the following.

Theorem 6.4. ([21], Theorem 4.2)
Let $f \in k[x_1, \dots, x_n] \setminus k[x_1^p, \dots, x_n^p]$. The following conditions are equivalent:

(1) $\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \sim 1$,

(2) *$k[x_1^p, \dots, x_n^p, f]$ is the ring of constants of a k-derivation,*

(3) *for every $b, c \in k[x_1^p, \dots, x_n^p]$ such that $\gcd(b, c) \sim 1$, the polynomial $bf + c$ is square-free and p-free.*

It is easy to see that

$$\gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \mid d(f)$$

for every k-derivation d of $k[x_1, \dots, x_n]$ and a polynomial $f \in k[x_1, \dots, x_n] \setminus k[x_1^p, \dots, x_n^p]$. If $d(f) = cf$ for some $c \in k \setminus \{0\}$, then

$$\gcd\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \sim \gcd\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right).$$

Hence, we obtain the following fact.

Corollary 6.5. *Let $f \in k[x_1, \dots, x_n] \setminus k[x_1^p, \dots, x_n^p]$. Assume that $d(f) = cf$ for some $c \in k \setminus \{0\}$. Then $k[x_1^p, \dots, x_n^p, f]$ is a ring of constants of a k-derivation if and only if the polynomial f is square-free and p-free.*

Finally, observe a list of monomial derivations in two variables with one-element p-bases of rings of constants. The motivation was connected with the paper of Okuda ([45]), who adapted van den Essen's algorithm ([6], see [7], 1.4, p. 37) to positive characteristic. Recall that k denotes a field of characteristic $p > 0$.

Example 6.6. ([18], Example 13)
Let m, n, r, s be nonnegative integers, $m, n \not\equiv -1 \pmod p$, and let $\alpha, \beta \in k \setminus \{0\}$. Consider the following examples:

$$\begin{cases} d_1(x) = \alpha x^{rp} \\ d_1(y) = \beta y^{sp}, \end{cases} \qquad k[x,y]^{d_1} = k[x^p, y^p, \beta x y^{sp} - \alpha x^{rp} y],$$

$$\begin{cases} d_2(x) = \alpha x \\ d_2(y) = -\alpha y, \end{cases} \qquad k[x,y]^{d_2} = k[x^p, y^p, xy],$$

$$\begin{cases} d_3(x) = \alpha y^n \\ d_3(y) = \beta x^m, \end{cases} \qquad k[x,y]^{d_3} = k[x^p, y^p, (n+1)\beta x^{m+1} - (m+1)\alpha y^{n+1}],$$

$$\begin{cases} d_4(x) = \alpha x^{rp} y^n \\ d_4(y) = \beta, \end{cases} \qquad k[x,y]^{d_4} = k[x^p, y^p, (n+1)\beta x - \alpha x^{rp} y^{n+1}],$$

$$\begin{cases} d_5(x) = 0 \\ d_5(y) = \beta, \end{cases} \qquad k[x,y]^{d_5} = k[x^p, y^p, x],$$

$$\begin{cases} d_6(x) = \alpha \\ d_6(y) = \beta x^m y^{sp}, \end{cases} \quad k[x,y]^{d_6} = k[x^p, y^p, \beta x^{m+1} y^{sp} - (m+1)\alpha y],$$

$$\begin{cases} d_7(x) = \alpha \\ d_7(y) = 0, \end{cases} \qquad k[x,y]^{d_7} = k[x^p, y^p, y].$$

Theorem 6.7. ([18], Theorem 16)
Let d be a monomial k-derivation of $k[x,y]$:

$$\begin{cases} d(x) = \alpha x^t y^u \\ d(y) = \beta x^v y^w, \end{cases}$$

where $\alpha, \beta \in k$. Then

$$k[x,y]^d = k[x^p, y^p, f]$$

for some $f \in k[x,y] \setminus k[x^p, y^p]$ if and only if $d = x^j y^l \cdot d_i$, where $j, l \geqslant 0$, $i \in \{1, 2, \ldots, 7\}$, and the derivation d_i is as in Example 6.6.

7. Eigenvector p-bases

Recall the Moore's determinant (see, for example, [14], Corollary 1.3.7, p. 8).

Lemma 7.1. *Let k be a field of characteristic $p > 0$, let $c_1, \ldots, c_m \in k$, $m > 1$. Then*

$$\begin{vmatrix} c_1 & c_1^p & \cdots & c_1^{p^{m-1}} \\ c_2 & c_2^p & \cdots & c_2^{p^{m-1}} \\ \vdots & \vdots & & \vdots \\ c_m & c_m^p & \cdots & c_m^{p^{m-1}} \end{vmatrix} = \prod_{i=1}^{m} \prod_{\alpha_1, \ldots, \alpha_{i-1} \in \mathbb{F}_p} (\alpha_1 c_1 + \ldots + \alpha_{i-1} c_{i-1} + c_i).$$

Recall also a notation

$$\operatorname{dgcd}(f_1, \ldots, f_m) = \gcd\left(\operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m},\ j_1, \ldots, j_m \in \{1, \ldots, n\}\right).$$

The following theorem, taking into consideration Theorem 5.4, is motivated by Corollary 6.5.

Theorem 7.2. ([24], Theorem 3.2)
Let k be a field of characteristic $p > 0$, consider polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n] \setminus \{0\}$, where $m > 1$. Assume that $f_1, \ldots, f_m$ are eigenvectors of some k-derivation of $k[x_1, \ldots, x_n]$ and their eigenvalues are linearly independent over the prime subfield $\mathbb{F}_p$. Then $f_1, \ldots, f_m$ are p-independent over $k[x_1^p, \ldots, x_n^p]$, and the following conditions are equivalent:

(1) *$k[x_1^p, \ldots, x_n^p, f_1, \ldots, f_m]$ is the ring of constants of some k-derivation,*

(2) $f_1, \ldots, f_m$ *are pairwise coprime, square-free and p-free,*

(3) $\operatorname{dgcd}(f_1, \ldots, f_m) \sim 1$,

(4) $\operatorname{dgcd}(f_{i_1}, f_{i_2}) \sim 1$ *for every* $i_1 \neq i_2$.

Proof. (Sketch.)
Let Δ be a k-derivation such that $\Delta(f_i) = c_i f_i$, where $c_i \in k$ for $i = 1, \ldots, m$, and $c_1, \ldots, c_m$ are linearly independent over $\mathbb{F}_p$. Consider k-derivations $d_j = \Delta^{p^{j-1}}$, $j = 1, \ldots, m$.

Consider the matrix

$$M = \begin{bmatrix} d_1(f_1) & d_2(f_1) & \cdots & d_m(f_1) \\ d_1(f_2) & d_2(f_2) & \cdots & d_m(f_2) \\ \vdots & \vdots & & \vdots \\ d_1(f_m) & d_2(f_m) & \cdots & d_m(f_m) \end{bmatrix}.$$

We have $d_j(f_i) = c_i^{p^{j-1}} f_i$ for $i, j \in \{1, \ldots, m\}$, so $\det M = c f_1 \ldots f_m$, where c is the value of the Moore's determinant from Lemma 7.1, $c \in k$. Since $c_1, \ldots, c_m$ are linearly independent over $\mathbb{F}_p$, we have $c \neq 0$ and $\det M \neq 0$.

On the other hand, one can show that

$$\det M = \sum_{j_1, \ldots, j_m \in \{1, \ldots, n\}} d_1(x_{j_1}) \ldots d_m(x_{j_m}) \operatorname{jac}^{f_1, \ldots, f_m}_{j_1, \ldots, j_m},$$

so $f_1, \ldots, f_m$ are p-independent over $k[x_1^p, \ldots, x_n^p]$ by Lemma 1.6. Moreover, we obtain that

$$\operatorname{dgcd}(f_1, \ldots, f_m) \mid f_1 \ldots f_m.$$

$\neg(3) \Rightarrow \neg(2)$ Assume that $\operatorname{dgcd}(f_1, \ldots, f_m)$ is divisible by an irreducible polynomial $g \in k[x_1, \ldots, x_n]$. Then $g \mid f_i$ for some i.

Now we change in the matrix M the derivation d_m to $d'_m = \frac{\partial}{\partial x_l}$, where $l \in \{1, \ldots, n\}$, and expand its determinant with respect to the last column. Again, using Lemma 7.1, we obtain the divisibility

$$\operatorname{dgcd}(f_1, \ldots, f_m) \mid \sum_{j=1}^{m} (-1)^{m+j} c_j f_1 \ldots f_{j-1} \frac{\partial f_j}{\partial x_l} f_{j+1} \ldots f_m,$$

where $c_j \in k \setminus \{0\}$. Hence, $g \mid f_1 \ldots f_{i-1} \frac{\partial f_i}{\partial x_l} f_{i+1} \ldots f_m$, so $g \mid f_j$ for some $j \neq i$ or $g \mid \frac{\partial f_i}{\partial x_l}$ for $l = 1, \ldots, n$, and then, by Lemma 4.5, $g^2 \mid f_i$ or $g \in k[x_1^p, \ldots, x_n^p]$.

$(4) \Rightarrow (2)$ For every $i_1 \neq i_2$, if $\operatorname{dgcd}(f_{i_1}, f_{i_2}) \sim 1$, then f_{i_1} and f_{i_2} are coprime, square-free and p-free by the implication $(1) \Rightarrow (3)$ of Theorem 5.4 (for $m = 2$).

The implications (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) follow directly from Theorem 5.4. The implication (3) $\Rightarrow$ (4) follows from Lemma 1.5. $\square$

8. Rings of constants of homogeneous derivations

The motivation to describe rings of constants of homogeneous derivations being polynomial algebras, comes from the following theorem.

Theorem 8.1. (Ganong, Daigle)
Let k be a field of characteristic $p > 0$, let A and R be polynomial k-algebras in two variables such that $A^p \subsetneqq R \subsetneqq A$. Then there exist $x, y \in A$ such that $A = k[x, y]$ and $R = k[x, y^p]$.

The above theorem was proved by Ganong in [11], in the case of algebraically closed field k and then by Daigle in [4] in the general case. Note also that Kimura and Niitsuma in [29] proved that, in the case of a perfect field k of characteristic $p > 0$, under these assumptions, A has a p-basis over R and R has a p-basis over A^p.

Nowicki and the author generalized the above theorem to n variables in the homogeneous case.

Theorem 8.2. ([28], Theorem 3.1, [27], Theorem 2.2)
Let p be a prime number. Let k be a field (of arbitrary characteristic) and let $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ be homogeneous polynomials such that

$$k[x_1^p, \ldots, x_n^p] \subset k[f_1, \ldots, f_n].$$

a) *If* $\operatorname{char} k \neq p$, *then*

$$k[f_1, \ldots, f_n] = k[x_1^{l_1}, \ldots, x_n^{l_n}]$$

for some $l_1, \ldots, l_n \in \{1, p\}$.

b) *If* $\operatorname{char} k = p$, *then*

$$k[f_1, \ldots, f_n] = k[y_1, \ldots, y_m, y_{m+1}^p, \ldots, y_n^p]$$

for some $m \in \{0, 1, \ldots, n\}$ and some k-linear basis $y_1, \ldots, y_n$ of $\langle x_1, \ldots, x_n \rangle$.

For proofs, we refer to two articles joint with Nowicki. The article [27] contains the proof of the above theorem. The article [28] contains a theorem about (polynomial graded) subalgebras containing $k[x_1^{p_1}, \ldots, x_n^{p_n}]$, where $p_1, \ldots, p_n$ are arbitrary prime numbers ([28], Theorem 2.1).

A k-derivation d of $k[x_1, \ldots, x_n]$ is called homogeneous of degree r if $d(x_i)$, if nonzero, is a homogeneous polynomial of degree $r + 1$ for $i = 1, \ldots, n$. In this case, for every homogeneous polynomial $f \in k[x_1, \ldots, x_n]$ of degree s, the polynomial $d(f)$, if nonzero, is homogeneous of degree $r+s$. The ring of constants of a homogeneous derivation is a graded subalgebra. As a consequence of Theorem 8.2 we obtain.

Theorem 8.3. ([28], Theorem 4.1)
Let d be a homogeneous k-derivation of $k[x_1,\dots,x_n]$, where k is a field of characteristic $p>0$. Then $k[x_1,\dots,x_n]^d$ is a polynomial k-algebra if and only if

$$(*)\qquad k[x_1,\dots,x_n]^d = k[y_1,\dots,y_m,y_{m+1}^p,\dots,y_n^p]$$

for some $m\in\{0,1,\dots,n\}$ and some k-linear basis $y_1,\dots,y_n$ of $\langle x_1,\dots,x_n\rangle$.

A homogeneous k-derivation of $k[x_1,\dots,x_n]$ of degree 0 is called linear. In this case a restriction of d to $\langle x_1,\dots,x_n\rangle$ is a k-linear endomorphism. The author obtained in [20], Theorem 3.2, a description of linear derivations with rings of constants of the form $(*)$ above. Finally, we have the following.

Theorem 8.4. ([28], Corollary 4.2)
Let d be a linear derivation of $k[x_1,\dots,x_n]$, where k is a field of characteristic $p>0$. Then $k[x_1,\dots,x_n]^d$ is a polynomial k-algebra if and only if the Jordan matrix of the endomorphism $d|_{\langle x_1,\dots,x_n\rangle}$ has one of the following forms:

$$\begin{bmatrix}\rho_1 & & 0\\ & \ddots & \\ 0 & & \rho_n\end{bmatrix},\quad \begin{bmatrix}\begin{bmatrix}\rho_1 & 1\\ 0 & \rho_1\end{bmatrix} & & & 0\\ & \rho_2 & & \\ & & \ddots & \\ 0 & & & \rho_{n-1}\end{bmatrix},\quad \underbrace{\begin{bmatrix}\begin{bmatrix}\rho_1 & 1 & 0\\ 0 & \rho_1 & 1\\ 0 & 0 & \rho_1\end{bmatrix} & & & 0\\ & \rho_2 & & \\ & & \ddots & \\ 0 & & & \rho_{n-2}\end{bmatrix}}_{\textit{only } p=2},$$

where nonzero ρ_i are linearly independent over the prime subfield $\mathbb{F}_p$.

Acknowledgements. The author would like to thank Professor Andrzej Nowicki for many helpful remarks that improved this article.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 81 – 94

ON COMBINATORIAL CRITERIA FOR ISOLATED SINGULARITIES

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Abstract. In this article we review combinatorial characterizations of isolated singularities. As a new result in two and three-dimensional case we give sufficient and necessary conditions for a nondegenerate singularity to be isolated in terms of its support. We also prove new sufficient conditions in the multidimensional case.

1. Introduction

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function. One of the problems in the theory of singularities is to check effectively that f is an isolated singularity. Many authors give different conditions to deal with this problem. For instance by the local Nullstellensatz f is an isolated singularity if and only if the Milnor number $\mu(f)$ is finite. Similarly the Łojasiewicz exponent $\mathcal{L}_0(f)$ is finite if and only if f is an isolated singularity. In this paper we review combinatorial conditions related to the support of an isolated singularity and give some new results in the nondegenerate class (for definitions see Preliminaries).

Kouchnirenko in [Ko77] gave for a set $M \subset \mathbb{N}^n$ a necessary and sufficient conditions that there exists an isolated singularity f with $\operatorname{supp} f \subset M$ (see Thm. 3.9). Other authors: Wall ([Wa96]), Orlik and Randell ([OR76]), Shcherbak ([Sh79]) obtained similar results. In Remark 3.11 we comment on the history of these results.

The quasihomogeneous case was considered by the authors named above as well as by Saito ([Sa71], [Sa87]), Krezuer and Skarke ([KS92]), Hertling and Kurbel ([HK12]). In this class of singularities we recall the necessary condition for the

2010 *Mathematics Subject Classification.* Primary 32S05, Secondary 14B05.
Key words and phrases. Isolated singularity, nondegenerate singularity, Kuchnirenko condition.

weights so that the singularity is isolated, which turns out sufficient in the two and three-dimensional case (see Thm. 4.2).

In section 5 we examine the problem in the class of nondegenerate singularities and give some new results. For dimension $n \leq 3$ we prove necessary and sufficient conditions for the support of a nondegenerate singularity so that the singularity is isolated (see Thm 5.4). It seems that for $n \geq 4$ Theorem 5.4 is also true (see Conj. 5.5). For higher dimensions we give only sufficient conditions (see Thm. 5.6). Wall considered another type of nondegeneracy than the Kouchnirenko nondegeneracy. He got similar results to the ones obtained in Section 5 (see Lem. 1.2 and Thm. 1.4 in [Wa98]).

In the last section using Remark 1.13 (ii) in [Ko76] we reformulate the results of the previous section in terms of the Newton number (see Cor. 6.2, Prop. 6.3, 6.4).

2. Preliminaries

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a nonzero holomorphic function in an open neighborhood of $0 \in \mathbb{C}^n$. We say that f is *a singularity* if $f(0) = 0$, $\nabla f(0) = 0$, where $\nabla f = (f'_{z_1}, \ldots, f'_{z_n})$. We say that f is *an isolated singularity* if f is a singularity, which has an isolated critical point in the origin i.e. additionally $\nabla f(z) \neq 0$ for $z \neq 0$ near 0. We note $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. We define the set $\operatorname{supp} f = \{\nu \in \mathbb{N}^n : a_\nu \neq 0\}$ and call it *the support of* f. Let $w_1, \ldots, w_n, d$ be positive integer numbers. The polynomial $f \in C[z_1, \ldots, z_n]$ is called *quasihomogeneous with weight system* $(w_1, \ldots, w_n, d)$ if

$$\sum_{i=1}^{n} \nu_i w_i = d \quad \text{for any } \nu \in \operatorname{supp} f.$$

We define

$$\Gamma_+(f) = \operatorname{conv}\{\nu + \mathbb{R}^n_+ : \nu \in \operatorname{supp} f\} \subset \mathbb{R}^n$$

and call it *the Newton diagram of* f. Let $u \in \mathbb{R}^n_+ \setminus \{0\}$. Put

$$l(u, \Gamma_+(f)) = \inf\{\langle u, v\rangle : v \in \Gamma_+(f)\},$$
$$\Delta(u, \Gamma_+(f)) = \{v \in \Gamma_+(f) : \langle u, v\rangle = l(u, \Gamma_+(f))\}.$$

We say that $S \subset \mathbb{R}^n$ is *a face of* $\Gamma_+(f)$ if $S = \Delta(u, \Gamma_+(f))$ for some $u \in \mathbb{R}^n_+ \setminus \{0\}$. The vector u is called *the primitive vector of* S. It is easy to see that S is a closed and convex set and $S \subset \operatorname{Fr}(\Gamma_+(f))$, where $\operatorname{Fr}(A)$ denotes the boundary of A. One can prove that a face $S \subset \Gamma_+(f)$ is compact if and only if all coordinates of its primitive vector u are positive. We call the family of all compact faces of $\Gamma_+(f)$ *the Newton boundary of* f and denote by $\Gamma(f)$. We denote by $\Gamma^k(f)$ the set of all compact k-dimensional faces of $\Gamma(f)$, $k = 0, \ldots, n-1$. For every compact face $S \in \Gamma(f)$ we define quasihomogeneous polynomial $f_S = \sum_{\nu \in S} a_\nu z^\nu$. We say that f is *nondegenerate on the face* $S \in \Gamma(f)$ if the system of equations

$$\frac{\partial f_S}{\partial z_1} = \ldots = \frac{\partial f_S}{\partial z_n} = 0$$

has no solution in $(\mathbb{C}^*)^n$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We say that f is *nondegenerate in the sense of Kouchnirenko* (shortly *nondegenerate*) if it is nondegenerate on each face of $\Gamma(f)$. We say that f is *convenient* if $\Gamma_+(f)$ has nonempty intersection with every coordinate axis. We say that f is *nearly convenient* if the distance of $\Gamma_+(f)$ to every coordinate axis does not exceed 1. Denote by $\mathcal{O}^n$ the local ring of germs of holomorphic functions in n-variables at $0 \in \mathbb{C}^n$. Let us recall that *the Milnor Number* $\mu(f)$ and *the Newton number* $\nu(f)$ are defined as

$$\mu(f) = \dim \mathcal{O}^n/(f'_{z_1}, \ldots, f'_{z_n}), \quad \nu(f) = n!V_n - (n-1)!V_{n-1} + \ldots + (-1)^n V_0,$$

where V_i denotes the sum of i-dimensional volumes of the intersection of the cone spanned by $\Gamma_+(f)$ with the coordinate subspace of dimension i.

3. Generic case

In this section we recall some known results dealing with support of isolated singularities. Kouchnirenko in [Ko77, Thm 1] gave for a set $M \subset \mathbb{N}^n$ necessary and sufficient conditions so that there exists an isolated singularity f with $\operatorname{supp} f \subset M$. Moreover, every singularity f with $\operatorname{supp} f \subset M$ and generic coefficients is isolated. Before giving his result we start with some notions and definitions.

Let $M \subset \mathbb{N}^n$. Define the sets $M_i = \{\nu \in \mathbb{N}^n : \nu + e_i \in M\}$, where $e_i, i = 1, \ldots, n$, is the standard basis in $\mathbb{R}^n$. Notice that if we take $f_M = \sum_{m \in M} z^m$ then $M_i = \operatorname{supp} \partial f_M/\partial z_i$ for every $i = 1, 2, \ldots, n$. Let $I \subset \{1, \ldots, n\}$. Set

$$OX_I = \{x \in \mathbb{R}^n \colon x_i = 0, i \notin I\}.$$

Observe that OX_I is the hyperplane spanned by axes $OX_i, i \in I$.

Let $I \subset \{1, 2, \ldots, n\}$. We say that *$M$ satisfies the Kouchnirenko condition for I* if there exist at least $|I|$ nonempty sets among the sets $M_1 \cap OX_I, \ldots, M_n \cap OX_I$. We say that *$M$ satisfies the Kouchnirenko condition* if M satisfies the Kouchnirenko condition for every $I \subset \{1, 2, \ldots, n\}$.

Remark 3.1. It is easy to check that M satisfies the Kouchnirenko condition if and only if a finite subset of M satisfies the Kouchnirenko condition.

Remark 3.2. If M satisfies the Kouchnirenko condition, it can happen that the singularity f_M is not an isolated singularity. For example let $f_M = (z_1+z_2)(z_3+z_1)$. It is easy to check that f is not isolated singularity and is degenerate on the face S determined by $f_S = z_3(z_1 + z_2)$.

Example 3.3. a) Let $f(z_1, z_2) = z_1^2 + z_1 z_2$. We show that $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Put $M = \operatorname{supp} f$. Then $M_1 = \{(0,1), (1,0)\}$, $M_2 = \{(1,0)\}$. If $I = \{1,2\}$ or $I = \emptyset$ we easily check that M satisfies the Kouchnirenko condition. If $I = \{1\}$, then $M_2 \cap OX_2 \neq \emptyset$. If $I = \{2\}$, then $M_1 \cap OX_1 \neq \emptyset$.

b) Let $f(z_1, z_2, z_3) = z_1(z_1 + z_2 + z_3)$. We show that $\operatorname{supp} f$ does not satisfy the Kouchnirenko condition. Indeed, take $I = \{2, 3\}$ then $|I| = 2$ but only $M_1 \cap OX_I \neq \emptyset$.

Now we explain the Kouchnirenko condition for I in the border cases $|I| = 1$ and $|I| = n$.

Property 3.4. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a singularity. We have the following properties:*

(i) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition for every $I = \{i\}, i = 1, 2, \dots, n$ if and only if f is nearly convenient,*
(ii) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition for $I = \{1, 2, \dots, n\}$ if and only if $f'_{z_i} \neq 0$, $i = 1, 2, \dots, n$.*

Proof.

(i) Put $M = \operatorname{supp} f$. Suppose that M satisfies the Kouchnirenko condition for every $I = \{i\}, i = 1, 2, \dots, n$. It is equivalent to saying that for every $i = 1, 2, \dots, n$, there exists j_i such that $M_{j_i} \cap OX_i \neq \emptyset$. This condition is equivalent to the condition that there exists a vertex of $\Gamma_+(f)$ lying on the plane $OX_{j_i}X_i$ at most at distance 1 to OX_i.

(ii) It is a direct consequence of the definition of the Kouchnirenko condition. ■

The following property shows that the Kouchnirenko condition for $\operatorname{supp} f$ implies that the Newton diagram of a singularity f has non-empty intersection with every coordinate hyperplane in $\mathbb{R}^n$, $n \geq 3$.

Property 3.5. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0), n \geq 3$, be a singularity. If $\operatorname{supp} f$ satisfies the Kouchnirenko condition then $\Gamma_+(f) \cap OX_I \neq \emptyset$ for every set $I \subset \{1, 2, \dots, n\}, |I| = n - 1$.*

Proof. Put $M = \operatorname{supp} f$. Suppose that M satisfies the Kouchnirenko condition. Without loss of generality it suffices to show $\Gamma_+(f) \cap OX_I \neq \emptyset$ for $I = \{2, 3, \dots, n\}$. Indeed, by the Kouchnirenko condition there exist at least $n - 1$ nonempty sets among the sets $M_1 \cap OX_I, \dots, M_n \cap OX_I$. Since $n \geq 3$ there exists $i \neq 1$ such that $M_i \cap OX_I \neq \emptyset$. Let $A \in M_i \cap OX_I$ for some $i \neq 1$. Since $i \neq 1$ then $A - e_i \in M \cap OX_I$. Hence $\Gamma_+(f) \cap OX_I \neq \emptyset$. It ends the proof. ■

The two following propositions give conditions equivalent to the Kouchnirenko condition for $\operatorname{supp} f$ in terms of the Newton diagram of singularity f in two and three variables.

Proposition 3.6. *Let $f : \left(\mathbb{C}^2, 0\right) \longrightarrow (\mathbb{C}, 0)$ be a singularity. Then the following conditions are equivalent:*

(i) *f is nearly convenient,*
(ii) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from Property 3.4(i). Now let us suppose that the condition (i) is satisfied. Let $I \subset \{1, 2\}$. For $I = \emptyset$ or $I = \{1, 2\}$ then it is easy to see that $\operatorname{supp} f$ satisfies the Kouchnirenko condition. If $I = \{1\}$

or $I = \{2\}$ then by Property 3.4(i) we get that $\operatorname{supp} f$ satisfies the Kouchnirenko condition for such I. ■

Proposition 3.7. *Let $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ be a singularity. Then the following conditions are equaivalent:*

(i) *f is nearly convenient and $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$ for every $i, j \in \{1, 2, 3\}$, $i \neq j$,*
(ii) *$\operatorname{supp} f$ satisfies the Kouchnirenko condition.*

Proof. Put $M = \operatorname{supp} f$. The implication $(ii) \Rightarrow (i)$ follows from Properties 3.4(i) and 3.5. Now let us suppose that the condition (i) is satisfied and take $I \subset \{1, 2, 3\}$. If $I = \emptyset$ or $I = \{1, 2, 3\}$ then it is easy to check that M satisfies the Kouchnirenko condition for such I. If $I = \{i\}$ for some $i \in \{1, 2, 3\}$ then by Property 3.4(i) M satisfies the Kouchnirenko condition for such I. Now let $I = \{1, 2, 3\} \setminus \{i\}$ for some $i \in \{1, 2, 3\}$. Without loss of generality we may assume that $i = 1$. Since f is nearly convenient we can choose points $A, B \in \operatorname{supp} f$ such that $\operatorname{dist}(A, OX_2) \leq 1$ and $\operatorname{dist}(B, OX_3) \leq 1$. Consider the following cases:

(a) $A, B \in OX_2X_3$. Then $M_2 \cap OX_2X_3 \neq \emptyset$ and $M_3 \cap OX_2X_3 \neq \emptyset$. Hence M satisfies the Kouchnirenko condition for I in this case.
(b) $A \in OX_2X_3$ and $B \notin OX_2X_3$. Since $A \in OX_2X_3$ and $\operatorname{dist}(A, OX_2) \leq 1$ then $M_2 \cap OX_2X_3 \neq \emptyset$. Since $B \notin OX_2X_3$ and $\operatorname{dist}(B, OX_3) \leq 1$then $B \in OX_1X_3$ and B is at distance 1 to OX_3. Therefore $M_1 \cap OX_2X_3 \neq \emptyset$. Summing up M satisfies the Kouchnirenko condition for I in this case. (We consider analogously the case $A \notin OX_2X_3$ and $B \in OX_2X_3$.)
(c) $A \notin OX_2X_3$ and $B \notin OX_2X_3$. Then $A, B \in OX_1X_3$ and are at distance 1 to OX_3. Hence $M_1 \cap OX_2X_3 \neq \emptyset$. Since $\Gamma_+(f) \cap OX_2X_3 \neq \emptyset$ then there exists $C \in \operatorname{supp} f \cap OX_2X_3$. Therefore $M_j \cap OX_2X_3 \neq \emptyset$ for some $j \in \{2, 3\}$. Summing up M satisfies the Kouchnirenko condition for I in this case.

■

There are some equivalent combinatorial conditions to the Kouchnirenko condition. Hertling and Kurbel collected such conditions for quasihomogeneous polynomial in [HK12, Lemma 2.1] but this lemma is also true without the assumption of quasihomogeneity. Now we give a refined version of their lemma.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define $|x| = |x_1| + \ldots + |x_n|$.

Lemma 3.8. *Let $M \subset \mathbb{N}^n$ and $|m| \geq 2$, $m \in M$. Then the following conditions are equaivalent.*

(K) *M satisfies the Kouchnirenko condition.*
(K') *M satisfies the Kouchnirenko condition for every $I \subset \{1, 2, \ldots, n\}$ such that $|I| \leq \frac{n+1}{2}$.*

(C1) *For every nonempty set* $I \subset \{1, 2, \dots, n\}$ *we have* $M \cap OX_I \neq \emptyset$ *or there exists* $K \subset \{1, 2, \dots, n\} \setminus I$ *with* $|K| = |I|$ *such that* $M_k \cap OX_I \neq \emptyset$ *for every* $k \in K$.
(C1') *As (C1), but only* I *with* $|I| \leq \frac{n+1}{2}$.
(C2) *For every* $I, J \subset \{1, 2, \dots, n\}$ *with* $|I| < |J|$ *there exists* $k \in \{1, 2, \dots, n\} \setminus I$ *such that* $M_k \cap OX_J \neq \emptyset$.

The proof is the same as the proof of [HK12, Lemma 2.1].

Now we give [Ko77, Thm. 1] in a slightly refined version.

Theorem 3.9. *Let* $M \subset \mathbb{N}^n$ *and* $|m| \geq 2$ *for every* $m \in M$. *Then the following conditions are equivalent.*

(ISe) *There exists an isolated singularity* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ *such that* $\operatorname{supp} f \subset M$.
(ISg) *A singularity* f, $\operatorname{supp} f \subset M$ *with generic coefficients is an isolated singularity.*
(K) M *satisfies the Kouchnirenko condition.*

Remark 3.10. f_M is a singularity if and only if $|m| \geq 2$ for every $m \in M$.

Remark 3.11. (This remark is a slightly refined part of [HK12, Remarks 2.3]) Several people discovered parts of Theorem 3.9. We will not prove this theorem here, but comment on its history and references.

(i) The implication $(ISe) \Rightarrow (K)$ is a consequence of [Ko76, Thm. I] and [Ko76, Remarque 1.13 (ii)], but the Kouchnirenko did not carry out the explanation of [Ko76, Remarque 1.13 (ii)] in detail. He gave a short proof of the refined version $(ISe) \Leftrightarrow (K')$ in [Ko77, Thm. 1]. This reference [Ko77] seems to have been cited up to now only in [Sh79], it seems to have been almost completely ignored.
(ii) Around the same time as Kouchnirenko, Orlik and Randell proved $(ISe) \Leftrightarrow (C2)$ in the preprint [OR76, Thm. 2.12], but the published paper [OR77] does not contain this result. It seems that they have not published this result.
(iii) O.P. Shcherbak stated a result for maps [Sh79, Thm. 1] from which one can extract $(ISe) \Leftrightarrow (C1)$, but he did not provide a proof. This was done by Wall [Wa96, Chap. 5], who also stated explicitly $(ISe) \Leftrightarrow (ISg) \Leftrightarrow (C1)$ for maps in [Wa96, Thm. 5-1] and quasihomogeneous version of $(ISe) \Leftrightarrow (ISg) \Leftrightarrow (C1)$ for maps in [Wa96, Thm. 5-3]. The hypersurface case was done by Wall explicitly in [Wa96, (5-7)].
(For details see Section 4.)
(iv) A short proof valid only in quasihomogeneous case of $(ISg) \Leftrightarrow (C1)$ is given by Kreuzer and Skarke [KS92, proof of Thm. 1]. Although it requires some work to see that the condition stated in [KS92, Thm. 1] is equivalent to $(C1)$.

As a direct consequence of Theorem 3.9 we have the following corollary.

Corollary 3.12. *The support of an isolated singularity f satisfies the Kouchnirenko condition.*

Proof. Put $M = \operatorname{supp} f$. Suppose to the contrary, there exists $I \subset \{1, \ldots, n\}$ such that there are exactly $p < |I|$ nonempty sets $M_{j_1} \cap OX_I, \ldots, M_{j_p} \cap OX_I$ among the sets $M_i \cap OX_i$, $i = 1, 2, \ldots, n$. Therefore $M_k \cap OX_I = \emptyset$ for $k \in \{1, 2, \ldots n\} \setminus \{j_1, \ldots, j_p\}$. For such k we obviously get

$$(1) \qquad \frac{\partial f}{\partial z_k} = \sum_{i \notin I} z_i h_i \quad \text{and hence} \quad \{z \in \mathbb{C}^n : z_i = 0, i \notin I\} \subset \left\{ \frac{\partial f}{\partial z_k} = 0 \right\},$$

for some $h_i \in \mathcal{O}^n$. Substitute $z_i = 0$ for $i \notin I$ to the system of equations:

$$\frac{\partial f}{\partial z_{j_1}} = \cdots = \frac{\partial f}{\partial z_{j_p}} = 0.$$

We get a system of p equations with $|I|$ variables. Therefore by (1) and Corollary 8 in [G, p. 81] we get

$$\dim\{\nabla f = 0\} \geq |I| - p > 0,$$

which contradicts the assumption that zero of ∇f is isolated. ■

Remark 3.13. Saito proved that a support of an isolated singularity f satisfies condition (C1), which by Lemma 3.8 is equivalent to the Kouchnirenko condition (see Lemma 1.5 in [Sa71]). It can also be extracted from Remark 3 in [Sh79].

As a direct consequence of the above corollary and Property 3.4(i) we give the following property.

Property 3.14. *Every isolated singularity f is nearly convenient.*

4. Quasihomogeneous case

Quasihomogeneous singularities are a special class of singularities. Obviously to determine when they are isolated we may check whether they satisfy the Kouchnirenko condition. However, we would like to give combinatorial conditions in terms of their weights instead. By Milnor-Orlik formula [MO70] for quasihomogeneous isolated singularities the Milnor number $\mu(f)$ is equal to $\prod_{i=1}^{n}[(d/w_i) - 1]$. Hence a first necessary condition is that $\prod_{i=1}^{n}[(d/w_i) - 1]$ is a positive integer number. It is not a sufficient condition which the example below shows.

Example 4.1. Let $f(z_1, z_2, z_3) = z_1^5 + z_2^4 + z_1^2 z_3^2$. It is a quasihomogeneous polynomial with weight system $(4, 5, 6, 20)$ and

$$\left(\frac{20}{4} - 1\right)\left(\frac{20}{5} - 1\right)\left(\frac{20}{6} - 1\right) = 28 \in \mathbb{N}.$$

On the other hand f is not nearly convenient. Hence by Property 3.14 the singularity f is not an isolated singularity. ■

A good tool to examine whether singularities are isolated is the Poincaré function. For quasihomogeneous polynomial with weight system $(w_1, \ldots, w_n, d)$, $w_i < d, i = 1, 2, \ldots, n$, the Poincaré function is a rational function

$$\rho_{w,d}(t) = \prod_{i=1}^{n} \frac{(t^d - t^{w_i})}{(t^{w_i} - 1)}.$$

It is well known that if there exists a quasihomogeneous isolated singularity with weight system $(w_1, \ldots, w_n, d)$ then $\rho_{w,d}(t) \in \mathbb{N}[t]$ (see [AGV] or [Bou, Chap. V, sec. 5.1). Hence we have a second necessary condition for quasihomogeneous singularities to be isolated. It turns out that for dimensions $n = 2, 3$, it is also a sufficient condition.

Theorem 4.2. [Sa87, Thm. 3] *Let $(w_1, \ldots, w_n, d), w_i < d, i = 1, 2, \ldots, n$ be a weight system and $n \leq 3$. Then $\rho_{w,d}(t) \in \mathbb{Z}[t]$ if and only if there exists an isolated quasihomogeneous singularity with weight system $(w_1, \ldots, w_n, d)$.*

Remark 4.3. The above theorem is also stated in [Ar74, remark after Cor. 4.13] and [AGV, 2nd remark in 12.3].

The condition $\rho_{w,d}(t) \in \mathbb{Z}[t]$ is equivalent to a simple numerical condition.

Lemma 4.4. ([HK12], Lemma 2.4) *Let $(w_1, \ldots, w_n, d), w_i < d, i = 1, 2, \ldots, n$ be a weight system. The following conditions are equivalent:*

(P) $\rho_{w,d}(t) \in \mathbb{Z}[t]$,

(GCD) *for every $J \subset \{1, \ldots, n\}$ the* $\gcd\{w_j : j \in J\}$ *divides at least $|J|$ of the numbers $d - w_k, k = 1, \ldots, n$.*

Example 4.5. For the quasihomogeneous singularity $f(z_1, z_2, z_3) = z_1^5 + z_2^4 + z_1^2 z_3^2$ with weight system $(4, 5, 6, 20)$ from Example 4.1 the condition (GCD) is not satisfied. Indeed, take $J = \{3\}$, then $w_3 = 6$ does not divide any of numbers: $d - w_1 = 15, d - w_2 = 16, d - w_3 = 14$. Hence by the above lemma $\rho_{w,d}(t) \notin \mathbb{Z}[t]$ and by Theorem 4.2 there is no isolated quasihomogeneous singularity with such weight system.

On the other hand for quasihomogeneous singularity $f(z_1, z_2, z_3) = z_1^5 + z_2^4 + z_1 z_3^2$ with weight system $(4, 5, 8, 20)$ we easily check the condition (GCD) is satisfied. Therefore by Theorem 4.2 and Theorem 3.9 a quasihomogeneous singularity with weight system $(4, 5, 8, 20)$ with generic coefficients is an isolated singularity. ■

For $n \geq 4$ the condition $\rho_{w,d}(t) \in \mathbb{Z}[t]$ is not a sufficient condition in Theorem 4.2. See the following example which comes from [AGV, 12.3] and was given by Ivlev.

Example 4.6. Let $f(z_1, z_2, z_3, z_4) = z_1^{265} + z_2^8 z_1 + z_3^4 z_2 + z_4^{11} z_1$. It is a quasihomogeneous singularity with weight system $(1, 33, 58, 24, 265)$. We easily check that f satisfies (GCD) condition and hence by Lemma 4.4 the Poincaré function

$\rho_{w,d}(t) \in \mathbb{Z}[t]$. On the other hand, $\operatorname{supp} f$ does not satisfy the Kouchnirenko condition for $I = \{2, 4\}$ since only $OX_I \cap \operatorname{supp} f'_{z_1} \neq \emptyset$. Therefore, by Corollary 3.12, f cannot be an isolated singularity.

5. Nondegenarate class

In the previous sections we examined the characterization of isolated singularities in the case of generic coefficients. In this section we will consider the same problem for fixed coefficients in the class of nondegenerate singularities. Precisely, we take a nondegenerate singularity $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ and ask if there exist combinatorial conditions for the support of f, which imply (or are equivalent) to f being an isolated singularity. For dimensions $n = 2, 3$ we give such equivalent conditions.

Theorem 5.1. *Let $f : (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}, 0)$ be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) *f is an isolated singularity,*
(b) *f is nearly convenient.*

Remark 5.2. The definition of near convenience for $n = 2$ appeared for the first time in [Len96] and Theorem 5.1 was stated in this paper. See also [Len08].

Theorem 5.3. [BKO] *Let $f : (\mathbb{C}^3, 0) \longrightarrow (\mathbb{C}, 0)$ be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) *f is an isolated singularity,*
(b) *f is nearly convenient and $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$, $i, j \in \{1, 2, 3\}$, $i \neq j$.*

By Properties 3.6, 3.7 we can merge Theorems 5.1 and 5.3 in one following theorem.

Theorem 5.4. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \leq 3$, be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition,*
(b) *f is an isolated singularity.*

The proof of the above theorem is given after the proof of Theorem 5.6. It seems that for $n \geq 4$ Theorem 5.4 is also true. Therefore we may state the following conjecture.

Conjecture 5.5. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate singularity. Then the following conditions are equivalent:*

(a) $\operatorname{supp} f$ *satisfies the Kouchnirenko condition,*
(b) *f is an isolated singularity.*

Now, we give some sufficient combinatorial conditions for nondegenerate singularity to be isolated.

Theorem 5.6. *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, *be a nondegenerate singularity such that*

(i) f *is nearly convenient,*
(ii) $\Gamma_+(f) \cap OX_iX_j \neq \emptyset$, $i, j \in \{1, \ldots, n\}$, $i \neq j$.

Then f *is an isolated singularity.*

Remark 5.7. Observe that condition (ii) only is not necessary for an isolated singularity. Indeed, take $f(z_1, z_2, z_3, z_4) = z_1z_2 + z_3z_4$. Of course, f is an isolated singularity, but does not satisfy the condition (ii).

Since every convenient singularity satisfies the conditions (i) and (ii), as a direct consequence of the above theorem we have the following corollary.

Corollary 5.8. *Every convenient nondegenarate singularity is an isolated singularity.*

To prove Theorem 5.6 we give some lemmas and properties. Most of them can be found in [O13] and [BKO] but we repeat them for the convenience of the reader in slightly refined versions. For a series $\phi \in \mathbb{C}\{t\}$, $\phi \neq 0$, by $\operatorname{info}\phi$ (resp. $\operatorname{inco}\phi$) we mean the initial form of ϕ (resp. the coefficient of $\operatorname{info}\phi$). Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a nonzero holomorphic function in an open neighborhood of $0 \in \mathbb{C}^n$ and let $\sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu$ be the Taylor expansion of f at 0. Let $w = (w_1, \ldots, w_n) \in (\mathbb{N}_+)^n$. We define the number

$$\operatorname{ord}_w f = \inf\{\nu_1w_1 + \ldots + \nu_nw_n : \nu = (\nu_1, \ldots, \nu_n) \in \operatorname{supp} f\}$$

and we call it the *order of* f *with respect to* w. The sum of such monomials $a_{\nu_1\ldots\nu_n} z_1^{\nu_1} \ldots z_n^{\nu_n}$ for which $\nu_1w_1 + \ldots + \nu_nw_n = \operatorname{ord}_w f$ is called the *initial form of* f *with respect to* w and is denoted by $\operatorname{info}_w f$. Now we give two simple and useful properties. We omit their easy proofs.

Property 5.9. (see Property 2.1 in [O13]) *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $f(0) = 0$ *and* $\phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n$ *be a parametrization such that* $\phi(0) = 0$, $\phi_i \neq 0$, $i = 1, \ldots, n$. *Put* $w = (\operatorname{ord}\phi_i)_{i=1}^n$. *If* $\operatorname{info}_w f \circ \operatorname{info}\phi \neq 0$, *then*

$$\operatorname{info}(f \circ \phi) = \operatorname{info}_w f \circ \operatorname{info}\phi, \quad \operatorname{ord}(f \circ \phi) = \operatorname{ord}_w f.$$

Property 5.10. (see Property 2.2 in [O13]) *Let* $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, $f(0) = 0$, $w \in (\mathbb{N} \setminus \{0\})^n$, $i \in \{1, \ldots, n\}$. *Suppose that* $\operatorname{info}_w f$ *depends on* z_i, *then*

$$(\operatorname{info}_w f)'_{z_i} = \operatorname{info}_w f'_{z_i}.$$

The following lemma is used in the proof of Lemma 5.14, which in turn is the main tool in the proof of Theorem 5.6.

Lemma 5.11. (see Lemma 2.3 in [O13]) *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, *be a singularity and* $\phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n$ *be a parameterization such that* $\phi(0) = 0$, $\phi_i \neq 0$, $i = 1, \ldots, n$. *Put* $w = (\operatorname{ord}\phi_i)_{i=1}^n$ *and*

$$K = \{i \in \{1, \ldots, n\} : f'_{z_i} \circ \phi = 0\} \neq \emptyset.$$

Then for the face $S = \Delta(w, \Gamma_+(f)) \in \Gamma(f)$ *we get that* $(f_S)'_{z_i} \circ \operatorname{info}\phi = 0$ *for* $i \in K$.

Proof. Put $J = \{j \in K : S \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_j = 0\}\}$. Then for every $i \in K \setminus J$ we can find a monomial in $\operatorname{info}_w f$ in which the variable z_i appears. Therefore by Property 5.10 we get $(\operatorname{info}_w f)'_{z_i} = \operatorname{info}_w f'_{z_i}$ for $i \in K \setminus J$. Therefore by Property 5.9 we get for $i \in K \setminus J$

$$0 = \operatorname{info}_w f'_{z_i} \circ \operatorname{info}\phi = (\operatorname{info}_w f)'_{z_i} \circ \operatorname{info}\phi = (f_S)'_{z_i} \circ \operatorname{info}\phi.$$

On the other hand $(f_S)'_{z_i} \circ \operatorname{info}\phi = 0$, for $i \in J$. ■

The following proposition is a direct consequence of the above lemma.

Proposition 5.12. (see Corollary 2.4 in [O13]) *Let* $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 2$, *be a singularity and* $\phi = (\phi_i)_{i=1}^n \in \mathbb{C}\{t\}^n$ *be a parametrization such that* $\phi(0) = 0$, $\phi_i \neq 0$, $i = 1, \dots, n$. *If* $(\nabla f) \circ \phi = 0$, *then there exists a face* $S \in \Gamma(f)$ *such that* $(\nabla f_S) \circ \operatorname{info}\phi = 0$. *Thus* f *is degenerate on the face* S.

The following well-known property says that the Newton boundary of the restriction $f|_{\{z_{k+1} = \dots = z_n = 0\}}$ is the restriction of the Newton boundary of f to the set $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{k+1} = \dots = x_n = 0\}$.

Property 5.13. *Let* $f \in \mathcal{O}^n$, $n \geq 2$. *Assume that* $g(z_1, \dots, z_k) = f(z_1, \dots, z_k, 0, \dots, 0) \in \mathcal{O}^k$, $k < n$, *is a nonzero germ. Then*

$$\Gamma(g) = \{S \in \Gamma(f) : S \subset \{x_{k+1} = \dots = x_n = 0\}\}. \tag{2}$$

Proof. "$\subset$". Let $S \in \Gamma(g)$, then $S = \Delta(u, \Gamma_+(g))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^k$. Of course, $S \subset \Gamma_+(f) \cap \{x_{k+1} = \dots = x_n = 0\}$. Set

$$u' = (u_1, \dots, u_k, l(u, \Gamma_+(g)) + 1, \dots, l(u, \Gamma_+(g)) + 1) \in \mathbb{R}^n.$$

We show that $S = \Delta(u', \Gamma_+(f))$. By definition of u' we have that $l(u', \Gamma_+(f))$ can be attained only for $v \in \Gamma_+(f) \cap \{x_{k+1} = \dots = x_n = 0\}$. On the other hand it is easy to check that

$$\Gamma_+(f) \cap \{x_{k+1} = \dots = x_n = 0\} = \Gamma_+(g).$$

So we get $l(u', \Gamma_+(f)) = l(u, \Gamma_+(g))$ and $\Delta(u', \Gamma_+(f)) = \Delta(u, \Gamma_+(g))$. Summing up we obtain $S = \Delta(u', \Gamma_+(f))$, so $S \in \Gamma(f)$.

"$\supset$". Let $S \in \Gamma(f)$ and $S \subset \{x_{k+1} = \dots = x_n = 0\}$. Then $S = \Delta(u, \Gamma_+(f))$ for some $u \in (\mathbb{R}_+ \setminus \{0\})^n$ and as we observed above $\Gamma_+(f) \cap \{x_{k+1} = \dots = x_n = 0\} = \Gamma_+(g)$. So $l(u, \Gamma_+(f)) = l(u', \Gamma_+(g))$, where $u' = (u_1, \dots, u_k)$. It follows that $\Delta(u', \Gamma_+(g)) = \Delta(u, \Gamma_+(f))$. Hence $S = \Delta(u', \Gamma_+(g))$, so $S \in \Gamma(g)$. That ends the proof. ■

Denote $OZ_iZ_j = \{z \in \mathbb{C}^n : z_k = 0, k \notin \{i, j\}\}$, $i \neq j$, $i, j = 1, 2, \dots n$. The following lemma is a stronger version of Proposition 5.12.

Lemma 5.14. (see Lemma 4.3 in [BKO]) *Let $f \in \mathcal{O}^n$, $n \geq 2$, be a singularity and $\nabla f \circ \phi = 0$ for some $\phi = (\phi_1, \ldots, \phi_n) \in \mathbb{C}\{t\}^n$, $\phi(0) = 0$. Assume there exist $i \neq j$, such that $\phi_i \neq 0, \phi_j \neq 0$ and $f_{|OZ_iZ_j} \not\equiv 0$. Then there exists $S \in \Gamma(f)$ on which f is degenerate.*

Proof. For simplicity we may assume that $\phi_1, \ldots, \phi_k \neq 0$, $\phi_{k+1} = \ldots = \phi_n = 0$ for some $k \geq 2$. We can represent f in the form

$$f(z_1, \ldots, z_n) = g(z_1, \ldots, z_k) + z_{k+1}h_{k+1}(z_1, \ldots, z_n) + \ldots + z_n h_n(z_1, \ldots, z_n)$$

By the assumption we get $g \neq 0$, $g(0) = 0$, $\nabla g(\phi_1, \ldots, \phi_k) = 0$. By Proposition 5.12 there exists $S \in \Gamma(g)$, such that $(\operatorname{ord}\phi_i)_{i=1}^k$ is a primitive vector of S and $\nabla g_S \circ \operatorname{info}\phi = 0$. By Property 5.13 we get $S \in \Gamma(f)$. Of course $f_S = g_S$. Therefore we have

$$(f_S)'_{z_i}(\operatorname{info}\phi_1(t), \ldots, \operatorname{info}\phi_k(t), t, \ldots, t) \equiv 0,\ i = k+1, \ldots, n$$

and since $(\nabla g_S) \circ \operatorname{info}\phi = 0$, then

$$(f_S)'_{z_i}(\operatorname{info}\phi_1(t), \ldots, \operatorname{info}\phi_k(t), t, \ldots, t) \equiv 0,\ i = 1, \ldots k.$$

Hence

$$(f_S)'_{z_i}(\operatorname{inco}\phi_1, \ldots, \operatorname{inco}\phi_k, 1, \ldots, 1) = 0, \quad i = 1, \ldots, n,$$

thus f is degenerate on S. ■

Proof of Theorem 5.6 Suppose to the contrary, that f is not an isolated singularity. Then by the Curve Selection Lemma there exists a non-zero parametization $\phi = (\phi_1, \ldots, \phi_n)$ such that $(\nabla f) \circ \phi = 0$. It is not possible for ϕ to have $n-1$ coordinates equal to zero. Indeed, if for example $\phi = (0, \ldots, 0, \phi_n)$, $\phi_n \neq 0$, then by Property 3.14 we get that $f = az_n^k z_i + \ldots$ for some $i \in \{1, \ldots, n\}$, $a \neq 0$ and $k \geq 1$. Hence $f'_{z_i}(0, \ldots, 0, \phi_n) \neq 0$, which contradicts the assumption $(\nabla f) \circ \phi = 0$. Therefore we may assume that $\phi_i \neq 0, \phi_j \neq 0$ for some $i \neq j$. Without loss of generality we may assume that $\phi_1 \neq 0, \phi_2 \neq 0$. Since $\Gamma_+(f) \cap OX_1X_2 \neq \emptyset$, by Lemma 5.14 we have that f is degenerate on some face $S \in \Gamma(f)$, which contradicts the assumption on f. ■

Now we can prove Theorem 5.4.

Proof of Theorem 5.4 If f is an isolated singularity then by Corollary 3.12 $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Now suppose that f satisfies the Kouchnirenko condition. Then by Properties 3.6, 3.7 and Theorem 5.6 we get that f is an isolated singularity. ■

Remark 5.15. Wall considered another type of nondegeneracy than the Kouchnirenko nondegeneracy. He got similar results to the ones obtained in this section, see Lemma 1.2 and Theorem 1.4 in [Wa98].

6. The Milnor and Newton numbers

By the main theorem of [Ko76] we always have $\mu(f) \geq \nu(f)$, with equality for nondegenerate isolated singularities. Hence, if $\mu(f)$ is finite, then $\nu(f)$ is also finite. The inverse implication is false, which shows the following simple example.

Example 6.1. *Let $f(z_1, \dots, z_n) = (z_1 + \dots + z_n)^2$. Obviously f is not an isolated singularity, but since f is convenient we have $\nu(f) < \infty$.*

It is well known by the local Nullstellensatz that $\mu(f)$ is finite if and only if f is an isolated singularity. On the other hand, Kouchnirenko writes in Remark 1.13 (ii) of his celebrated paper [Kou76] that the Newton number of a singularity f is finite if and only if $\operatorname{supp} f$ satisfies the Kouchnirenko condition. Summing up, we can reformulate the results of the previous sections in terms of the Newton and Milnor numbers. By Theorem 3.9 we have the following corollary.

Corollary 6.2. *Let $M \subset \mathbb{N}^n$, $|m| \geq 2$ for every $m \in M$. Assume that $\nu(f_M) < \infty$. Then a singularity f, $\operatorname{supp} f \subset M$ with generic coefficients is an isolated singularity i.e. $\mu(f) < \infty$.*

We can also reformulate the results of Section 5. Observe that the singularity from Example 6.1 is degenerate. However the implication $\nu(f) < \infty \Rightarrow \mu(f) < \infty$ is true in the class of nondegenarate singularities in dimensions $n \leq 3$. Indeed, using Remarque 1.13 (ii) in [Ko76] we can reformulate Theorem 5.4, Corollary 5.8 and Conjecture 5.5 in terms of the Newton and Milnor numbers in the following way.

Proposition 6.3. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \leq 3$, be a nondegenerate singularity. Then*

$$\nu(f) < \infty \Leftrightarrow \mu(f) < \infty$$

Proposition 6.4. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate convenient singularity. Then*

$$\nu(f) < \infty \Leftrightarrow \mu(f) < \infty$$

Conjecture 6.5. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate singularity. Then*

$$\nu(f) < \infty \Leftrightarrow \mu(f) < \infty$$

Using Proposition 6.4 we may slightly weaken the assumptions of part (ii) of Theorem I in [Ko76] in the following way.

Corollary 6.6. *Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$, be a nondegenerate convenient singularity. Then $\mu(f)$, $\nu(f)$ are finite and $\mu(f) = \nu(f)$.*

Remark 6.7. Wall obtained a result analogous to the above corollary in the class of singularities nondegenerate in his sense, see Theorem 1.6 in [Wa98].

Acknowledgements. I would like to thank T. Krasiński and Sz. Brzostowski for their support and discussions during preparation of this paper. We also thank the anonymous referee for valuable remarks.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 95 – 113

ON $\mathcal{C}^0$-SUFFICIENCY OF JETS

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Abstract. The paper presents some details of the proofs by Kuiper and Kuo, and Bochnak and Łojasiewicz that refer to the impact of the Łojasiewicz exponent of gradient mappings on $\mathcal{C}^0$-sufficiency of jets.

Introduction

Let $\omega : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a k-jet and $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ - one of its $\mathcal{C}^k$-realizations. We say that f is $\mathcal{C}^0$-sufficient in the $\mathcal{C}^k$ class if, for any other $\mathcal{C}^k$-realization $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of ω there exist homeomorphisms $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ and $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that

$$g \circ \varphi = \psi \circ f \quad \text{in a neighbourhood of the origin.}$$

If this is the case, we say that f and g are $\mathcal{C}^0$-right-left equivalent, and if $\psi = \mathrm{id}_{\mathbb{R}}$ we say that f and g are $\mathcal{C}^0$-right equivalent. We say that f and g are V-equivalent if $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic as germs at 0.

The sufficiency of jets was studied by many authors, among them: Kuiper, Kuo, Bochnak and Łojasiewicz. In their, nowadays considered classical papers, the sufficiency of k-jets with respect to $\mathcal{C}^0$-right equivalence and the sufficiency of k-jets with respect to V-equivalence were studied, and necessary and sufficient conditions for sufficiency were given. In these cases the necessary and sufficient condition was formulated in terms of the Łojasiewicz inequality.

The present article presents some details of the proofs by Kuiper and Kuo and Bochnak and Łojasiewicz.

2010 *Mathematics Subject Classification.* Primary 14B05, Secondary 58A20.

Key words and phrases. Jet, sufficiency of jets, Łojasiewicz exponent.

The third-named author was partially supported by the Polish National Science Centre (NCN), grant 2012/07/B/ST1/03293.

1. $\mathcal{C}^r$-equivalence of functions

One of the major problems of catastrophe theory proposed by René Thom [30] is the classification of singularities of mappings and smooth functions at a point. If $f : (\mathbb{R}^n, a) \to (\mathbb{R}^s, b)$ will stand for the mapping f defined in a neighbourhood of the point $a \in \mathbb{R}^n$ with values in $\mathbb{R}^s$ such that $f(a) = b$, this problem can be formulated as follows:

Problem 1. *What conditions must be satisfied by smooth mappings $f, g : (\mathbb{R}^n, a) \to (\mathbb{R}^s, b)$ (of class $\mathcal{C}^k$; analytic), for the existence of diffeomorphisms $\varphi : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a)$, $\psi : (\mathbb{R}^s, b) \to (\mathbb{R}^s, b)$ (of class $\mathcal{C}^r$; analytic isomorphisms) such that*

$$g \circ \varphi = \psi \circ f \quad \text{in a neighbourhood of the point } a. \tag{1}$$

The mappings f, g satisfying (1) are called *equivalent at the point a* (respectively *$\mathcal{C}^r$-equivalent; analytically equivalent*), if φ, ψ are smooth diffeomorphisms (respectively of class $\mathcal{C}^r$; analytic isomorphisms).

We will illustrate the above problem by the following examples.

Example 1. *Let $k \in \mathbb{Z}$, $k > 0$. All functions $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$, defined by the formula*

$$f(x) = a_k x^k + a_{k+1} x^{k+1} + a_{k+2} x^{k+2} + \cdots, \qquad a_k \neq 0,$$

are analytically equivalent at zero. Indeed, it is sufficient to show that any such function is analytically equivalent at zero to the function $g(x) = x^k$, $x \in \mathbb{R}$. Taking $\psi(t) = t \operatorname{sgn} a_k$, $t \in \mathbb{R}$, and

$$\varphi(x) = x \sqrt[k]{|a_k + a_{k+1} x + a_{k+2} x^2 + \cdots|} \quad \text{in a neighbourhood of zero,}$$

we see that φ and ψ are analytic isomorphisms and $\psi \circ f = g \circ \varphi$ in a neighbourhood of zero.

For the functions of several variables, Problem 1 is not so simple as in Example 1 for one variable.

Example 2. *Let*

$$f(x_1, x_2) = x_1^2 x_2 + a x_2^5, \qquad g(x_1, x_2) = x_1^2 x_2 + x_2^5, \qquad (x_1, x_2) \in \mathbb{R}^2,$$

where $a \in \mathbb{R}$ is a parameter. Then the polynomials f and g have the same Taylor polynomial of order 3 at zero, equals to $x_1^2 x_2$, however

- *For $a > 0$, the functions f and g are analytically equivalent at zero, because for the analytic isomorphism*

  $$\varphi(x_1, x_2) = \left(\frac{1}{\sqrt[10]{a}} \cdot x_1,\ \sqrt[5]{a} \cdot x_2 \right), \qquad (x_1, x_2) \in \mathbb{R}^2,$$

  we have $f = g \circ \varphi$ in $\mathbb{R}^2$.

- *For $a \leqslant 0$, the functions f and g are not even $\mathcal{C}^0$-equivalent at zero, because by simple calculation we check that their sets of zeros have different numbers of topological components in each neighbourhood of the point $(0,0) \in \mathbb{R}^2$. Thus they can not be homeomorphic in any neighbourhood of the point $(0,0)$.*

In Examples 1, 2 we received analytic equivalence of analytic functions. There are analytic functions which are $\mathcal{C}^0$-equivalent at a point but are not analytically equivalent, as the following example shows.

Example 3. (Whitney). *Let*

$$f(x_1, x_2) = x_1 x_2 (x_1 + x_2)(x_1 - a x_2), \quad g(x_1, x_2) = x_1 x_2 (x_1 + x_2)(x_1 - b x_2),$$

where $a, b > 0$ are parameters. According to Corollary 1 in Section 2, for every $a, b > 0$ functions f and g are $\mathcal{C}^0$-equivalent at zero. For $a \neq b$, the functions f and g are not even $\mathcal{C}^1$-equivalent. If there were diffeomorphisms $\varphi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $\psi : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ of class $\mathcal{C}^1$ such that $\psi \circ f = g \circ \varphi$ in a neighbourhood of zero, then the differential $d_0\varphi$ at zero would transform the tangent spaces at zero of the components of $f^{-1}(0)$ to the corresponding tangent spaces of the components of $g^{-1}(0)$. Then identify the tangent spaces to $\mathbb{R}^2$ at 0 with $\mathbb{R}^2$ we would get $d_0\varphi(f^{-1}(0)) = g^{-1}(0)$, which is impossible.

In view of this example, we see that the analytic classification of functions leads to a very rich family of different classes. This redirected the study of equivalence of functions to the study of $\mathcal{C}^r$-equivalence, especially to study of $\mathcal{C}^0$-equivalence at a point. In this paper we concentrate on study the $\mathcal{C}^0$-equivalence of $\mathcal{C}^k$ functions.

2. $\mathcal{C}^0$-SUFFICIENCY OF JETS

Examples 1 and 2 impose the following particularly important case of the Problem 1.

Problem 2. *What conditions should be imposed on the Taylor polynomials of functions f and g such that these functions were $\mathcal{C}^0$-equivalent at zero?*

This problem leads to the notion of $\mathcal{C}^0$-sufficiency of jets.

By a *k-jet of $\mathcal{C}^k$ function* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ we mean a family v of all functions $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of class $\mathcal{C}^k$ with the same k-th Taylor polynomial centered at zero as a Taylor polynomial of function f:

$$\sum_{j=1}^{k} \frac{1}{j!} \sum_{i_1, \ldots, i_j = 1}^{n} \frac{\partial^j f}{\partial x_{i_1} \cdots \partial x_{i_j}}(0) x_{i_1} \cdots x_{i_j}.$$

The function f is called then *$\mathcal{C}^k$-realization* of the jet v. By $J^k(n)$ we denote the set of all k-jets of $\mathcal{C}^k$ functions in n variables. The k-jet of a function f can be identified with the k-th Taylor polynomial of the function. So $J^k(n)$ is isomorphic to $\mathbb{R}^N$, where $N = \binom{n+k}{n} - 1$.

A k-jet is called *$\mathcal{C}^0$-sufficient in* the $\mathcal{C}^k$ class, if any two of its $\mathcal{C}^k$-realizations are $\mathcal{C}^0$-equivalent at zero.

R. Thom [30] (see also [13]) proved that by adding to any polynomial a "generic" form of "high degrees" we get a $\mathcal{C}^0$-sufficient k-jet in an appropriate class (the same is also true for the k-jets of mappings). Precisely, we have

Theorem 1. (R. Thom). *Let us denote by $\pi_s : J^{k+s}(n) \to J^k(n)$ the natural projection. Let $v \in J^k(n)$. Then there is an integer $s > 0$ and there is a proper algebraic subset $\Sigma \subset \pi_s^{-1}(v)$ such that every $(k+s)$-jet $w \in \pi_s^{-1}(v) \setminus \Sigma$ is $\mathcal{C}^0$-sufficient in the $\mathcal{C}^{k+s}$ class.*

Bochnak and Łojasiewicz generalized this theorem (see [1]) showing that $s = 1$ (see Proposition 1 in Section 3).

In the language of k-jets Problem 2 can be written as follows.

Problem 3. *What conditions should be imposed on the k-jet to make it $\mathcal{C}^0$-sufficient in the $\mathcal{C}^k$ class?*

The $\mathcal{C}^0$-sufficiency of jet implies a topological equivalence (in a neighbourhood of zero) of sets of zeros of its realizations. This leads to the following definition:

A k-jet is called *V-sufficient in* $\mathcal{C}^k$ class, if for any two its $\mathcal{C}^k$-realizations f and g, the sets $f^{-1}(0)$ and $g^{-1}(0)$ are homeomorphic in a neighbourhood of zero.

The following beautiful theorem is a solution of Problem 3.

Theorem 2. (Kuiper-Kuo, Bochnak-Łojasiewicz). *Let v be a k-jet with f as its $\mathcal{C}^k$-realization, where $k \in \mathbb{Z}$, $k > 0$. The following conditions are equivalent:*

(a) *v is $\mathcal{C}^0$-sufficient in $\mathcal{C}^k$ class,*
(b) *v is V-sufficient in $\mathcal{C}^k$ class,*
(c) *$|\nabla f(x)| \geqslant C|x|^{k-1}$ in a neighbourhood of the point $0 \in \mathbb{R}^n$ for some constant $C > 0$, where ∇f is the gradient of the function f.*

In the above theorem the implication (a)⇒(b) is obvious; the implication (b)⇒(c) was proved by Bochnak and Łojasiewicz [1]; the implication (c)⇒(a) was proved by Kuiper [11] and Kuo [12]. The proof of Bochnak and Łojasiewicz (by contradiction) is based on the construction of an appropriate $\mathcal{C}^k$-realization of the jet, whose set of zeros is not a topological manifold in any neighbourhood of the point 0. It is known that for every $\mathcal{C}^k$-realization f of V-sufficient k-jet, the set of zeros $f^{-1}(0)$ is a topological manifold in some neighbourhood of zero or an empty set (see Lemma 2 in Section 4). The proofs of Kuiper and Kuo are based on the construction of a homeomorphism φ (see definition of $\mathcal{C}^0$-equivalence) using the general solution of an appropriate system of ordinary differential equations. The proof of Theorem 2 is discussed further in Section 4.

In Section 4, as the implication (c)⇒(a) of Theorem 2, we similarly prove the following

Corollary 1. *Let $f, g \in \mathbb{R}[x_1, x_2]$ be homogeneous forms that are decomposed in the products of linear forms without multiple factors. If $\deg f = \deg g$, then f and g are $\mathcal{C}^0$-equivalent at zero.*

Of course, the implication (a)⇒(b) in Theorem 2 holds also in the complex domain, where instead of the $\mathcal{C}^k$ functions it should be considered the class of holomorphic functions. It is easy to check that the proof of the implication (c)⇒(a) is transferred without any changes to the complex case. Unfortunately, the Bochnak and Łojasiewicz proof of the implication (b)⇒(c) is typically real and cannot be transferred to the case of holomorphic function. This implication over $\mathbb{C}$ was generalized by Teissier [29], who showed that for the holomorphic functions $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, the smallest integer k such that k-jet of function f is $\mathcal{C}^0$-sufficient in the class of holomorphic functions, satisfies the inequality $k \geqslant \lceil \mathcal{L}_0(\nabla f) \rceil + 1$, where $\lceil x \rceil$ denotes the smallest integer $k \geq x$ and $\mathcal{L}_0(\nabla f)$ – the Łojasiewicz exponent of ∇f at zero (see Section 3). The inequality $k \leqslant \lceil \mathcal{L}_0(\nabla f) \rceil + 1$ was proved by Chang and Lu [3], who based on the article of Kuo [12].

The problem of sufficiency of jets is of interest to many mathematicians, besides the mentioned above, inter alia: Kirschenbaum and Lu [8]; Koike [9]; Kucharz [10]; Kuo [13]; Kuo and Lu [15]; Lu [17]; Pelczar [21], [22]; Płoski [24]; Randall [25]; Takens [28]; Trotman [32].

3. The Łojasiewicz exponent

Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a function of class $\mathcal{C}^k$. In view of Theorem 2, a special importance is imposed on the optimal (i. e. the smallest) exponent α in *the Łojasiewicz inequality* [20]

(Ł) $\qquad |\nabla f(x)| \geqslant C|x|^\alpha \quad$ in a neighbourhood of zero for some $C > 0$.

This exponent is called *the Łojasiewicz exponent of gradient* ∇f at a zero and is denoted by $\mathcal{L}_0(\nabla f)$. This is obviously an invariant of singularities, that is, it stays invariant under a diffeomorphic change of variables. The knowledge of the exponent and its connections to other invariants of singularities helps in a more accurate characterization of different classes of singularities. This fact caused a great interest and an intense study of the exponent $\mathcal{L}_0(\nabla f)$. It was of interest to many scientists, among others: Chądzyński [4], Chądzyński and Krasiński [6]; Khadiri and Tougeron [7]; Kuo and Lu [14]; Lejeune-Jalabert and Teissier [19]; Płoski [23]; Teissier [29]; Tougeron [31].

Bochnak and Łojasiewicz generalized Theorem 1 (see [1], page 259) showing that $s = 1$. In the proof of this generalization they use Theorem 2 (c)⇒(a) to the following fact.

Proposition 1. *For a polynomial $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ of degree at most k there is a proper algebraic subset $\Sigma \subset \mathbb{R}^N$, where $N = \binom{n+k}{n-1}$, such that for every polynomial*

$$H_c(x) = \sum_{i_1+\cdots+i_n=k+1} c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $c = (c_{i_1,\ldots,i_n}; i_1 + \cdots + i_n = k+1) \in \mathbb{R}^N \setminus \Sigma$, we have

$$\mathcal{L}_0 \left(\nabla(f + H_c)\right) \leqslant k, \tag{2}$$

so then $|\nabla(f + H_c)(x)| \geqslant C|x|^k$ in a neighbourhood of the point $0 \in \mathbb{R}^n$ for some constant $C > 0$ (that is $f + H_c$ satisfies the condition (c) *of Theorem 2 for $k+1$).*

Proof. Since for every proper algebraic subset $V \subset \mathbb{C}^N$, a set $V \cap \mathbb{R}^N$ is a proper algebraic subset of $\mathbb{R}^N$, then it suffices to prove the proposition over $\mathbb{C}$. Let

$$\begin{aligned}
\Omega =& \{c \in \mathbb{C}^N : \exists_{r>0} \ \nabla(f + H_c)(x) \neq 0 \ \text{ for } \ 0 < |x| < r\},\\
\Delta =& \{c \in \mathbb{C}^N : \exists_{r>0} \ \nabla(f + H_b)(x) \neq 0 \ \text{ for } \ 0 < |x| < r, \ |b - c| < r\}.\\
G =& \{c \in \mathbb{C}^N : \exists_{C,r>0} \ |\nabla(f + H_c)(x)| \geqslant C|x|^k \ \text{ for } \ |x| < r\}.
\end{aligned}$$

Note first that the set Ω has a nonempty interior. Indeed, let us consider an algebraic set:

$$\Gamma = \{(c, x) \in \mathbb{C}^N \times \mathbb{C}^n : \nabla H_c(x) = 0\}.$$

Let $\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_l$ be a decomposiotion of Γ into irreducible components. Of course, $\mathbb{C}^N \times \{0\} \subset \Gamma$. Take any component Γ_{i_0} of the set Γ such that $\mathbb{C}^N \times \{0\} \subset \Gamma_{i_0}$. We will show that $\mathbb{C}^N \times \{0\} = \Gamma_{i_0}$. Suppose to the contrary, that $\mathbb{C}^N \times \{0\} \subsetneq \Gamma_{i_0}$, then $\dim_{\mathbb{C}} \Gamma_{i_0} > N$. Since $\nabla H_c(x) = 0$ is a system of homogenous equations, it is easy to check that for each $c \in \mathbb{C}^N$ there is $x \neq 0$, such that $(c, x) \in \Gamma_{i_0}$. However, it is impossible, because for $c \in \mathbb{C}^N$ such that $H_c(x) = x_1^{k+1} + \cdots + x_n^{k+1}$ there is no $x \neq 0$ satisfying $\nabla H_c(x) = 0$. Summing up $\Gamma_{i_0} = \mathbb{C}^N \times \{0\}$. Denoting by A the set $\bigcup_{i \neq i_0} \{c \in \mathbb{C}^N : (c, 0) \in \Gamma_i\}$, we see that this is a proper algebraic subset of $\mathbb{C}^N$. Moreover, for $c \in \mathbb{C}^N \setminus A$ the gradient $\nabla(f + H_c)$ has no zeros at infinity. Thus, the set of zeros of $\nabla(f + H_c)$ is finite. This gives that $\mathbb{C}^N \setminus \Omega \subset A$ and prove the announced remark.

Taking into account the above remark we will prove that $\mathbb{C}^N \setminus \Delta$ is contained in a proper algebraic subset Σ of space $\mathbb{C}^N$. In fact, let

$$\Omega_j = \{c \in \mathbb{C}^N : \nabla(f + H_c)(x) \neq 0 \ \text{ for } \ 0 < |x| < \frac{1}{j}\}, \quad j \in \mathbb{N}.$$

Then $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. From the previous observation $\operatorname{Int} \Omega \neq \emptyset$, so from the Baire theorem, there is $j_0 \in \mathbb{N}$ such that $\operatorname{Int} \Omega_{j_0} \neq \emptyset$. Let

$$T = \{(c, x) \in \mathbb{C}^N \times \mathbb{C}^n : \nabla(f + H_c)(x) = 0\}$$

and let $T = T_1 \cup \ldots \cup T_m$ be a decomposiotion of T into irreducible components. If $\mathbb{C}^N \times \{0\} \not\subset T$, then by setting $\Sigma = \{c \in \mathbb{C}^N : (c, 0) \in T\}$ we get the mentioned remark in this case. So, assume that $\mathbb{C}^N \times \{0\} \subset T$. Then there is i_0 such that $\mathbb{C}^N \times \{0\} \subset T_{i_0}$. We will show that $\mathbb{C}^N \times \{0\} = T_{i_0}$. Assuming the contrary, we get $\dim_{\mathbb{C}} T_{i_0} > N$. Thus, each point $(c, 0)$ is an accumulation point of the set $T_{i_0} \setminus [\mathbb{C}^N \times \{0\}]$. In particular, each point $(c, 0)$, where $c \in \Omega_{j_0}$ is an accumulation point of the set $T_{i_0} \setminus [\mathbb{C}^N \times \{0\}]$. This is impossible, because Ω_{j_0} has nonempty interior. As a consequence $\mathbb{C}^N \times \{0\} = T_{i_0}$. Now, setting $\Sigma = \bigcup_{i \neq i_0} \{c \in \mathbb{C}^N : (c, 0) \in T_i\}$ we get the mentioned remark, too.

Finally we will show that $\mathbb{C}^N \setminus \Sigma \subset G$, which finishes the proof of the proposition. We will base on the original Bochnak and Łojasiewicz proof [1], p. 259. Suppose to the contrary, that there exists $c \in \mathbb{C}^N \setminus \Sigma$ such that $c \notin G$. Then there exists a sequence $(a_\nu) \subset \mathbb{C}^n \setminus \{0\}$, $a_\nu \to 0$ such that

$$\frac{|\nabla(f + H_c)(a_\nu)|}{|a_\nu|^k} \to 0 \quad \text{as} \quad \nu \to \infty. \tag{3}$$

We will prove that there exists a sequence $b_\nu \in \mathbb{C}^N$ such that

$$\nabla(f + H_c)(a_\nu) = \nabla H_{b_\nu}(a_\nu) \quad \text{and} \quad b_\nu \to 0. \tag{4}$$

Indeed, let $\delta_\nu = \nabla(f + H_c)(a_\nu)$ and $L_\nu : \mathbb{C}^n \to \mathbb{C}^n$ be an isometry such that $L_\nu(\frac{a_\nu}{|a_\nu|}) = (1, 0, \ldots, 0)$ and $L_\nu(0) = 0$. Denote by M_ν the matrix of mapping L_ν. Then all the coefficients of the matrices M_ν and M_ν^{-1} are bounded by 1. Let $\delta_\nu \cdot M_\nu^{-1} = (\theta_{\nu,1}, \ldots, \theta_{\nu,n})$. Then from (3) we have

$$\frac{\theta_{\nu,i}}{|a_\nu|^k} \to 0 \quad \text{as} \quad \nu \to \infty \quad \text{for} \quad i = 1, \ldots, n. \tag{5}$$

Take polynomials

$$G_\nu(x) = \frac{\theta_{\nu,1}}{k+1} x_1^{k+1} + \sum_{i=2}^{n} \theta_{\nu,i} x_1^k x_i$$

and

$$H_{b_\nu} = \frac{1}{|a_\nu|^k} G_\nu \circ L_\nu.$$

Then

$$\nabla G_\nu(x) = \left(x_1^{k-1} \left(\theta_{\nu,1} x_1 + k\theta_{\nu,2} x_2 + \cdots + k\theta_{\nu,n} x_n\right), \theta_{\nu,2} x_1^k, \ldots, \theta_{\nu,n} x_1^k\right),$$

so $\nabla G_\nu(1, 0, \ldots, 0) = (\theta_{\nu,1}, \ldots, \theta_{\nu,n})$. Hence

$$\nabla H_{b_\nu}(a_\nu) = \frac{1}{|a_\nu|^k} \nabla G_\nu(L_\nu(\frac{a_\nu}{|a_\nu|} |a_\nu|)) \cdot M_\nu = (\theta_{\nu,1}, \ldots, \theta_{\nu,n}) \cdot M_\nu = \delta_\nu.$$

Moreover, (5) implies that $b_\nu \to 0$ as $\nu \to \infty$, because b_ν are made of points $\left(\frac{\theta_{\nu,1}}{|a_\nu|^k (k+1)}, \frac{\theta_{\nu,2}}{|a_\nu|^k}, \ldots, \frac{\theta_{\nu,n}}{|a_\nu|^k}\right)$ by the linear transformations with the uniformly bounded coefficients. As a result, (4) has been proved. In summary, from (4) and the definition of sequence δ_ν we get

$$\nabla(f + H_{c-b_\nu})(a_\nu) = \nabla(f + H_c)(a_\nu) - \nabla H_{b_\nu}(a_\nu) = 0$$

and $c - b_\nu \in \mathbb{C}^N \setminus \Sigma \subset \Delta$ for sufficiently large ν (because $c - b_\nu \to c$ as $\nu \to \infty$). This contradicts the definition of set Δ and completes the proof. □

From the Proposition 1 we deduce immediately its generalization.

Corollary 2. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be an analytic function and $k \in \mathbb{Z}$, $k > 0$. Then there is a proper algebraic subset $\Sigma \subset \mathbb{R}^N$, where $N = \binom{n+k}{n-1}$, such that for each $c = (c_{i_1,\dots,i_n}; i_1 + \dots + i_n = k+1) \in \mathbb{R}^N \setminus \Sigma$ we have $\mathcal{L}_0(\nabla(f + H_c)) \leqslant k$, where $H_c(x) = \sum_{i_1+\dots+i_n=k+1} c_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}$.*

Proof. Let $f = g + h + u$, where $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ denotes the polynomial of degree at most k, $h : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ denotes the homogeneous polynomial of degree $k+1$ and $u : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ denotes the analytic function such that $\operatorname{ord}_0 u > k+1$. According to Proposition 1, there exists a proper algebraic subset $\Sigma_1 \subset \mathbb{R}^N$ such that the inequality $\mathcal{L}_0(\nabla(g + H_c)) \leqslant k$ holds for every $c \in \mathbb{R}^N \setminus \Sigma_1$. If $c_0 \in \mathbb{R}^N$ is a system of coefficients of h, then $\Sigma_2 = \{c - c_0 : c \in \Sigma_1\}$ is a proper algebraic subset of $\mathbb{R}^N$ and $\mathcal{L}_0(\nabla(g + h + H_c)) \leqslant k$ for every $c \in \mathbb{R}^N \setminus \Sigma_2$. Since $\operatorname{ord}_0 u > k+1$, we obtain $|\nabla u(x)| \leqslant C|x|^{k+1}$ in a neighbourhood of zero, for some $C > 0$. This and the previous one implies the inequality $\mathcal{L}_0(\nabla(f + H_c)) \leqslant k$. □

The example which follows will illustrate the preceding results: Theorem 1 and Proposition 1 .

Example 4. *Let $f \in J^2(2)$ be of the form $f(x_1, x_2) = x_1^2$.*

Then the 2-jet f is not $\mathcal{C}^0$-sufficient in $\mathcal{C}^2$ class, because, for example, a set of zeros of its $\mathcal{C}^2$-realization $g(x_1, x_2) = x_1^2 - x_2^4$ is not homeomorphic to $f^{-1}(0)$ in any neighbourhood of zero.

Let $\Sigma = \mathbb{R}^3 \times \{0\}$, for every $c = (c_1, c_2, c_3, c_4) \in \mathbb{R}^4 \setminus \Sigma$ and let $H_c(x) = c_1x_1^3 + c_2x_1^2x_2 + c_3x_1x_2^2 + c_4x_2^3$. Then the sets of zeros of $\frac{\partial(f+H_c)}{\partial x_1}$ and $\frac{\partial(f+H_c)}{\partial x_2}$ have no common tangents at a point zero. Thus $\mathcal{L}_0(\nabla(f + H_c)) \leqslant 2$ and according to the Theorem 2, the 3-jet $f + H_c$, $c \in \mathbb{R}^4 \setminus \Sigma$, is $\mathcal{C}^0$-sufficient in the $\mathcal{C}^3$ class.

Remark 1. *It is worth going back for a moment to the polynomial $g(x_1, x_2) = x_1^2x_2 + x_2^5$ in Example 2. We will calculate $\mathcal{L}_0(\nabla g)$. In these calculations, it is convenient to pass to the complex case. In this case, the Łojasiewicz exponent of gradient ∇g is defined in the same way as above and denoted by $\mathcal{L}_0^{\mathbb{C}}(\nabla g)$. Using the results of Chądzyński and Krasiński (*Theorem 1 *in* [6], *see also* [5]) *we get that the exponent $\mathcal{L}_0^{\mathbb{C}}(\nabla g)$ is attained on the set*

$$S = \{z \in \mathbb{C}^2 : \frac{\partial g}{\partial z_1}(z)\frac{\partial g}{\partial z_2}(z) = 0\}.$$

It is easy to check that $S = S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$S_1 = \mathbb{C} \times \{0\}, \qquad S_2 = \{0\} \times \mathbb{C},$$
$$S_3 = \{(-i\sqrt{5}t^2, t) \in \mathbb{C}^2 : t \in \mathbb{C}\}, \quad S_4 = \{(i\sqrt{5}t^2, t) \in \mathbb{C}^2 : t \in \mathbb{C}\}.$$

Then

$$\nabla g|_{S_1}(t, 0) = (0, t^2), \qquad \nabla g|_{S_2}(0, t) = (0, 5t^4),$$
$$\nabla g|_{S_3}(-i\sqrt{5}t^2, t) = (-2i\sqrt{5}t^3), \quad \nabla g|_{S_1}(i\sqrt{5}t^2, t) = (-2i\sqrt{5}t^3).$$

Hence, we get $\mathcal{L}_0^{\mathbb{C}}(\nabla g) = 4$. *In particular* $\mathcal{L}_0(\nabla g) \leqslant 4$. *Since* $\nabla g(0,t) = (0, 5t^4)$ *for* $t \in \mathbb{R}$, *we deduce that* $\mathcal{L}_0(\nabla g) = 4$.

The polynomial $f = x_1^2 x_2 + a x_2^5$, $a \in \mathbb{C}$, *is a* $\mathcal{C}^4$*-realization of* 4*-jet* v *of polynomial* g. *Since* $\mathcal{L}_0(\nabla g) = 4 = 5-1$, *the Łojasiewicz inequality* (Ł) *and Theorem 2 implies that the* 4*-jet* v *is not* $\mathcal{C}^0$*-sufficient. It agrees with the statement in Example 2, that for* $a \leqslant 0$ *the functions* f *and* g *are not equivalent at zero. By Theorem 2,* 5*-jet of function* g *is* $\mathcal{C}^0$*-sufficient in* $\mathcal{C}^5$ *class. This means that the addition to* g *any terms of degree at least* 6, *leads to an equivalent at zero function* g.

4. Proof of Theorem 2

Implication (c)⇒(a). Let us begin with a simple lemma.

Lemma 1. *Let* $G \subset \mathbb{R} \times \mathbb{R}^n$ *be an open set and* $W : G \to \mathbb{R}^n$ *be a continuous mapping. If a system*

$$\frac{dy}{dt} = W(t, y) \tag{6}$$

has a global uniqueness of solutions property in $G \setminus (\mathbb{R} \times \{0\})$ *and if*

$$|W(t,x)| \leqslant C|x| \quad \text{for} \quad (t,x) \in U, \tag{7}$$

for some constant $C > 0$ *and some neighbourhood* $U \subset G$ *of* $(\mathbb{R} \times \{0\}) \cap G$, *then* (6) *has a global uniqueness of solutions property in* G.

Proof. By the uniqueness of solutions of (6) in $G \setminus (\mathbb{R} \times \{0\})$, it suffices to prove that there exists a locally unique solution of a system (6) that passes through the point 0. Assume that $(t_0, 0) \in G$. Condition (7) implies that the mapping $y_0(t) = 0$, defined in some neighbourhood of t_0, is a solution of (6). Suppose that there exists another solution $y_1 : (a,b) \to \mathbb{R}^n$ of (6) such that $y_1(t_0) = 0$. Then y_0 and y_1 fulfill the following system of integral equations

$$y(t) = \int_{t_0}^{t} W(\xi, y(\xi)) d\xi. \tag{8}$$

Let $0 < \varepsilon < \frac{1}{C}$ be small enough to guarantee that graphs of $y_0, y_1 : I \to \mathbb{R}^n$, where $I = [t_0 - \varepsilon, t_0 + \varepsilon] \subset (a,b)$ lie in U. Then there exists $\eta \in I$ such that

$$\varrho := \sup_{t \in I} |y_0(t) - y_1(t)| = |y_0(\eta) - y_1(\eta)|.$$

In view of the assumption we get that $\varrho > 0$. Therefore (8) and assumption (7) give

$$\varrho = \left| \int_{t_0}^{\eta} [W(\xi, y_0(\xi)) - W(\xi, y_1(\xi))] d\xi \right| \leqslant \left| \int_{t_0}^{\eta} C |y_1(\xi))| d\xi \right| \leqslant C\varrho\varepsilon < \varrho,$$

which is impossible. □

Proof of implication (c)⇒(a). In the case $k = 1$ this is a consequence of the inverse function theorem. Let us assume that $k > 1$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be the k-th

Taylor polynomial of a k-jet v and let g be a $\mathcal{C}^k$-realization of jet v. It suffices to show that mappings f and g are $\mathcal{C}^0$-equivalent. From the choice of g we have

$$\lim_{x\to 0}\frac{g(x)-f(x)}{|x|^k}=0,$$

which implies that for every $\varepsilon_0>0$ there exists $\delta_0>0$ such that

$$|g(x)-f(x)|\leqslant \varepsilon_0|x|^k \qquad \text{for}\quad |x|<\delta_0. \tag{9}$$

We may assume that g is defined in $\mathbb{R}^n$. Therefore we have a well-defined mapping $F:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, where

$$F(\xi,x)=f(x)+\xi(g(x)-f(x)),\qquad \xi\in\mathbb{R},\quad x\in\mathbb{R}^n.$$

We note that (cf. Kuo [12], Lemma 1, p. 168) there exist ε and $\delta>0$ such that

$$|\nabla F(\xi,x)|\geqslant \varepsilon|x|^{k-1}\quad \text{for}\quad |x|<\delta\quad\text{and}\quad -2<\xi<2. \tag{10}$$

Indeed, since f and g are $\mathcal{C}^k$ functions, $\nabla(g-f)$ is a $\mathcal{C}^{k-1}$ mapping. The choice of g shows that the $(k-1)$-th Taylor polynomial centered at zero of mapping $\nabla(g-f)$ vanishes identically. Hence

$$\lim_{x\to 0}\frac{|\nabla(g-f)(x)|}{|x|^{k-1}}=0.$$

Therefore there exists $\delta>0$ such that

$$|\nabla(g-f)(x)|\leqslant\frac{C}{4}|x|^{k-1}\quad\text{for}\quad |x|<\delta,$$

where C comes from the condition (c) of Theorem 2. Since

$$\nabla F(\xi,x)=[(g-f)(x),\nabla f(x)+\xi\nabla(g-f)(x)], \tag{11}$$

then by taking $\varepsilon=\frac{C}{2}$, we have from assumption (c)

$$|\nabla F(\xi,x)|\geqslant|\nabla f(x)+\xi\nabla(g-f)(x)|\geqslant|\nabla f(x)|-2|\nabla(g-f)(x)|\geqslant\varepsilon|x|^{k-1}$$

provides $|x|<\delta$ and $-2<\xi<2$. This gives (10). One can of course assume that $\varepsilon=\varepsilon_0$ and $\delta=\delta_0<\frac{1}{2}$.

Define $G=\{(\xi,x)\in\mathbb{R}\times\mathbb{R}^n:|x|<\delta,\ -2<\xi<2\}$, where ε and δ are as above. Let $X:G\to\mathbb{R}^n\times\mathbb{R}$ be a mapping of the form

$$X(\xi,x)=(X_1,\dots,X_{n+1})=\frac{(g(x)-f(x))}{|\nabla F(\xi,x)|^2}\nabla F(\xi,x),\qquad\text{provided}\quad x\neq 0$$

and $X(\xi,0)=0$. By (9) and (10), we have

$$|X(\xi,x)|\leqslant\frac{\varepsilon|x|^k}{|\nabla F(\xi,x)|}\leqslant\frac{\varepsilon|x|^k}{\varepsilon|x|^{k-1}}=|x|\quad\text{for}\quad(\xi,x)\in G,x\neq 0. \tag{12}$$

It is easy to see that the above inequality holds also for $x=0$, so X is continuous.

Let us define a vector field $W:G\to\mathbb{R}^n$ by

$$W(\xi,x)=\frac{1}{X_1(\xi,x)-1}[X_2(\xi,x),\dots,X_{n+1}(\xi,x)].$$

Inequality (12) implies that

$$|X_1(\xi,x)-1| \geqslant 1-|X(\xi,x)| \geqslant 1-|x| > 1-\delta > \frac{1}{2} \quad \text{for} \quad (\xi,x)\in G,$$

whence W is well-defined. Moreover it is continuous and

$$|W(\xi,x)| \leqslant 2|x| \qquad \text{for} \quad (\xi,x)\in G. \tag{13}$$

Consider now a system of differential equations

$$\frac{dy}{dt} = W(t,y). \tag{14}$$

Since $k>1$, then W is at least of class $\mathcal{C}^1$ on $G\setminus(\mathbb{R}\times\{0\})$, so it is a lipschitzian vector field. As a consequence, the above system has a uniqueness of solutions property in $G\setminus(\mathbb{R}\times\{0\})$. Hence, inequality (13) and Lemma 1 implies the global uniqueness of solutions of the system (14) throughout G. Since $y_0(t)=0$, $t\in(-2,2)$ is one of the solutions of (14), then the above implies the existence of a neighbourhood $U\subset\mathbb{R}^n$ of 0 such that every integral solution y_x of (14) with $y_x(0)=x$, where $x\in U$, is defined at least in $[0,1]$.

Now, let us define a mapping $\varphi: U\to\mathbb{R}^n$ by the formula

$$\varphi(x)=y_x(1),$$

where y_x stands for an integral solution of (14) with $y_x(0)=x$. This mapping is continuous and bijective. It gives a homeomorphism of some neighbourhoods of the origin. Indeed, considering solution $\overline{y}_x:[0,1]\to\mathbb{R}^n$ of (14) with $\overline{y}_x(1)=x$, where x is from some neighbourhood of the origin, we get that $\varphi(\overline{y}_x(0))=x$. Similar reasoning shows that the mapping $x\mapsto\overline{y}_x(0)$ is continuous in the neighbourhood of the origin. Consequently $\varphi:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ maps homeomorphically a neighbourhood of the origin onto a neighbourhood of the origin.

Finally, note that for every $x\in U$,

$$F(t,y_x(t)) = \text{const.} \quad \text{in} \quad [0,1]. \tag{15}$$

Indeed, from definition of W we derive the formula

$$[1,W(\xi,x)] = \frac{1}{X_1(\xi,x)-1}(X(\xi,x)-e_1) \qquad \text{for} \quad (\xi,x)\in G,$$

where $e_1=[1,0,\dots,0]\in\mathbb{R}^{n+1}$ and $[1,W]: G\to\mathbb{R}\times\mathbb{R}^n$. Thus, if we denote by $\langle a,b\rangle$ the scalar product of two vectors a, b, then according to (11) for $t\in[0,1]$, we have

$$\begin{aligned}\frac{dF(t,y_x(t))}{dt} &= \langle(\nabla F)(t,y_x(t)),[1,W(t,y_x(t))]\rangle\\ &=\frac{1}{X_1(t,y_x(t))-1}\left(\langle(\nabla F)(t,y_x(t)),X(t,y_x(t))\rangle-\frac{\partial F}{\partial\xi}(t,y_x(t))\right)\\ &=\frac{1}{X_1(t,y_x(t))-1}\big(g(y_x(t))-f(y_x(t))-g(y_x(t))+f(y_x(t))\big)=0.\end{aligned}$$

This gives (15). Finally, (15) yields

$$f(x) = F(0, x) = F(0, y_x(0)) = F(1, y_x(1)) = F(1, \varphi(x)) = g(\varphi(x))$$

for $x \in U$. This ends the proof of the implication (c)⇒(a) in Theorem 2. □

Proof of Corollary 1. Let $k = \deg f$. It suffices to prove the corollary assumming that

$$f(x) = (\alpha_1 x_1 + \alpha_2 x_2)h(x) \quad \text{i} \quad g(x) = (\beta_2 x_1 + \beta_2 x_2)h(x),$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ and $h \in \mathbb{R}[x_1, x_2]$ is a form of degree $k-1$. Moreover, it can be assumed that f and g differ only by a constant factor and that the region $\{(x_1, x_2) \in \mathbb{R}^n : \alpha_1 x_1 + \alpha_1 x_2 > 0,\ \beta_1 x_1 + \beta_2 x_2 > 0\}$ is disjoint from $h^{-1}(0)$. Then there is an interval (a, b) containing the interval $[0, 1]$ such that for every $\xi \in (a, b)$ a linear mapping

$$L_\xi(x) = (\alpha_1 x_1 + \beta_1 x_2) + (1 - \xi)[(\alpha_2 - \alpha_1)x_1 + (\beta_2 - \beta_1)x_2]$$

does not divide h. Let $F(\xi, x) = f(x) + \xi(g(x) - f(x))$. Then $F(\xi, x) = L_\xi(x)h(x)$, so for every $\xi \in (a, b)$, function F does not have multiple factors. Therefore after eventually diminishing the interval (a, b) such that still $[0, 1] \subset (a, b)$, and using the curve selection lemma, we easily show that F satisfies (10) for $\xi \in (a, b)$. Since $g - f$ is a form of degree k, it satisfies (9) for some $\varepsilon_0 > 0$. Repeating now the rest of the proof of the implication (c)⇒(a) in Theorem 2, we get the assertion. □

Implication (b)⇒(c). In developing this proof we used the original Bochnak and Łojasiewicz proof [1]. Assuming that the implication fails, the proof consists in the construction of an appropriate $\mathcal{C}^k$-realization of jet, whose set of zeros is not a topological manifold in any neighbourhood of the point 0. In fact there is the following

Lemma 2. *Let v be a k-jet and let f be its $\mathcal{C}^k$-realization. If v is V-sufficient in $\mathcal{C}^k$, then there is a neighbourhood $U \subset \mathbb{R}^n$ of 0 such that $f^{-1}(0) \cap (U \setminus \{0\})$ is a $(n-1)$-dimensional topological manifold or an empty set.*

Proof. Let g be a k-th Taylor polynomial of jet v. Then

$$h = g + x_1^{k+1} + \cdots + x_n^{k+1}$$

is a $\mathcal{C}^k$-realization of jet v. Moreover ∇h has no zeros at infinity (even over $\mathbb{C}$), so its set of zeros is finite. Therefore the assertion follows from the implicit function theorem and from the definition of V-sufficiency. □

A key point in the proof of considered implication is Proposition 2 given below. In the proof of mentioned proposition we will use the following Morse lemma, which follows from the previously proven implication (c)⇒(a) in Theorem 2 (cf. [18] Lemma 2.2).

Corollary 3. (Morse lemma). *Let f be a function of class $\mathcal{C}^2$ in a neighbourhood of $a \in \mathbb{R}^n$, $n > 1$, such that*

$$(16) \qquad f(a) = 0, \qquad \nabla f(a) = 0 \qquad \text{and} \qquad \det\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right] \neq 0.$$

Then there is a homeomorphism $\varphi : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a)$ and there is an integer $0 \leqslant l \leqslant n$ such that

$$f \circ \varphi(x) = \sum_{i=1}^{l}(x_i - a_i)^2 - \sum_{i=l+1}^{n}(x_i - a_i)^2 \quad \textit{in a neighbourhood of } a.$$

Proof. It suffices to consider the case $a = 0$. Then, from (16), 2-nd Taylor polynomial of function f is a quadratic form: $h(x) = \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(0)x^i x^j$. It can be assumed, from the assumption (16), by the appropriate selection of linear coordinate system, that

$$h(x) = \sum_{i=1}^{l} x_i^2 - \sum_{i=l+1}^{n} x_i^2 \quad \text{for some} \quad l \in \mathbb{Z}, \quad 0 \leqslant l \leqslant n.$$

We can directly verify that $|\nabla h(x)| = 2|x|^{2-1}$ for $x \in \mathbb{R}^n$. Hence and from the implication (c)$\Rightarrow$(a) in Theorem 2, 2-jet of function h is $\mathcal{C}^0$-sufficient in $\mathcal{C}^2$. Since f is $\mathcal{C}^2$-realization of this jet, there is a homeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that $f \circ \varphi = h$ in a neighbourhood of 0. $\square$

In the proof of Proposition 2 we will also need two known topological facts. Let's start with the definition.

The set $S^l = \{(x_1, \ldots, x_{l+1}) \in \mathbb{R}^{l+1} : x_1^2 + \cdots + x_{l+1}^2 = 1\}$ as well as any set homeomorphic to S^l will be called a *sphere* of dimension l.

Let A be a topological manifold and S — a sphere in A. The mappings φ, $\psi : S \to A$ will be called *homotopic in A*, if there is a continuous mapping $H : S \times [0, 1] \to A$ such that

$$H(x, 0) = \varphi(x) \qquad \text{and} \qquad H(x, 1) = \psi(x) \qquad \text{for} \qquad x \in S.$$

The mapping H will be called *a homotopy of φ and ψ in A.*

We will say that a sphere S is *contractible in A*, if there is a point $a \in A$ such that the mapping $\varphi : S \ni x \mapsto x \in A$ is homotopic in A to a constant map $\psi : S \ni x \mapsto a$. The homotopy of mappings φ and ψ will be called *a null-homotopy in A.*

Lemma 3. *Let A be a topological manifold of dimension k and $a \in A$. If $1 \leqslant l \leqslant k - 2$, then there exists a neighbourhood $U \subset A$ of a such that every l-dimensional sphere $S \subset U \setminus \{a\}$ is contractible in $U \setminus \{a\}$.*

Proof. We may assume, by choosing a neighbourhood $U \subset A$ of a homeomorphic with $\mathbb{R}^k$, that $U = \mathbb{R}^k$ and $a = 0$. Let $S \subset \mathbb{R}^k \setminus \{0\}$ be an arbitrary l-dimensional sphere and $\varphi : S^l \to S$ be a homeomorphism. Approximating φ

by a polynomial mapping $\psi : S^l \to \mathbb{R}^k \setminus \{0\}$, we may assume that φ and ψ are homotopic in $\mathbb{R}^k \setminus \{0\}$. It is easy to find a line $E \subset \mathbb{R}^k \setminus \psi(S^l)$ such that $0 \in E$. The mappings ψ and $a+\psi$ are homotopic in $\mathbb{R}^k \setminus \{0\}$ for every $a \in E$. Moreover there is $a \in E$ such that 0 is not in the convex hull of $(a+\psi(S^l))$. Therefore $a+\psi$ is contractible in $\mathbb{R}^k \setminus \{0\}$. □

Lemma 4. *The sphere $S = \{(x_1, \dots, x_l) \in \mathbb{R}^l : x_1^2 + \dots + x_l^2 = r^2\}$, where $r > 0$ is not contractible in $\mathbb{R}^l \setminus \{0\}$.*

Proof. Assume to the contrary that there is a null-homotopy $H : S \times [0,1] \to \mathbb{R}^l \setminus \{0\}$. It can be assumed that $r = 1$ and that $H(S \times [0,1]) \subset S$. Therefore a mapping h defined by $h(x) = H(\frac{x}{|x|}, 1-|x|)$ for $0 < |x| \leqslant 1$ and $h(0) = H(y,1)$, where $y \in S$, is a continuous mapping of a ball $D = \{x \in \mathbb{R}^l : |x| \leqslant r\}$ onto a sphere S, whereas $h(x) = x$ for $x \in S$. Thus S is a deformation retract of ball D, which is impossible. □

Proposition 2. *Let $n > 1$ and $f : (\mathbb{R}^n, a) \to (\mathbb{R}, 0)$ be a function of class $\mathcal{C}^2$ fulfilling the assumptions* (16) *of Morse lemma. Then $f^{-1}(0)$ is not a topological manifold of dimension $n-1$ in any neighbourhood of point a.*

Proof. In view of Corollary 3 (Morse lemma), it suffices to reduce our considerations to the case $a = 0$,

$$f(x) = \sum_{i=1}^{l} x_i^2 - \sum_{i=l+1}^{n} x_i^2$$

and $f^{-1}(0) \neq \{0\}$. Then $1 \leqslant l < n$. It can be assumed, of course, that $l \leqslant \frac{n}{2}$.

The theorem is clearly true for $l = 1$, since then a set $f^{-1}(0) \setminus \{0\}$ has at least four topological components in every neighbourhood of the origin for $n = 2$, and at least two such components for $n > 2$. It can therefore be assumed that $n > 2$ and $l > 1$ and then

$$1 \leqslant l-1 \leqslant (n-1)-2. \tag{17}$$

Assume now that for some neighbourhood $\Omega \subset \mathbb{R}^n$ of the point $0 \in \mathbb{R}^n$,

$$A = f^{-1}(0) \cap \Omega \text{ is a topological manifold of dimension } n-1.$$

Therefore (17) and Lemma 3 implies that there is a neighbourhood $U \subset A$ of the origin such that every $(l-1)$-dimensional sphere $S \subset U \setminus \{0\}$ is contractible in $U \setminus \{0\}$. However, by taking a $(l-1)$-dimensional sphere

$$S = \{(x_1, \dots, x_l) \in \mathbb{R}^l : x_1^2 + \dots + x_l^2 = r^2\}$$

for sufficiently small $r > 0$ and a point $\overset{o}{x} = (\overset{o}{x}_{l+1}, \dots, \overset{o}{x}_n) \in \mathbb{R}^{n-l}$ such that $\overset{o}{x}{}^2_{l+1} + \dots + \overset{o}{x}{}^2_n = r^2$, we see that $S \times \{\overset{o}{x}\} \subset U \setminus \{0\}$. The sphere $S \times \{\overset{o}{x}\}$ is contractible in $U \setminus \{0\}$ by the assumption. Let $H = (h_1, \dots, h_n) : S \times \{\overset{o}{x}\} \times [0,1] \to U \setminus \{0\}$ be a null-homotopy of $S \times \{\overset{o}{x}\}$ in $U \setminus \{0\}$. Then

$$h_1^2 + \dots + h_l^2 = h_{l+1}^2 + \dots + h_n^2 \quad \text{in} \quad S \times \{\overset{o}{x}\} \times [0,1].$$

Hence $h_1^2 + \cdots + h_l^2$ does not vanish anywhere in $S \times \{\overset{o}{x}\} \times [0,1]$, so $(h_1, \ldots, h_l)$ is a null-homotopy of S in $\mathbb{R}^l \setminus \{0\}$. This contradicts the assertion of Lemma 4. $\square$

Remark 2. *The assumption* $\det \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a)\right] \neq 0$ *in Corollary 2 may not be omitted, because a polynomial* $f(x_1, x_2) = x_1^3 - x_2^3$ *does not satisfy this assumption for* $a = 0$ *and* $f^{-1}(0) = \{(t,t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$ *is a topological manifold of dimension* 1.

In the proof of the considered implication the well known Bochnak and Łojasiewicz inequality [1] play the dominant role.

Lemma 5. (Bochnak-Łojasiewicz inequality) *Let* $0 < \theta < 1$. *If the function* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ *is analytic, then*

$$|x||\nabla f(x)| \geqslant \theta |f(x)| \qquad \text{in some neighbourhood of } 0.$$

Proof of implication (b)⇒(c). The assumption (b) implies that k-th Taylor polynomial h of function f is nonzero. Otherwise the functions $f_1(x) = 0$, $f_2(x) = x_1^{k+1}$ would be the $\mathcal{C}^k$-realizations of a k-jet which is V-sufficient in the class $\mathcal{C}^k$, which is impossible. Hence, in case $n = 1$, $\mathcal{L}_0(\nabla f) = \operatorname{ord}_0 f' \leqslant k - 1$. This gives (c) in this case. Assume therefore that $n > 1$.

In the case $k = 1$ from (b) it follows $\nabla f(0) \neq 0$. In fact, otherwise for the two $\mathcal{C}^1$ realizations $f_1(x) = x_1^2$ and $f_2(x) = x_1 x_2$ of the 1-jet v the sets $f_1^{-1}(0)$ and $f_2^{-1}(0)$ would be homeomorphic, in some neighbourhoods of zero, which is impossible. The condition $\nabla f(0) \neq 0$ obviously implies (c). Therefore we may assume that $k > 1$.

Since

$$\lim_{x \to 0} \frac{\nabla f(x) - \nabla h(x)}{|x|^{k-1}} = 0,$$

$\mathcal{L}_0(\nabla f) \leqslant k - 1$ if and only if $\mathcal{L}_0(\nabla h) \leqslant k - 1$. Hence, it is sufficient to verify the implication for $f = h$.

Assume to the contrary that (c) is not satisfied. Then, for a sequence $(a_\nu) \subset \mathbb{R}^n \setminus \{0\}$ such that $a_\nu \to 0$ as $\nu \to \infty$, we have

$$\frac{|\nabla f(a_\nu)|}{|a_\nu|^{k-1}} \to 0 \qquad \text{as} \quad \nu \to \infty. \tag{18}$$

Therefore, the Bochnak-Łojasiewicz inequality (Lemma 5) gives

$$\frac{|f(a_\nu)|}{|a_\nu|^{k}} \to 0 \qquad \text{as} \quad \nu \to \infty. \tag{19}$$

Taking a subsequnce of (a_ν), we may suppose that $|a_{\nu+1}| \leqslant \frac{1}{2}|a_\nu|$ for $\nu \in \mathbb{N}$. Then

$$B_\nu = \{x \in \mathbb{R}^n : |x - a_\nu| \leqslant \frac{1}{4}|a_\nu|\}, \quad \nu \in \mathbb{N}, \quad \text{is a family of disjoint closed balls.}$$

Let us take an arbitrary sequence $(\lambda_\nu) \subset \mathbb{R}$ such that

$$\frac{\lambda_\nu}{|a_\nu|^{k-2}} \to 0 \qquad \text{as} \quad \nu \to \infty. \tag{20}$$

Since $k > 1$, we may assume that

$$\lambda_\nu \quad \text{is not an eigenvalue of the matrix} \quad \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a_\nu)\right]. \tag{21}$$

Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ such that $\alpha(x) = 0$ for $|x| \geqslant \frac{1}{4}$ and $\alpha(x) = 1$ in some neighbourhood of 0. Consider a mapping $F : \mathbb{R}^n \to \mathbb{R}$ defined by the formulas

$$F(x) = \alpha\left(\frac{x - a_\nu}{|a_\nu|}\right)\left(f(a_\nu) + d_{a_\nu} f(x - a_\nu) + \frac{1}{2}\lambda_\nu |x - a_\nu|^2\right) \quad \text{for} \quad x \in B_\nu$$

and $F(x) = 0$ for $x \notin \bigcup_{\nu=1}^\infty B_\nu$. Then F is of class $\mathcal{C}^k$ (even of class $\mathcal{C}^\infty$) and $F(0) = 0$. Moreover $f(a_\nu) = F(a_\nu)$ and $\nabla f(a_\nu) = \nabla F(a_\nu)$, so

$$(f - F)(a_\nu) = 0 \quad \text{i} \quad \nabla(f - F)(a_\nu) = 0 \qquad \text{for} \quad \nu \in \mathbb{N}. \tag{22}$$

Let $M > 0$ be a constant such that $|\alpha(x)| \leqslant M$ for $x \in \mathbb{R}^n$. Then for $x \in B_\nu$,

$$\begin{aligned}\frac{|F(x)|}{|x|^k} &\leqslant M \frac{|f(a_\nu) + d_{a_\nu} f(x - a_\nu) + \frac{1}{2}\lambda_\nu |x - a_\nu|^2|}{|x|^k} \\ &\leqslant 2^k M \frac{|f(a_\nu)| + |\nabla f(a_\nu)||a_\nu| + \frac{1}{2}|\lambda_\nu||a_\nu|^2}{|a_\nu|^k}.\end{aligned}$$

Hence, (18), (19) and (20) implies

$$\frac{|F(x)|}{|x|^k} \to 0 \qquad \text{as} \qquad x \to 0.$$

In consequence, $f - F$ is a $\mathcal{C}^k$-realization of k-jet v. In view of (22) and the assumption (b), Lemma 2 implies that $(f - F)^{-1}(0)$ is a $(n-1)$-dimensional topological manifold in every sufficiently small neighbourhood of the point $0 \in \mathbb{R}^n$. On the other hand, (21) gives

$$\det\left[\frac{\partial^2 (f - F)}{\partial x_i \partial x_j}(a_\nu)\right] \neq 0 \qquad \text{for} \quad \nu \in \mathbb{N}.$$

This with (22) and Proposition 2 implies that $(f - F)^{-1}(0)$ is not a topological manifold of dimension $n - 1$ in any neighbourhood of a_ν. In particular it is not a topological manifold in any neighbourhood of 0 (because $a_\nu \to 0$). This contradiction yields the truth of the considered implication. $\square$

5. Equivalence of mappings at infinity

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and let $f : \mathbb{K}^n \to \mathbb{K}$. By the *Łojasiewicz exponent at infinity* of gradient ∇f, denoted by $\mathcal{L}_\infty(\nabla f)$, we mean the supremum of exponents $\nu \in \mathbb{R}$ in the following *Łojasiewicz inequality*:

$$|\nabla f(x)| \geq C|x|^\nu \text{ as } |x| > R \text{ for some constants } C > 0 \text{ and } R > 0.$$

It is known that for a polynomial function f we have $\mathcal{L}_\infty(\nabla f) \in \mathbb{Q} \cup \{-\infty\}$ and $\mathcal{L}_\infty(\nabla f) > -\infty$ if and only if the set $(\nabla f)^{-1}(0)$ is finite.

Similar considerations (as in the above sections of this paper) may be carried out for functions in neighbourhoods of infinity. In the case of polynomials in two complex variables P. Cassou-Noguès and H. H. Vui [2, Theorem 5] proved that:

Let $f \in \mathbb{C}[z_1, z_2]$, $\mathcal{L}_\infty(\nabla f) \geq 0$ and $k \in \mathbb{Z}$, $k \geq 1$. The following conditions are equivalent:
(i) $\mathcal{L}_\infty(\nabla f) \geq k-1$,
(ii) there exists $\varepsilon > 0$, such that for every polynomial $P \in \mathbb{C}[z_1, z_2]$ of degree $\deg P \leq k$, whose modules of coefficients of monomials of degree k are less or equal ε, the links at infinity of almost all fibers $f^{-1}(\lambda)$ and $(f+P)^{-1}(\lambda)$, $\lambda \in \mathbb{C}$ are isotopic.

Recall that by link at infinity of the fiber $P^{-1}(\lambda)$ of a polynomial $P : \mathbb{C}^2 \to \mathbb{C}$ we mean the set $P^{-1}(\lambda) \cap \{(x, y) \in \mathbb{C}^2 :\ |x|^2 + |y|^2 = r^2\}$ for sufficiently large r.

The above result of P. Cassou-Noguès and H. H. Vui was generalized by G. Skalski [27, Theorems 3, 7]:

Let $f \in \mathbb{K}[x_1, \ldots, x_n]$, let $k \in \mathbb{Z}$, $k \geq 0$, and let $\mathcal{L}_\infty(\nabla f) \geqslant k-1$. Then there exists $\varepsilon > 0$, such that for each polynomial $P \in \mathbb{K}[x_1, \ldots, x_n]$ of degree $\deg P \leq k$, whose modules of coefficients of monomials of degree k does not exceed ε, polynomials f and $f+P$ are analytically equivalent at infinity.

We say that functions $f, g : \mathbb{K}^n \to \mathbb{K}$ are *analytically equivalent at infinity* when there exists an analytic diffeomorphism φ of neighbourhoods of infinity, such that $|\varphi(x)| \to \infty$ if and only if $|x| \to \infty$ and there exists an analytic diffeomorphism $\psi : \mathbb{K} \to \mathbb{K}$, such that

$$f \circ \varphi = \psi \circ g \quad \text{in a neighbourhood of infinity.}$$

The inverse to the Skalski theorem is false (see [27, Remark 2]) .

The method of proof of this theorem is slightly similar to the proof of Theorem 2 in this article. It consists in integrating the appropriate vector field

$$W(\xi, x) = \frac{1}{X_1(\xi, x) - 1}[X_2(\xi, x), \ldots, X_{n+1}(\xi, x)],$$

where

$$X(\xi, x) = (X_1, \ldots, X_{n+1}) = \frac{P(x)}{|\nabla F(\xi, x)|^2}\nabla F(\xi, x)$$

and $F(\xi, x) = f(x) + \xi P(x)$ with $\overline{\nabla F(\xi, x)}$ instead of $\nabla F(\xi, x)$ in the complex case.

The method of integration of the field was used also in the result by Rodak and Spodzieja [26, Theorem 1]:

Let $f \colon \mathbb{K}^n \to \mathbb{K}^m$, where $m \leq n$, be a C^2 mapping (holomorphic if $\mathbb{K} = \mathbb{C}$). Assume that there exist $k \in \mathbb{R}$ and positive constants C, R such that

$$\nu(df(x)) \geq C|x|^{k-1}, \quad |x| \geq R. \tag{23}$$

Then there exists $\varepsilon > 0$ such that for any $P \in \mathcal{P}_{k,\varepsilon}$ the mappings f and $f+P$ are isotopic at infinity, where the symbol $\mathcal{P}_{k,\varepsilon}$ (for $k \in \mathbb{R}$, $\varepsilon > 0$) denotes all C^2 mappings $P\colon \mathbb{K}^n \to \mathbb{K}^m$, for which there exists $R > 0$ such that

$$|P(x)| \le \varepsilon |x|^k \text{ and } |dP(x)| \le \varepsilon |x|^{k-1} \text{ for any } |x| \ge R, \tag{24}$$

where $dP(x)$ is the diferential of P at $x \in \mathbb{K}^n$. The symbol ν stands for

$$\nu(A) = \inf\{\|A^*\varphi\| : \varphi \in Y', \|\varphi\| = 1\},$$

where A^* is the adjoint operator in the space of linear continuous mappings from Y' to X' and X', Y' are the dual spaces of Banach spaces X an Y respectively.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 115 – 134

INTRODUCTION TO THE LOCAL THEORY OF PLANE ALGEBRAIC CURVES

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Abstract. We consider the algebroid plane curves defined by formal power series of two variables with coefficients in an algebraically closed field. Using quadratic transformations we prove the local normalization theorem. Then we study the intersection multiplicity of algebroid curves and give an introduction to the Newton diagrams.

These notes are intended as a concise introduction to the local theory of plane algebraic curves. We consider the algebroid plane curves defined by formal power series of two variables with coefficients in an algebraically closed field. Using quadratic transformations we prove the local normalization theorem. Then we study the intersection multiplicity of algebroid curves and give an introduction to the Newton diagrams. We assume known the basic theorems on formal power series: the Weierstrass Preparation Theorem, the Implicit Function Theorem and Hensel's Lemma. A standard reference for this material is Abhyankar [1] (see also Hefez [5]). The book [8] by Seidenberg was very helpful when preparing this text. For further study of algebroid curves we refer the reader to Campillo [2].

In what follows $\mathbb{K}$ is an algebraically closed field of arbitrary characteristic. The ring of formal power series in two variables x, y with coefficients in the field $\mathbb{K}$ will be denoted $\mathbb{K}[[x,y]]$ and its field of fractions $\mathbb{K}((x,y))$. If $f = \sum_{i \geqslant k} f_i$ is a nonzero formal power series represented as the sum of homogeneous forms f_i with $f_k \neq 0$ then we write $\operatorname{ord} f = k$ and $\operatorname{in} f = f_k$. Additionally we put $\operatorname{ord} 0 = \infty$ and $\operatorname{in} 0 = 0$. We use the usual conventions on the symbol ∞. A power series $u \in \mathbb{K}[[x,y]]$ is a unit if $uv = 1$ for a power series $v \in \mathbb{K}[[x,y]]$. Note that u is a unit if and only if its constant term $u(0)$ is nonzero. If $f, g \in \mathbb{K}[[x,y]]$ are such that

2010 *Mathematics Subject Classification.* Primary 32S55, Secondary 14H20.
Key words and phrases. Plane algebraic curve, branch, intersection multiplicity, Newton diagram.

$f = gu$ for a unit u then we write $f \sim g$. The principal ideal of $\mathbb{K}[[x,y]]$ generated by f is denoted $(f)\mathbb{K}[[x,y]]$. The reader will find the description of prime ideals of the ring $\mathbb{K}[[x,y]]$ in Appendix C.

1. Algebroid curves, quadratic transformations

Let $f \in \mathbb{K}[[x,y]]$ be a nonzero formal power series without constant term. The algebroid curve $f = 0$ is by definition the principal ideal $(f)\mathbb{K}[[x,y]]$ generated by f. We also denote $\{f = 0\}$ the algebroid curve of equation $f = 0$. Thus we have $\{f = 0\} = \{g = 0\}$ if and only if $f \sim g$. The curve $\{f = 0\}$ is reduced (resp. irreducible) if the power series f does not have multiple factors (resp. is irreducible). If $f = f_1^{m_1} \dots f_s^{m_s}$ in $\mathbb{K}[[x,y]]$ with f_i irreducible and coprime then the curves $\{f_i = 0\}$ are called irreducible components of $\{f = 0\}$ with multiplicities m_i.

The order (multiplicity) of the curve $\{f = 0\}$ is the number $\operatorname{ord} f$. The definition is correct because from $f \sim g$ it follows $\operatorname{ord} f = \operatorname{ord} g$. The curves of order 1 are called regular or non-singular. The curves of order strictly greater than 1 are called singular. If $f \sim g$ then $\operatorname{in} f = c \operatorname{in} g$ for a constant $c \in \mathbb{K} \setminus \{0\}$. The affine curve $\operatorname{in} f = 0$ (see Fulton [4]) is called the tangent cone to the curve $f = 0$. From the Factorization Lemma (see Appendix A) we get

Property 1.1. *The tangent cone to the irreducible curve $\{f = 0\}$ is an affine line, i.e.* $\operatorname{in} f = l^{\operatorname{ord} f}$, *where $l = bx - ay$ is a non-zero linear form.*

Let $\Phi(x,y) = (ax + by + \cdots, cx + dy + \cdots)$ be a pair of formal power series such that $ad - bc \neq 0$. Then $f \mapsto f \circ \Phi$ is an isomorphism of the ring $\mathbb{K}[[x,y]]$ (every $\mathbb{K}$-isomorphism of $\mathbb{K}[[x,y]]$ is of this form). We have $\operatorname{ord} f = \operatorname{ord}(f \circ \Phi)$ and $\operatorname{in}(f \circ \Phi) = \operatorname{in} f \circ \operatorname{in} \Phi$, where $\operatorname{in} \Phi = (ax + by, cx + dy)$.

The algebroid curves $\{f = 0\}$ and $\{g = 0\}$ are equivalent if $f \circ \Phi = gu$ for a pair Φ satisfying the above conditions and for a unit u. Equivalent curves are of the same orders and their tangent cones are affine isomorphic. Any two regular curves are formally equivalent.

Let $f = f(x,y) \in \mathbb{K}[[x,y]]$ be an irreducible power series of order $n > 0$. From Property 1.1 it follows that $\operatorname{ord} f(x,0) = n$ or $\operatorname{ord} f(0,y) = n$.

Definition 1.2. *Suppose that $f \in \mathbb{K}[[x,y]]$ is a power series such that* $\operatorname{ord} f(0,y) = \operatorname{ord} f = n$ *(in this case we say that f is y-general). Let y_1 be a new variable. A power series $f_1 \in \mathbb{K}[[x,y_1]]$ is a strict quadratic transformation of $f \in \mathbb{K}[[x,y]]$ if $f_1(0,0) = 0$ and $f(x, ax + xy_1) = x^n f_1(x,y_1)$ in $\mathbb{K}[[x,y_1]]$ for an $a \in \mathbb{K}$. We write then $f_1 = Q(f)$.*

Let us note the basic properties of quadratic transformations. We keep the notations introduced in Definition 1.2

Lemma 1.3. *Suppose that the irreducible power series $f \in \mathbb{K}[[x,y]]$ is y-general of order n and put $f_1 = Q(f)$. Then*

(i) *the line* $y - ax = 0$ *is tangent to the curve* $f(x, y) = 0$ *(so the constant* $a \in \mathbb{K}$ *is uniquely determined by* f*) and* $\operatorname{ord} f_1(0, y_1) = n$. *If* $a \neq 0$ *then* $\operatorname{ord} f(x, 0) = n$.

(ii) *If* $f \sim g$ *in* $\mathbb{K}[[x, y]]$ *and* $f_1 = Q(f)$, $g_1 = Q(g)$ *then* $f_1 \sim g_1$ *in* $\mathbb{K}[[x, y_1]]$.

(iii) *If* $f \in \mathbb{K}[[x]][y]$ *is a distinguished polynomial in* y *then* $f_1 \in \mathbb{K}[[x]][y_1]$ *and* f_1 *is a distinguished polynomial in* y_1.

Proof. Since f is y-general and irreducible we have $f(x, y) = c(y - a_0 x)^n + \cdots +$*(terms of order* $> n$*)* in $\mathbb{K}[[x, y]]$ for a constant $c \neq 0$ (see Property 1.1). Therefore we get $f(x, ax + xy_1) = x^n f_1(x, y_1)$ in $\mathbb{K}[[x, y_1]]$ with $f_1(x, y_1) = (a - a_0 + y_1)^n + \cdots +$ *(terms of order* $> n$*)*. Thus $f_1(0, 0) = 0$ if and only if $a = a_0$ and in this case $\operatorname{ord} f_1(0, y_1) = n$. The remaining properties follow directly from Definition 1.2. □

Lemma 1.4. *If* $f \in \mathbb{K}[[x, y]]$ *is a* y*-general irreducible power series then* $f_1 = Q(f) \in \mathbb{K}[[x, y]]$ *is an irreducible power series.*

Proof. By Lemma 1.3 *(iii)* we may assume that $f = f(x, y)$ is a y-distinguished polynomial of degree n. Then the power series $f_1 = f_1(x, y_1)$ is a y_1-distinguished polynomial of degree n and it suffices to check that f_1 is irreducible in the ring $\mathbb{K}[[x]][y_1]$. Suppose the contrary

$$f_1(x, y_1) = \left(y_1^k + b_1(x) y_1^{k-1} + \cdots + b_k(x)\right)\left(y_1^l + c_1(x) y_1^{l-1} + \cdots + c_l(x)\right)$$

in $\mathbb{K}[[x]][y_1]$, where $k, l > 0$.

Clearly $k + l = n$ and consequently

$$\begin{aligned} f(x, ax + xy_1) &= x^n f_1(x, y_1) = \\ &= \left((xy_1)^k + b_1(x) x (xy_1)^{k-1} + \cdots + b_k(x) x^k\right) \cdot \\ &\qquad \cdot \left((xy_1)^l + c_1(x) x (xy_1)^{l-1} + \cdots + c_l(x) x^l\right). \end{aligned}$$

Let z be a new variable. From the above identity it follows that

$$\begin{aligned} & f(x, ax + z) = \\ &= \left(z^k + x b_1(x) z^{k-1} + \cdots + x^k b_k(x)\right)\left(z^l + x c_1(x) z^{l-1} + \cdots + x^l c_l(x)\right). \end{aligned}$$

This shows that the power series $f(x, ax + z) \in \mathbb{K}[[x, z]]$ is reducible. We get a contradiction because it is irreducible as the image of the irreducible power series $f(x, y)$ by an isomorphism $\mathbb{K}[[x, y]] \to \mathbb{K}[[x, z]]$. □

Lemma 1.5. *Let* $f = f(x, y) \in \mathbb{K}[[x, y]]$ *be an irreducible* y*-general power series of order* $n = \operatorname{ord} f > 1$. *Then there exists a sequence of power series* $f_i = f_i(x, y_i) \in \mathbb{K}[[x, y_i]]$, $i = 0, 1, \ldots, m$ *such that* $f_0 = f$ *(and* $y_0 = y$*),* $f_{i+1} = Q(f_i)$, $\operatorname{ord} f_i = n$ *for* $i < m$ *and* $\operatorname{ord} f_m < n$.

Proof. Let $y_0 = y$ and $f_0 = f$ and let us consider $f_1 = Q(f_0)$. If $\operatorname{ord} f_1 < n$ then we put $m = 1$ and the sequence f_0, f_1 verifies the condition. If $\operatorname{ord} f_1 = n$ (we have always $\operatorname{ord} f_1 \leqslant \operatorname{ord} f$ since $\operatorname{ord} f_1(0, y_1) = n$) then we put $f_2 = Q(f_1)$. If $\operatorname{ord} f_2 < n$ we are done. We have to show that after a finite number of steps we get a sequence $f_0, \dots, f_m$ such that $f_{i+1} = Q(f_i)$, $\operatorname{ord} f_i = n$ for $i < m$ and $\operatorname{ord} f_m < n$. Otherwise there would exist an infinite sequence $f_0, \dots, f_m, \dots$ such that $f_{i+1} = Q(f_i)$ and $\operatorname{ord} f_i = n$ for all $i \geqslant 0$. Let $y_i - a_i x = 0$ be the tangent to the curve $f_i(x, y_i) = 0$. It is easy to check that $f(x, y(x)) = 0$, where $y(x) = \sum_{i=1}^{+\infty} a_{i-1} x^i$. We get a contradiction because f is irreducible, $\operatorname{ord} f > 1$ and the condition $f(x, y(x)) = 0$ implies that $y - y(x)$ divides $f(x, y)$ in $\mathbb{K}[[x, y]]$. $\square$

Now we can construct the transformation reducing the order of an irreducible power series.

Proposition 1.6. *Let $f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible y-general power series of order $n = \operatorname{ord} f > 1$. Let $\tilde{y}$ be a new variable.*

Then there exist an integer $m > 0$ and a polynomial $P(x) = \sum_{i=1}^{m} a_{i-1} x^i$ of degree $\leqslant m$ such that

(i) $f(x, P(x) + x^m \tilde{y}) = x^{mn} \tilde{f}(x, \tilde{y})$ *in* $\mathbb{K}[[x, \tilde{y}]]$,
(ii) $\tilde{f} = \tilde{f}(x, \tilde{y}) \in \mathbb{K}[[x, \tilde{y}]]$ *is an irreducible power series such that* $\operatorname{ord} \tilde{f} < n$,
(iii) *we have* $\operatorname{ord} \tilde{f}(0, \tilde{y}) = n$. *If* $P(x) \neq 0$ *then* $\operatorname{ord} f(x, 0) = \operatorname{ord} P(x) \cdot n$,
(iv) *if* $f \sim W$ *and* $f \sim \tilde{W}$, *where* W *and* $\tilde{W}$ *are distinguished polynomials, then* $W(x, P(x) + x^m \tilde{y}) = x^{mn} \tilde{W}(x, \tilde{y})$.

Proof. Let $f_0, f_1, \dots, f_m$ be a sequence of power series from Lemma 1.5. Thus we get $f_i(x, a_i x + x y_{i+1}) = x^n f_{i+1}(x, y_{i+1})$ $(i = 0, 1, \dots, m-1)$ for some $a_i \in \mathbb{K}$. Let $P(x) = \sum_{i=1}^{m} a_{i-1} x^i$, $\tilde{y} = y_m$ and $\tilde{f}(x, \tilde{y}) = f_m(x, \tilde{y})$. Since f_{i+1} is the strict transformation of f_i $(i = 0, \dots, m-1)$ we get *(i)* of Proposition 1.6. Part *(ii)* follows from Lemma 1.4.

To check *(iii)* suppose that $k = \operatorname{ord} P(x) < \infty$. Hence we have $a_{k-1} \neq 0$ and $a_{i-1} = 0$ for $i < k$. Consequently we get $f_i(x, x y_{i+1}) = x^n f_{i+1}(x, y_{i+1})$ for $i < k-1$ and $f_{k-1}(x, a_{k-1} x + x y_k) = x^n f_k(x, y_k)$. Since $a_{k-1} \neq 0$, from the last identity we obtain $\operatorname{ord} f_{k-1}(x, 0) = n$ by Lemma 1.3 *(i)*. From $\operatorname{ord} f_i(x, 0) = n + \operatorname{ord} f_{i+1}(x, 0)$ for $i < k-1$ we infer that $\operatorname{ord} f(x, 0) = \operatorname{ord} f_0(x, 0) = nk$.

Property *(iv)* follows from the fact that $f \sim W$ and $f_1 \sim W_1$ imply $W_1 = Q(W)$. $\square$

Remark 1.7 In the above considerations the power series $f \in \mathbb{K}[[x, y]]$ is y-general and for such a power series we define quadratic transformation. If $f \in \mathbb{K}[[x, y]]$ is x-general then we can easily reformulate the definition. In particular if $\operatorname{ord} f(x, 0) = \operatorname{ord} f = n$ then the quadratic transformation is of the form $f(by + y x_1, y) = y^n f_1(x_1, y)$, $f_1(0, 0) = 0$. If $\operatorname{ord} f(x, 0) = \operatorname{ord} f(0, y) = n$ and $ab \neq 0$ then the obtained strict quadratic transformations of f are equivalent.

2. Parametrizations

Let t be a variable. A paramerization is a pair $(\phi(t), \psi(t)) \in \mathbb{K}[[t]]^2$ such that $\phi(0) = \psi(0) = 0$ and $\phi(t) \neq 0$ or $\psi(t) \neq 0$ in $\mathbb{K}[[t]]$. Two parametrizations $(\phi(t), \psi(t))$ and $(\phi_1(t), \psi_1(t))$ are equivalent if there exists $\tau(t) \in \mathbb{K}[[t]]$, $\operatorname{ord} \tau(t) = 1$ such that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$. A parametrization $(\phi(t), \psi(t))$ is good if there does not exist $\tau(t)$, $\operatorname{ord} \tau(t) > 1$ and a parametrization $(\phi_1(t_1), \psi_1(t_1))$ such that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$.

Theorem 2.1 (Normalization Theorem)**.** *Let $f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible power series. Then there exists a good parametrization $(\phi(t), \psi(t))$ such that $f(\phi(t), \psi(t)) = 0$, $\operatorname{ord} f(x, 0) = \operatorname{ord} \psi(t)$ and $\operatorname{ord} f(0, y) = \operatorname{ord} \phi(t)$. If $(\phi^*(u), \psi^*(u))$ is a parametrization such that $f(\phi^*(u), \psi^*(u)) = 0$ then there exists a series $\sigma(u) \in \mathbb{K}[[u]]$, $\sigma(0) = 0$ such that $\phi^*(u) = \phi(\sigma(u))$ and $\psi^*(u) = \psi(\sigma(u))$.*

A good parametrization $(\phi(t), \psi(t))$ such that $f(\phi(t), \psi(t)) = 0$ is called a normalization of the curve $f(x, y) = 0$. From Theorem 2.1 it follows that every irreducible curve has a normalization unique up to equivalence.

Proof. (of Theorem 2.1) We use induction on $\operatorname{ord} f$.

If $\operatorname{ord} f = 1$ the theorem easily follows from the Implicit Function Theorem. Suppose that $n > 1$ is an integer and that the theorem is true for all irreducible power series of order $< n$. Fix an irreducible power series f such that $\operatorname{ord} f = n$. Without diminishing the generality we may assume that $\operatorname{ord} f(0, y) = n$. Let $\tilde{f}(x, \tilde{y}) \in \mathbb{K}[[x, \tilde{y}]]$ be a power series from Proposition 1.6. Thus we get $f(x, P(x) + x^m \tilde{y}) = x^{mn} \tilde{f}(x, \tilde{y})$, where $P(x)$ is a polynomial of degree $\leqslant m$, $\operatorname{ord} \tilde{f}(0, \tilde{y}) = n$ and $\operatorname{ord} \tilde{f} < n$. By induction hypothesis there is a normalization $(\phi(t), \tilde{\psi}(t))$ of the curve $\tilde{f}(x, \tilde{y}) = 0$ such that $\operatorname{ord} \phi(t) = \operatorname{ord} \tilde{f}(0, \tilde{y})$ and $\operatorname{ord} \tilde{\psi}(t) = \operatorname{ord} \tilde{f}(x, 0)$. Let us put $\psi(t) = P(\phi(t)) + \phi(t)^m \tilde{\psi}(t)$ and consider the parametrization $(\phi(t), \psi(t))$. Obviously we have $f(\phi(t), \psi(t)) = 0$.

To check that the parametrization $(\phi(t), \psi(t))$ is good suppose that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$ for a parametrization $(\phi_1(t_1), \psi_1(t_1))$ and for a series $\tau(t) \in \mathbb{K}[[t]]$, $\operatorname{ord} \tau(t) \geqslant 1$. Thus $\psi_1(\tau(t)) - P(\phi_1(\tau(t))) = \phi_1(\tau(t))^m \tilde{\psi}(t)$ and consequently $\operatorname{ord} \Big(\psi_1(t_1) - P(\phi_1(t_1))\Big) \geqslant \operatorname{ord} \phi_1(t_1)^m$. Let us put $\tilde{\psi}_1(t_1) := \frac{\psi_1(t_1) - P(\phi_1(t_1))}{\phi_1(t_1)^m}$. We get then $\operatorname{ord} \tilde{\psi}_1(t_1) \geqslant 0$ and $\tilde{\psi}(t) = \tilde{\psi}_1(\tau(t))$. From the equalities $\phi(t) = \phi_1(\tau(t))$ and $\tilde{\psi}(t) = \tilde{\psi}_1(\tau(t))$ it follows that $\operatorname{ord} \tau(t) = 1$ since the parametrization $(\phi(t), \tilde{\psi}(t))$ is good. This proves that $(\phi(t), \psi(t))$ is a normalization of the curve $f(x, y) = 0$.

Let us recall that $\operatorname{ord} \phi(t) = \operatorname{ord} \tilde{f}(0, \tilde{y}) = n = \operatorname{ord} f(0, y)$. To calculate $\operatorname{ord} \psi(t)$ let us suppose first $P(x) \neq 0$. Then $\operatorname{ord} P(\phi(t)) = (\operatorname{ord} P)(\operatorname{ord} \phi) \leqslant m(\operatorname{ord} \phi) = \operatorname{ord} \phi^m < \operatorname{ord} \phi^m \tilde{\psi}$ and $\operatorname{ord} \psi(t) = \operatorname{ord} \Big(P(\phi(t)) + \phi(t)^m \tilde{\psi}(t)\Big) = \operatorname{ord} P(\phi(t)) = (\operatorname{ord} P)(\operatorname{ord} \phi) = (\operatorname{ord} P)n = \operatorname{ord} f(x, 0)$ by Proposition 1.6 *(iii)*. If $P(x) = 0$ then

$\operatorname{ord}\psi(t) = \operatorname{ord}\phi(t)^m\tilde{\psi}(t) = mn + \operatorname{ord}\tilde{\psi} = mn + \operatorname{ord}\tilde{f}(x,0) = \operatorname{ord} f(x,0)$. Summing up we have checked that $\operatorname{ord}\phi(t) = \operatorname{ord} f(0,y)$ and $\operatorname{ord}\psi(t) = \operatorname{ord} f(x,0)$.

Now let $(\phi^*(u), \psi^*(u))$ be a parametrization such that $f(\phi^*(u), \psi^*(u)) = 0$. Put $\tilde{\psi}^*(u) = \frac{\psi^*(u) - P(\phi^*(u))}{\phi^*(u)^m} \in \mathbb{K}((u))$. Let $W(x,y)$ be a distinguished polynomial associated with $f(x,y)$. We get

$$\begin{aligned} 0 = W\left(\phi^*(u), \psi^*(u)\right) &= W\left(\phi^*(u), P(\phi^*(u)) + \phi^*(u)^m\tilde{\psi}^*(u)\right) = \\ &= \left(\phi^*(u)\right)^{mn} \tilde{W}\left(\phi^*(u), \tilde{\psi}^*(u)\right) \end{aligned}$$

and hence $\tilde{W}\left(\phi^*(u), \tilde{\psi}^*(u)\right) = 0$.

From the last equality it follows that $\operatorname{ord}\tilde{\psi}^*(u) > 0$ since $\tilde{\psi}^*(u)$ is a root of the distinguished $\tilde{W}(\phi^*(u), y) \in \mathbb{K}[[u]][y]$ (see Remark 2.2 given below). Let $(\phi(t), \tilde{\psi}(t))$ be a normalization of the curve $\tilde{f}(x,\tilde{y}) = 0$. By assumption we get $\phi^*(u) = \phi(\tau(u))$ and $\tilde{\psi}^*(u) = \tilde{\psi}(\tau(u))$, which implies $\phi^*(u) = \phi(\tau(u))$ and $\psi^*(u) = \psi(\tau(u))$. $\square$

Remark 2.2 If $\zeta(u)^n + \alpha_1(u)\zeta(u)^{n-1} + \cdots + \alpha_n(u) = 0$ in $\mathbb{K}((u))$ then it is easy to check that $\operatorname{ord}\zeta(u) \geqslant \inf_i\{\frac{1}{i}\operatorname{ord}\alpha_i(u)\}$. In particular if the polynomial $y^n + \alpha_1(u)y^{n-1} + \cdots + \alpha_n(u)$ is distinguished then $\operatorname{ord}\alpha_i(u) > 0$ for $i = 1, \ldots, n$ and consequently $\operatorname{ord}\zeta(u) > 0$.

Corollary 2.3. *If $f(x,y) \in \mathbb{K}[[x,y]]$ with $n = \operatorname{ord} f(0,y) < \infty$ then there exist power series $\alpha(s), \beta_1(s), \ldots, \beta_n(s) \in \mathbb{K}[[s]]$ (s is a variable) without constant term such that*

$$f(\alpha(s), y) \sim \prod_{j=1}^{n}(y - \beta_j(s)) \text{ in } \mathbb{K}[[s,y]].$$

Proof. Using the Weierstrass Preparation Theorem we may assume that $f(x,y) \in \mathbb{K}[[x]][y]$ is a distinguished polynomial of degree n. We prove the corollary by induction on $n = \deg_y f$. If $n = 1$ the corollary is obvious. Suppose that $n > 1$ and the corollary is true for polynomials of degree $n-1$. Let $f(x,y)$ be a distinguished polynomial of degree n. Using Theorem 2.1 to an irreducible factor of the series $f(x,y)$ we find a parametrization $(\alpha(s), \beta(s))$ such that $f(\alpha(s), \beta(s)) = 0$. We get then $f(\alpha(s), y) = (y - \beta(s))g(s,y)$ in $\mathbb{K}[[s]][y]$, where $g(s,y) = y^{n-1} + \ldots$ is a distinguished polynomial of degree $n-1$. We apply the induction hypothesis to $g(s,y)$. $\square$

Let us note

Corollary 2.4 (Puiseux Theorem)**.** *Let $\mathbb{K}$ be an algebraically closed field of characteristic l. Let $n > 0$ be an integer such that $n \not\equiv 0 \pmod{l}$. Then for every* *distinguished and irreducible polynomial* $P(x, y) = y^n + \sum_{i=1}^{n} a_i(x)y^{n-i}$ *there exists a series* $y(s) \in \mathbb{K}[[s]]$*,* $y(0) = 0$ *such that*

$$P(s^n, y) = \prod_{\epsilon^n=1} (y - y(\epsilon s)).$$

Proof. Let $(\phi(t), \psi(t))$be a normalization of the curve $P(x, y) = 0$. Then $\operatorname{ord} \phi(t) = \operatorname{ord} P(0, y) = n$ and there exists a series $\sigma(t)$ such that $\phi(t) = \sigma(t)^n$ in $\mathbb{K}[[t]]$ since $n \not\equiv 0 \pmod{l}$ (we use the Implicit Function Theorem or Hensel's Lemma to the equation $y^n - \phi(t) = 0$). Clearly $\operatorname{ord} \sigma(t) = 1$ and $\psi(t) = y(\sigma(t))$ for a power series $y(s) \in \mathbb{K}[[s]]$. The parametrization $(s^n, y(s))$ is good. Therefore we have $\mathrm{GCD}\,(\{n\} \cup \operatorname{supp}\, y(s)) = 1$ and $y(\epsilon_1 s) \neq y(\epsilon_2 s)$ if $\epsilon_1^n = \epsilon_2^n = 1$ and $\epsilon_1 \neq \epsilon_2$. Hence we get the corollary because $P(s^n, y(\epsilon s)) = 0$ for all ϵ such that $\epsilon^n = 1$. □

Lemma 2.5. *Let* $\phi(t) \in \mathbb{K}[[t]]$ *be a nonzero power series of order* $n > 0$*. Then any power series* $g(t) \in \mathbb{K}[[t]]$ *can be expressed in the following form*

$$g(t) = \sum_{i=0}^{n-1} a_i(\phi(t))t^i, \qquad \text{where } a_i = a_i(x) \in \mathbb{K}[[x]] \text{ for } i = 0, \ldots, n-1.$$

The coefficients $a_i = a_i(x)$ *are uniquely determined by* $\phi(t)$ *and* $g(t)$*.*

Proof. Let us fix $g(t) \in \mathbb{K}[[t]]$ and put $F(x, t) = \phi(t) - x$. Then we get $\operatorname{ord} F(0, t) = \operatorname{ord} \phi(t) = n$ and the Weierstrass Division Theorem gives $g(t) = q(x, t)F(x, t) + \sum_{i=0}^{n-1} a_i(x)t^i$. Substituting $\phi(t)$ for x we obtain $g(t) = \sum_{i=0}^{n-1} a_i(\phi(t))t^i$. To show the uniquess it suffices to observe that if we had a relation as above with $g(t) = 0$ and with some nonzero $a_i(x)$, then two terms $a_i(\phi(t))t^i$ and $a_j(\phi(t))t^j$, $i \neq j$ would necessarily have the same finite order. This obviously cannot be the case. □

Now we can prove a theorem partialy converse to Theorem 2.1.

Theorem 2.6. *For every parametrization* $(\phi(t), \psi(t))$ *there exists an irreducible power series* $f = f(x, y)$ *such that* $f(\phi(t), \psi(t)) = 0$*. It is determined uniquely by the parametrization up to a unit of the ring* $\mathbb{K}[[x, y]]$*.*

Proof. Suppose that $\phi(t) \neq 0$ and put $n = \operatorname{ord} \phi(t)$. By Lemma 2.5 we get that $\mathbb{K}[[t]] = \mathbb{K}[[\phi(t)]] + \mathbb{K}[[\phi(t)]]t + \cdots + \mathbb{K}[[\phi(t)]]t^{n-1}$, which implies that the ring $\mathbb{K}[[t]]$ is a finite module over $\mathbb{K}[[\phi(t)]]$. Therefore the ring $\mathbb{K}[[t]]$ is integral over $\mathbb{K}[[\phi(t)]]$. In particular, the series $\psi(t)$ is integral over $\mathbb{K}[[\phi(t)]]$ and there exists $f(x, y) \in \mathbb{K}[[x]][y]$ monic with respect to y such that $f(\phi(t), \psi(t)) = 0$. Replacing $f(x, y)$ by its irreducible factor we get the first part of the theorem. The uniqueness follows from the fact that the ideal I of power series $g(x, y) \in \mathbb{K}[[x, y]]$ such that $g(\phi(t), \psi(t)) = 0$ is a prime ideal and it is not maximal since $(\phi(t), \psi(t)) \neq (0, 0)$ (see Appendix C). □

Lemma 2.7. *Suppose that the domain* A *is a subring of the domain* B *such that* B *is a free* A*-module of rank* $n > 0$*. Let* K *be the field of fractions of* A *and* L *the field of fractions of* B*. Then* $(L : K) = n$*.*

Proof. By assumption there exists a sequence $e_1, \dots, e_n$ of elements of B such that every element $b \in B$ can be written uniquely in the form $b = a_1e_1 + \cdots + a_ne_n$ for some $a_1, \dots, a_n \in A$. In particular B is a finite A-module and consequently B is integral over A. Therefore for every $b \in B$, $b \neq 0$ there exists $b' \in B$ such that $bb' \in A \setminus \{0\}$. In fact if $b \notin A$ and $b^k + a_1b^{k-1} + \cdots + a_k = 0$ is the equation of integral dependence of minimal degree $k > 0$ then $a_k \neq 0$ and $bb' = -a_k$ for $b' = b^{k-1} + a_1b^{k-2} + \cdots + a_{k-1}$. Thus every element of the field L may be written in the form $\frac{b}{a}$, where $a \in A \setminus \{0\}$ and $b \in B$. If $b = a_1e_1 + \cdots + a_ne_n$ then $\frac{b}{a} = \left(\frac{a_1}{a}\right)e_1 + \cdots + \left(\frac{a_n}{a}\right)e_n$ and $(L : K) \leqslant n$. The equality follows from the fact that $e_1, \dots, e_n$ are linearly independent over K. $\square$

We denote by $\mathbb{K}((\phi(t)))$ the field of fractions of the domain $\mathbb{K}[[\phi(t)]]$.

Theorem 2.8. *Let $(\phi(t), \psi(t))$ be a good parametrization such that $\phi(t) \neq 0$. Let $n = \operatorname{ord}\phi(t)$. Then*

(a) $\Big(\mathbb{K}((t)) : \mathbb{K}((\phi(t)))\Big) = n$,
(b) $\mathbb{K}((t)) = \mathbb{K}((\phi(t)))(\psi(t))$.

Proof. By Lemma 2.5 the ring $\mathbb{K}[[t]]$ is a free module over $\mathbb{K}[[\phi(t)]]$ of rank n. Therefore Property (a) follows from Lemma 2.7. On the other hand by Theorems 2.6 and 2.1 there exists an irreducible power series $f = f(x, y) \in \mathbb{K}[[x, y]]$ such that $f(\phi(t), \psi(t)) = 0$ and $\operatorname{ord} f(0, y) = \operatorname{ord}\phi(t) = n$. Using the Weierstrass Preparation Theorem we may assume that f is a distinguished polynomial in y of degree n with coefficients in $\mathbb{K}[[x]]$. Furthermore, $f(x, y)$ is irreducible in $\mathbb{K}[[x]][y]$ and consequently in $\mathbb{K}((x))[y]$ since the ring $\mathbb{K}[[x]]$ is normal. Thus $f(\phi(t), y)$ is a minimal polynomial of $\psi(t)$ over $\mathbb{K}((\phi(t)))$ and $\Big(\mathbb{K}((\phi(t)))(\psi(t)) : \mathbb{K}((\phi(t)))\Big) =$ the degree of $f(\phi(t), y)$ in the indeterminate y, which is equal to $n = \Big(\mathbb{K}((t)) : \mathbb{K}((\phi(t)))\Big)$. This shows that $\mathbb{K}((\phi(t)))(\psi(t)) = \mathbb{K}((t))$. $\square$

For any parametrization $(\phi(t), \psi(t)) \in \mathbb{K}[[t]]^2$ we denote by $\mathbb{K}((\phi(t), \psi(t)))$ the field of fractions of the ring $\mathbb{K}[[\phi(t), \psi(t)]]$.

Theorem 2.9. *A parametrization $(\phi(t), \psi(t))$ is good if and only if $\mathbb{K}((\phi(t), \psi(t))) = \mathbb{K}((t))$.*

Proof. Suppose that $\phi(t) \neq 0$. It is easy to see that $\mathbb{K}((\phi(t)))(\psi(t)) \subset \mathbb{K}((\phi(t), \psi(t)))$. Therefore if $(\phi(t), \psi(t))$ is good then $\mathbb{K}((\phi(t), \psi(t))) = \mathbb{K}((t))$ by Theorem 2.8. Suppose that $\mathbb{K}((\phi(t), \psi(t))) = \mathbb{K}((t))$ and let $\tau(t) \in \mathbb{K}[[t]]$ be a power series without constant term such that $\phi(t) = \phi_1(\tau(t))$, $\psi(t) = \psi_1(\tau(t))$ for a parametrization $(\phi_1(s), \psi_1(s))$. Then $t \in \mathbb{K}((\phi(t), \psi(t))) \subset \mathbb{K}((\tau(t)))$, which implies $\operatorname{ord}\tau(t) = 1$. Therefore $(\phi(t), \psi(t))$ is a good parametrization. $\square$

Here is another application of Theorem 2.8.

Theorem 2.10. *There exists a nonzero power series $d(t) \in \mathbb{K}[[\phi(t), \psi(t)]]$ ("a universal denominator") such that $d(t)\mathbb{K}[[t]] \subset \mathbb{K}[[\phi(t), \psi(t)]]$.*

Proof. Suppose that $\phi(t) \neq 0$. Since $\mathbb{K}((t)) = \mathbb{K}((\phi(t)))(\psi(t))$ is an extension of $\mathbb{K}((\phi(t)))$ of degree n, the elements $1, \psi(t), \dots, \psi(t)^{n-1}$ form a linear basis of $\mathbb{K}((t))$ over $\mathbb{K}((\phi(t)))$.

Therefore, we may write

$$t^i = \alpha_{i,0}(\phi(t)) + \alpha_{i,1}(\phi(t))\psi(t) + \cdots + \alpha_{i,n-1}(\phi(t))\psi(t)^{n-1}, \tag{1}$$

where $i = 0, 1, \dots, n-1$.

Let $d(t) \in \mathbb{K}[[\phi(t)]]$ be a common denominator of the elements $\alpha_{i,j}(\phi(t))$, where $i = 0, 1, \dots, n-1$ and $j = 0, 1, \dots, n-1$. The relation (1) implies

$$d(t)t^i \in \mathbb{K}[[\phi(t)]][\psi(t)] \text{ for } i = 0, 1, \dots, n-1. \tag{2}$$

Since $\mathbb{K}[[t]] = \mathbb{K}[[\phi(t)]] + \cdots + \mathbb{K}[[\phi(t)]]t^{n-1}$ by Lemma 2.5 we get by (2) $d(t)\mathbb{K}[[t]] \subset \mathbb{K}[[\phi(t)]][\psi(t)]$. □

3. Intersection multiplicity

Let $f = f(x, y) \in \mathbb{K}[[x, y]]$ be an irreducible power series. Let us fix a normalization $(\phi(t), \psi(t))$ of the curve $f(x, y) = 0$. For every $g = g(x, y) \in \mathbb{K}[[x, y]]$ we define:

$$v_f(g) = \operatorname{ord} g(\phi(t), \psi(t)) \in \mathbb{N} \cup \{\infty\}.$$

Proposition 3.1. *For any $g, g' \in \mathbb{K}[[x, y]]$ the following properties hold:*

(i) *$v_f(g) = 0$ if and only if $g(0) \neq 0$, $v_f(g) = \infty$ if and only if f divides g in $\mathbb{K}[[x, y]]$,*
(ii) *$v_f(g + g') \geqslant \inf\{v_f(g), v_f(g')\}$. If $v_f(g) \neq v_f(g')$ then the equality holds,*
(iii) *$v_f(gg') = v_f(g) + v_f(g')$,*
(iv) *$v_f(g + hf) = v_f(g)$ for $h \in \mathbb{K}[[x, y]]$.*

Proof. To check part *(i)* note that the ideal $I = \{h(x, y) \in \mathbb{K}[[x, y]] : h(\phi(t), \psi(t)) = 0\}$ is a prime non-maximal ideal. This implies (see Appendix C) that $I = (f)$ which proves that $v_f(g) = \infty$ if and only if f divides g. The remaining properties follow directly from the definition. □

Remark 3.2 With every irreducible curve $\{f = 0\}$ we associate the field $\mathcal{M}_f$ of meromorphic fractions on $\{f = 0\}$. For this purpose we consider fractions $\frac{g}{h}$, where $g, h \in \mathbb{K}[[x, y]]$ and $h \not\equiv 0 \bmod f$. We write $\frac{g}{h} \equiv \frac{g_1}{h_1}$ if f divides $gh_1 - g_1h$. The cosets of the relation $\equiv$ form in a natural way a field denoted $\mathcal{M}_f$. The function v_f extends to the valuation $v_f : \mathcal{M}_f \to \mathbb{Z} \cup \{\infty\}$ defined by $v_f\left(\frac{g}{h}\right) = v_f(g) - v_f(h)$.

Proposition 3.3 (Basic Inequality). *We have $v_f(g) \geqslant (\operatorname{ord} f)(\operatorname{ord} g)$. The equality holds if and only if $\{f = 0\}$ and $\{g = 0\}$ don't have a common tangent.*

We need

Lemma 3.4. *Let $(\phi(t),\psi(t))$ be a parametrization, $n=\inf\{\operatorname{ord}\phi(t), \operatorname{ord}\psi(t)\}<\infty$, $\phi(t)=at^n+\cdots$, $\psi(t)=bt^n+\cdots$, where $a\neq 0$ or $b\neq 0$. Then for every power series $g=g(x,y)\in\mathbb{K}[[x,y]]$: $\operatorname{ord} g(\phi(t),\psi(t))\geqslant(\operatorname{ord} g)n$ with equality if and only if $(\operatorname{in} g)(a,b)\neq 0$.*

Proof. (of Lemma 3.4) Let us write $g(x,y)=\sum_{\alpha+\beta=m} g_{\alpha\beta}(x,y)x^\alpha y^\beta$, where $m=\operatorname{ord} g$ and $\sum_{\alpha+\beta=m} g_{\alpha\beta}(0,0)x^\alpha y^\beta=\operatorname{in} g$ ("Hadamard's Lemma").

We get $g(\phi(t),\psi(t)) = t^{mn}\sum_{\alpha+\beta=m} g_{\alpha\beta}(\phi(t),\psi(t))\left(\frac{\phi(t)}{t^n}\right)^\alpha\left(\frac{\psi(t)}{t^n}\right)^\beta = t^{mn}\left((\operatorname{in} g)(a,b)+\textit{terms of order}>0\right)$ which proves the lemma. □

Proof. (of Proposition 3.3) Let $(\phi(t),\psi(t))$ be a normalization of the irreducible curve $f(x,y)=0$. Then $\inf\{\operatorname{ord}\phi(t),\operatorname{ord}\psi(t)\}=\inf\{\operatorname{ord} f(0,y),\operatorname{ord} f(x,0)\}=\operatorname{ord} f$ since $f=0$ has exactly one tangent. Let $n=\operatorname{ord} f$, $\phi(t)=at^n+\cdots$, $\psi(t)=bt^n+\cdots$. Thus $a\neq 0$ or $b\neq 0$. Since $\operatorname{ord} f(\phi(t),\psi(t))=\operatorname{ord} 0=\infty$ we get from Lemma 3.4 that $(\operatorname{in} f)(a,b)=0$ and consequently the unique tangent to $f=0$ is given by the equation $bx-ay=0$.

Now we get $v_f(g)=\operatorname{ord} g(\phi(t),\psi(t))\geqslant(\operatorname{ord} g)\inf\{\operatorname{ord}\phi(t),\operatorname{ord}\psi(t)\}=(\operatorname{ord} g)(\operatorname{ord} f)$ by the first part of Lemma 3.4. The equality $v_f(g)=(\operatorname{ord} g)(\operatorname{ord} f)$ holds if and only if $(\operatorname{in} g)(a,b)\neq 0$, which takes place exactly when the system of equations $\operatorname{in} g=\operatorname{in} f=0$ has the unique solution $x=0$, $y=0$ that is if $f=0$ and $g=0$ don't have a common tangent. □

Proposition 3.5. *For any irreducible $f,g\in\mathbb{K}[[x,y]]$ we get $v_f(g)=v_g(f)$.*

To prove Proposition 3.5 we check the following lemma.

Lemma 3.6. *Suppose that f is irreducible, $n=\operatorname{ord} f(0,y)<\infty$ and $f(\alpha(s),y)\sim\prod_{j=1}^n(y-\beta_j(s))$ in $\mathbb{K}[[s]][y]$. Then for any $g(x,y)\in\mathbb{K}[[x,y]]$:*

$$\sum_{j=1}^n \operatorname{ord} g(\alpha(s),\beta_j(s))=(\operatorname{ord}\alpha(s))v_f(g).$$

Proof. (of Lemma 3.6) Let $(\phi(t),\psi(t))$ be a normalization of the curve $f(x,y)=0$. Then $\alpha(s)=\phi(\sigma_j(s))$, $\beta_j(s)=\psi(\sigma_j(s))$ for a power series $\sigma_j(s)$, $\sigma_j(0)=0$.

We get then

$$\sum_{j=1}^n \operatorname{ord} g(\alpha(s),\beta_j(s))=\sum_{j=1}^n \operatorname{ord} g(\phi(t),\psi(t))\operatorname{ord}\sigma_j(s)=v_f(g)\sum_{j=1}^n\operatorname{ord}\sigma_j(s).$$

To calculate the last sum let us note that $\operatorname{ord}\alpha(s) = \operatorname{ord}\phi(t)\operatorname{ord}\sigma_j(s) = n\operatorname{ord}\sigma_j(s)$ and consequently $\sum_{j=1}^{n}\operatorname{ord}\sigma_j(s) = \operatorname{ord}\alpha(s)$, which proves the lemma. $\square$

Proof. (of Proposition 3.5) Let $f, g \in \mathbb{K}[[x,y]]$ be irreducible. Suppose that f, g are y-general; $n = \operatorname{ord} f(0,y)$, $p = \operatorname{ord} g(0,y)$. By Corollary 2.3 we get

$$f(\alpha(s), y) \sim \prod_{j=1}^{n}(y - \beta_j(s)),$$

$$g(\alpha(s), y) \sim \prod_{j=1}^{p}(y - \gamma_j(s)).$$

Using Lemma 3.6 twice we get:

$$\operatorname{ord}\alpha(s)\, v_f(g) = \sum_{j=1}^{n}\operatorname{ord} g(\alpha(s), \beta_j(s)) = \sum_{j=1}^{n}\operatorname{ord}\prod_{k=1}^{p}(\beta_j(s) - \gamma_k(s)) =$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{p}\operatorname{ord}\,(\beta_j(s) - \gamma_k(s)) = \sum_{k=1}^{p}\operatorname{ord} f(\alpha(s), \gamma_k(s)) = (\operatorname{ord}\alpha(s))\, v_g(f).$$

Then $v_f(g) = v_g(f)$.

Suppose that $\operatorname{ord} f(0,y) = n < \infty$ and $\operatorname{ord} g(0,y) = \infty$. The last conditions imply that $g \sim x$ and $v_f(g) = v_f(x) = \operatorname{ord}\phi(t) = \operatorname{ord} f(0,y) = v_x(f) = v_g(f)$.

Similarly we check the proposition when $\operatorname{ord} f(0,y) = \infty$ and $\operatorname{ord} g(0,y) = p < \infty$. If $\operatorname{ord} f(0,y) = \operatorname{ord} g(0,y) = \infty$ then f and g are divisible by x and $v_f(g) = \infty = v_g(f)$.

Let us note the formula for the order of the resultant of two polynomials.

Proposition 3.7. *Let $R_{f,g}(x)$ be the resultant of two polynomials $f(x,y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ and $g(x,y) = b_0(x)y^p + b_1(x)y^{p-1} + \cdots + b_p(x)$. Assume that f is irreducible and distinguished. Then*

$$\operatorname{ord} R_{f,g}(x) = v_f(g).$$

Proof. By Corollary 2.3 there exist power series $\alpha(s), b_1(s), \ldots, \beta_n(s) \in \mathbb{K}[[s]]$ without constant term such that $f(\alpha(s), y) = \prod_{j=1}^{n}(y - \beta_j(s))$. From the definition of resultant we get $R_{f,g}(\alpha(s)) = \pm\prod_{j=1}^{n} g(\alpha(s), \beta_j(s))$ and consequently $\operatorname{ord} R_{f,g}(\alpha(s)) = \sum_{j=1}^{n}\operatorname{ord} g(\alpha(s), \beta_j(s)) = (\operatorname{ord}\alpha(s))v_f(g)$ by Lemma 3.6 and $\operatorname{ord} R_{f,g} = v_f(g)$ since $\operatorname{ord} R_{f,g}(\alpha(s)) = (\operatorname{ord} R_{f,g})\operatorname{ord}\alpha(s)$. $\square$

Now let $f \in \mathbb{K}[[x,y]]$ be an arbitrary non-zero power series without constant term and let $f = \prod_{i=1}^{r} f_i$ be the decomposition of f into irreducible factors. We define $i_0(f,g) = \sum_{i=1}^{r} v_{f_i}(g)$. Moreover if $f(0) \neq 0$ then we put $i_0(f,g) = 0$ and if $f \equiv 0$: $i_0(f,g) = \infty$. From the properties of v_f (Propositions 3.1, 3.3, 3.5) we get the fundamental properties of $i_0(f,g)$ (if $f(0)=g(0)=0$ then $i_0(f,g)$ is called intersection multiplicity of the curves $f=0$ and $g=0$).

Proposition 3.8. *For any $f,g,g' \in \mathbb{K}[[x,y]]$:*

(i) $0 \leqslant i_0(f,g) \leqslant \infty$, $i_0(f,g)=0$ *if and only if* $f(0)\neq 0$ *or* $g(0)\neq 0$; $i_0(f,g)=\infty$ *if and only if* f,g *have a common factor in* $\mathbb{K}[x,y]]$,
(ii) $i_0(f,gg')=i_0(f,g)+i_0(f,g')$,
(iii) $i_0(f,g+hf)=i_0(f,g)$ *for every* $h\in\mathbb{K}[[x,y]]$,
(iv) $i_0(f,g)=i_0(g,f)$,
(v) $i_0(f,g) \geqslant (\operatorname{ord} f)(\operatorname{ord} g)$; *the equality holds if and only if the curves* $f=0$ *and* $g=0$ *do not have a common tangent.*

From Proposition 3.7 we get easily the following:

Proposition 3.9. *If* $f(x,y)=y^n+a_1(x)y^{n-1}+\cdots+a_n(x)$ *is distinguished,* $g(x,y)=b_0(x)y^p+b_1(x)y^{p-1}+\cdots+b_p(x)$ *and* $R_{f,g}(x)$ *is their* y*-resultant, then* $\operatorname{ord} R_{f,g}(x)=i_0(f,g)$.

We can give here an axiomatic characterization of the intersection multiplicity (see Kałużny-Spodzieja [6]).

Theorem 3.10. *Let* $I:\mathbb{K}[[x,y]]\times\mathbb{K}[[x,y]]\to\mathbb{N}\cup\{\infty\}$ *be a function with properties*

(1) $I(f,g)=I(g,f)$,
(2) $I(f,g_1g_2)=I(f,g_1)+I(f,g_2)$,
(3) $I(f,g)=I(f,g+hf)$,
(4) $I(x,y)\neq 0,\infty$

Then $I(f,g)=i_0(f,g)I(x,y)$.

Clearly properties *(1)* and *(2)* imply

(2') $I(f_1f_2,g)=I(f_1,g)+I(f_2,g)$.

To prove Theorem 3.10 we need the following lemma.

Lemma 3.11. *If* I *is a function such as in Theorem 3.10 then the following properties hold:*

(5) *if* f *or* g *is a unit then* $I(f,g)=0$,
(6) *if* f *and* g *have a common divisor of positive order then* $I(f,g)=\infty$.

Proof. (of Lemma 3.11) To check property *(5)* note that using properties *(2')* and *(3)* we get

$$I(x,y)=I(1,y)+I(x,y)=I(1,y+(-y)1)+I(x,y)=I(1,0)+I(x,y)$$

and

$$I(1,0)+I(x,y)=I(1,g+(-g)1)+I(x,y)=I(1,g)+I(x,y).$$

Using the above equalities we get $I(x,y)=I(1,g)+I(x,y)$ hence $I(1,g)=0$ since $I(x,y)\neq 0,\infty$.

If $f(0) \neq 0$ then we have

$$0 = I(1, g) = I\left(f\left(\frac{1}{f}\right), g\right) = I\left(g, f\left(\frac{1}{f}\right)\right) = I(g, f) + I\left(g, \frac{1}{f}\right).$$

Hence $I(g, f) = 0$ and consequently $I(f, g) = 0$.

To check *(6)* consider a power series h such that $h(0) = 0$. We can write $h = xh_1 + yh_2$ in $\mathbb{K}[[x, y]]$ and

$$I(h, 0) = I(h, 0 \cdot x) = I(h, 0) + I(h, x) = I(h, 0) + I(xh_1 + yh_2, x).$$

From properties *(1)* and *(3)* we get that $I(xh_1 + yh_2, x) = I(yh_2, x)$ and

$$\begin{aligned} I(h, 0) &= I(h, 0) + I(yh_2, x) = \\ &= I(h, 0) + I(y, x) + I(h_2, x) = I(h, 0) + I(x, y) + I(h_2, x). \end{aligned}$$

Hence $I(h, 0) = \infty$ since $I(x, y) \neq 0, \infty$.

Now suppose that f and g have a common divisor h, $h(0) = 0$. So we have $f = f_1 h$, $g = g_1 h$ in $\mathbb{K}[[x, y]]$ and we get

$$I(f, g) = I(f_1, g_1 h) + I(h, g_1 h) = I(f_1, g_1 h) + I(h, 0) = \infty.$$

□

Remark 3.12 From property *(5)* it follows that $I(f, g) = I(uf, vg)$ for any units u, v.

Now we can give the proof of Theorem 3.10.

Proof. (of Theorem 3.10.) If $i_0(f, g) = \infty$ then f and g have a common factor of positive order and $I(f, g) = \infty$ by property *(6)*.

It suffices to check that if f, g are coprime then $I(f, g) = i_0(f, g)I(x, y)$. We will prove this equality by induction with respect to $i_0(f, g)$. If $i_0(f, g) = 0$ then f or g is a unit and $I(f, g) = 0$ by property *(5)*.

Let $k > 0$ be an integer and suppose that the equality $I(f, g) = i_0(f, g)I(x, y)$ is true for every pair f, g such that $i_0(f, g) < k$. If the series f or g is reducible then the equality $I(f, g) = i_0(f, g)I(x, y)$ is true: we use properties *(2)* and *(2')* of function I and the induction hypothesis. Thus it suffices to consider the case where f, g are irreducible and $i_0(f, g) = k$. If a power series h is irreducible then $h \sim x$ or $h \sim y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$, where $y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ is a distinguished polynomial. We have to consider three cases:

(1) $f(x, y) = x$, $g(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ is a distinguished polynomial. Then $i_0(f, g) = n$ and $I(f, g) = I(x, y^n) = nI(x, y) = i_0(f, g)I(x, y)$.
(2) $f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$, $g(x, y) = x$. We use the first case and symmetry of I, i_0.
(3) $f(x, y) = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$, $g(x, y) = y^p + b_1(x)y^{p-1} + \cdots + b_p(x)$ are distinguished polynomials of degrees $n, p > 0$. Without diminishing the

generality we may suppose that $p \geqslant n$. Then we may write $g = y^{p-n}f + xh$ in $\mathbb{K}[[x,y]]$ and consequently

$$I(f,g) = I(f, y^{p-n}f + xh) = I(f,x) + I(f,h) = nI(x,y) + I(f,h)$$

since $I(f,x) = nI(x,y)$ by Case 2.
To finish the proof it suffices to check the formula $I(f,h) = i_0(f,h)I(x,y)$. If $h(0) = 0$ then this equality follows from the induction hypothesis since $i_0(f,h) < i_0(f,g) = k$. If $h(0) \neq 0$ then the both sides of this equality are 0.

□

As the first application of the theorem proved above we give the following property.

Proposition 3.13. *Let f, g be coprime power series without constant term. Then for any power series $\Phi, \Psi \in \mathbb{K}[[u,v]]$ we have:*

$$i_0(\Phi(f,g), \Psi(f,g)) = i_0(\Phi, \Psi) i_0(f,g).$$

Proof. Let us consider the function I given by formula $I(\Phi,\Psi) = i_0(\Phi(f,g), \Psi(f,g))$. It is easy to see that the function I satisfies the conditions *(1)*, *(2)*, *(3)* and *(4)* of Theorem 3.10. Thus $I(\Phi,\Psi) = i_0(\Phi,\Psi)I(u,v) = i_0(\Phi,\Psi)i_0(f,g)$. □

For any power series $f, g \in \mathbb{K}[[x,y]]$ the ideal (f,g) generated by f and g is a $\mathbb{K}$-linear subspace of the algebra $\mathbb{K}[[x,y]]$.

Theorem 3.14 (Macauley's Formula)**.** *For every $f, g \in \mathbb{K}[[x,y]]$:*

$$i_0(f,g) = \dim_{\mathbb{K}} \mathbb{K}[[x,y]] / (f,g).$$

Proof. Let us denote by $I(f,g)$ the right side of the above equality (the codimension of the ideal generated by f, g). It is easy to see that the function I satisfies *(1)*, *(3)* and *(4)* of Theorem 3.10 and $I(x,y) = 1$. Thus to check the theorem it suffices to prove property *(2)*: $I(f, g_1g_2) = I(f,g_1) + I(f,g_2)$. If $I(f,g_1g_2) = \infty$ then f, g_1g_2 have a common prime divisor (see Appendix B). Then f, g_1 or f, g_2 have a common divisor and consequently $I(f,g_1) = \infty$ or $I(f,g_2) = \infty$.

Suppose that $I(f,g_1g_2) < \infty$ i.e. f, g_1g_2 are coprime. Recall the following fact of Linear Algebra. If U, V, W are $\mathbb{K}$-linear spaces such that $W \subset V \subset U$ and W have a finite codimension in U then

$$\dim_{\mathbb{K}} U/W = \dim_{\mathbb{K}} U/V + \dim_{\mathbb{K}} V/W.$$

Applying the above formula to $W = (f, g_1g_2)$, $V = (f, g_1)$ and $U = \mathbb{K}[[x,y]]$ we get $I(f,g_1g_2) = I(f,g_1) + I(f,g_2)$ since $\dim_{\mathbb{K}} V/W = I(f,g_2)$. □

Let $f, g \in \mathbb{K}[[x,y]]$ be power series without constant term. Let $\mathbb{K}((f,g))$ be the field of fractions of the ring $\mathbb{K}[[f,g]]$. Then $\mathbb{K}((f,g))$ is a subfield of the field $\mathbb{K}((x,y))$.

Theorem 3.15 (Weil's Formula). *If power series f, g without constant term are coprime then*

$$i_0(f,g) = \Big(\mathbb{K}((x,y)) : \mathbb{K}((f,g))\Big).$$

Proof. By Palamodov's Theorem (see Appendix D) the extension $\mathbb{K}[[x,y]] \supset \mathbb{K}[[f,g]]$ is a free module of rank $\dim_{\mathbb{K}} \mathbb{K}[[x,y]]/(f,g)$. Thus Theorem 3.15 follows from Theorem 3.14 and Lemma 2.7. □

4. Newton diagrams and parametrizations of algebroid curves

In this section we sketch an approach to Newton's study of plane curve singularities valid in arbitrary characteristic. A lucid and interesting introduction to Newton's method is due to Teissier [9]. See also Teissier [10] where a systematic treatment of the subject is given and Cassou-Noguès, Płoski [3] for applications to invariants of singularities.

Let $\mathbb{R}_+ = \{a \in \mathbb{R} : a \geqslant 0\}$. For any subsets $\Delta, \Delta' \subset \mathbb{R}^2_+$ we consider the Minkowski sum $\Delta + \Delta' = \{u + v : u \in \Delta \text{ and } v \in \Delta'\}$. For any subset $E \subset \mathbb{N}^2$ we denote by $\Delta(E)$ the convex hull of the set $E + \mathbb{R}^2_+$. The sets od the form $\Delta(E)$, where $E \subset \mathbb{N}^2$ are called Newton diagrams. We use Teissier's notation: $\left\{\frac{k}{l}\right\} = \Delta(\{(k,0),(0,l)\})$, $\left\{\frac{k}{\infty}\right\} = \Delta(\{(k,0)\}) = (k,0) + \mathbb{R}^2_+$, $\left\{\frac{\infty}{l}\right\} = \Delta(\{(0,l)\}) = (0,l) + \mathbb{R}^2_+$ for any integers $k, l > 0$. For any power series $f = \sum c_{\alpha\beta} x^\alpha y^\beta \in \mathbb{K}[[x,y]]$ we put $\operatorname{supp} f = \{(\alpha,\beta) \in \mathbb{N}^2 : c_{\alpha,\beta} \neq 0\}$. It is easy to check that $\operatorname{supp} fg \subset \operatorname{supp} f + \operatorname{supp} g$. The Newton diagram $\Delta_{x,y}(f)$ of a power series f is by definition $\Delta(\operatorname{supp} f)$. Note that if the coordinates (x,y) are generic i.e. $\operatorname{ord} f(x,0) = \operatorname{ord} f(0,y) = \operatorname{ord} f$ then $\Delta_{x,y}(f) = \left\{\frac{\operatorname{ord} f}{\operatorname{ord} f}\right\}$. The property of order: $\operatorname{ord} fg = \operatorname{ord} f + \operatorname{ord} g$ may be generalized as follows:

Lemma 4.1. $\Delta_{x,y}(fg) = \Delta_{x,y}(f) + \Delta_{x,y}(g)$.

Proof. The rule of multiplication of formal power series implies the following two properties:

(a) if $(\alpha,\beta) \in \operatorname{supp} fg$ then $(\alpha,\beta) = (\alpha_1,\beta_1) + (\alpha_2,\beta_2)$, where $(\alpha_1,\beta_1) \in \operatorname{supp} f$ and $(\alpha_2,\beta_2) \in \operatorname{supp} g$,

(b) if $(\alpha,\beta) \in \mathbb{N}^2$ has a unique representation $(\alpha,\beta) = (\alpha_1,\beta_1) + (\alpha_2,\beta_2)$ for some $(\alpha_1,\beta_1) \in \operatorname{supp} f$ and $(\alpha_2,\beta_2) \in \operatorname{supp} g$ then $(\alpha,\beta) \in \operatorname{supp} fg$.

To abbreviate the notation we write Δ instead of $\Delta_{x,y}$. Note first that the set $\Delta(f) + \Delta(g)$ being the sum of two convex subsets of $\mathbb{R}^2_+$ is convex. From (a) we get $\operatorname{supp} fg + \mathbb{R}^2_+ \subset (\operatorname{supp} f + \mathbb{R}^2_+) + (\operatorname{supp} g + \mathbb{R}^2_+) \subset \Delta(f) + \Delta(g)$ and consequently $\Delta(fg) \subset \Delta(f) + \Delta(g)$ since $\Delta(fg)$ is the smallest convex subset which contains $\operatorname{supp} fg + \mathbb{R}^2_+$.

On the other hand if (α,β) is a vertex of $\Delta(f)+\Delta(g)$ then (α,β) has property (b) and $(\alpha,\beta)\in \operatorname{supp} fg \subset \Delta(fg)$. Since the vertices of $\Delta(f)+\Delta(g)$ belong to $\Delta(fg)$ we get $\Delta(f)+\Delta(g)\subset\Delta(fg)$.

Summing up, we have $\Delta(fg)=\Delta(f)+\Delta(g)$. □

Proposition 4.2. *Let $f\in\mathbb{K}[[x,y]]$ be an irreducible power series. Then*

$$\Delta_{x,y}(f)=\left\{\frac{i_0(f,y)}{i_0(f,x)}\right\}.$$

Proof. If $f\sim x$ or $f\sim y$ then the proposition is obvious. Let $f(x,0)f(0,y)\neq 0$ and put $m=\operatorname{ord} f(x,0)$, $n=\operatorname{ord} f(0,y)$. Since $\Delta_{x,y}(f)=\Delta_{x,y}(fu)$ for any unit u we may assume by the Weierstrass Preparation Theorem that $f=y^n+a_1(x)y^{n-1}+\cdots+a_n(x)$ is a distinguished polynomial. Let $(\phi(t),\psi(t))$ be a normalization of the branch $f=0$. Then $\operatorname{ord}\phi(t)=i_0(f,x)=n$ and $\operatorname{ord}\psi(t)=i_0(f,y)=m$. By Corollary 2.3 there are nonzero power series $\alpha(s),\beta_1(s),\dots,\beta_n(s)\in\mathbb{K}[[s]]$ without constant term such that

$$y^n+a_1(\alpha(s))y^{n-1}+\cdots+a_n(\alpha(s))=(y-\beta_1(s))\cdots(y-\beta_n(s)).$$

We have $\alpha(s)=\phi(\sigma_j(s))$, $\beta_j(s)=\psi(\sigma_j(s))$ for a $\sigma_j(s)$ without constant term. Thus we get $\operatorname{ord}\beta_j(s)=\frac{\operatorname{ord}\psi}{\operatorname{ord}\phi}\operatorname{ord}\alpha=\frac{m}{n}\operatorname{ord}\alpha$ for $j=1,\dots,n$. Let $k\in[1,n]$ be such that $a_k(x)\neq 0$. Then $a_k(\alpha(s))=(-1)^k(\beta_1(s)\cdots\beta_k(s)+\cdots)$ and $\operatorname{ord} a_k(\alpha(s))\geqslant \inf\{\operatorname{ord}\beta_{j_1}\cdots\beta_{j_k} : 1\leqslant j_1<\cdots<j_k\leqslant n\}=k\frac{m}{n}\operatorname{ord}\alpha$, which implies $\frac{\operatorname{ord} a_k}{k}\geqslant\frac{m}{n}=\frac{i_0(f,y)}{i_0(f,x)}$ with equality for $k=n$. This proves the proposition. □

Now we can pass to the main result of this section

Theorem 4.3. *Let $f\in\mathbb{K}[[x,y]]$ be a nonzero formal power series without constant term and let $f=f_1\cdots f_r$ in $\mathbb{K}[[x,y]]$ with irreducible f_i, $i=1,\dots,r$. Let $(\phi_i(t_i),\psi_i(t_i))$ be a normalization of the branch $f_i=0$ for $i=1,\dots,r$. Then*

$$\Delta_{x,y}(f)=\sum_{i=1}^{r}\left\{\frac{\operatorname{ord}\psi_i}{\operatorname{ord}\phi_i}\right\}.$$

Proof. By Lemma 4.1 we get $\Delta_{x,y}(f)=\sum_{i=1}^{r}\Delta_{x,y}(f_i)$. On the other hand by Proposition 4.2 and the Normalization Theorem we have $\Delta_{x,y}(f_i)=\left\{\frac{\operatorname{ord}\psi_i}{\operatorname{ord}\phi_i}\right\}$ for $i=1,\dots,r$. □

Appendix

Let $\mathbb{K}$ be an arbitrary field not necessarily algebraically closed.

A. Factorization Lemma. *Suppose that a power series $f \in \mathbb{K}[[x,y]]$ satisfies the condition* $\operatorname{in} f = \phi\psi$*, where ϕ, ψ are coprime homogeneous forms of positive degree. Then there exist $g, h \in \mathbb{K}[[x,y]]$ such that $f = gh$ in $\mathbb{K}[[x,y]]$, where* $\operatorname{in} g = \phi$, $\operatorname{in} h = \psi$.

The proof of the lemma is based on the following property:

Macauley's property *If $\phi, \psi \in \mathbb{K}[x,y]$ are coprime homogeneous forms of degree $m > 0$ and $n > 0$ then every homogeneous form of degree $\geqslant m+n-1$ can be written as $\alpha\phi + \beta\psi$, where α, β are homogeneous forms.*

Proof. Every homogeneous form χ of degree $\geqslant m+n-1$ can be written as $\sum_{i+j=m+n-1} \chi_{ij} x^i y^j$, so it suffices to check Macaulay's property for forms of degree $m+n-1$. Let H_k be the $\mathbb{K}$-linear space of homogeneous forms of degree k (by convention the zero is a homogeneous form of degree k for all k). The mapping

$$H_{n-1} \times H_{m-1} \ni (\alpha, \beta) \mapsto \alpha\phi + \beta\psi \in H_{m+n-1}$$

is a linear mapping of vector spaces of the same dimension $m+n$. Since the forms ϕ, ψ are coprime the mapping is injective. Hence, the mapping is also surjective. $\square$

Proof of Factorization Lemma. Write $f = f_{m+n} + f_{m+n+1} + \cdots$. We are looking for power series g and h in the form $g = \phi_m + \phi_{m+1} + \cdots$ and $h = \psi_n + \psi_{n+1} + \cdots$, where $\phi_m = \phi$ and $\psi_n = \psi$. The equality $f = gh$ holds if and only if the following conditions are fulfilled

$$\begin{aligned}
&\phi_m\psi_n = f_{m+n}\\
&\phi_{m+1}\psi_n + \phi_m\psi_{n+1} = f_{m+n+1}\\
&\phi_{m+2}\psi_n + \phi_{m+1}\psi_{n+1} + \phi_m\psi_{n+2} = f_{m+n+2}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Applying Macauley's property to the given $\phi_m = \phi$, $\psi_n = \psi$ and utilizing the above equations, first we find the forms ϕ_{m+1}, ψ_{n+1}, then the forms ϕ_{m+2}, ψ_{n+2},.... Proceeding in this way we get step by step the homogeneous components of g and h. $\square$

B. Elimination Lemma. *Let $f, g \in \mathbb{K}[[x,y]]$ be non-zero power series without constant term. Then f, g are coprime if and only if the following condition holds*

(∗) *there exist integers $d, d' > 0$ such that the monomials x^d, $y^{d'}$ lie in the ideal (f,g) generated by f and g in $\mathbb{K}[[x,y]]$.*

Proof. If $x^d, y^{d'} \in (f,g)$ then every divisor of f and g divides x^d and $y^{d'}$ so f, g are coprime. Suppose that f and g are coprime. Then $f(0,y) \neq 0$ or $g(0,y) \neq 0$

since if $f(0,y) = g(0,y) = 0$ in $\mathbb{K}[[y]]$ then x divides f and g. Suppose that $f(0,y) \neq 0$. Using the Weierstrass Preparation Theorem we may assume that $f = y^n + a_1(x)y^{n-1} + \cdots + a_n(x)$ is a distinguished polynomial. Replacing g by the remainder of division by f, we get $g = b_0(x)y^{n-1} + \cdots + b_{n-1}(x)$. Let $R(x)$ be the y-resultant of polynomials f, g. Then f, g are coprime as elements of $\mathbb{K}[[x]][y]$ and consequently $R(x) \neq 0$. Let $d = \operatorname{ord} R(x)$. We get $x^d \in (f,g)$ since the resultant lies in the ideal generated by f and g. Similarly we check that $y^{d'} \in (f,g)$ for an integer $d' > 0$. □

C. Prime ideals in the ring $\mathbb{K}[[x,y]]$. *Prime ideals in the ring $\mathbb{K}[[x,y]]$ are: (0), maximal ideal $\mathcal{M} = (x,y)$ and principal ideals (f) generated by irreducible power series $f \in \mathbb{K}[[x,y]]$.*

Proof. Let I be a non-zero prime ideal of the ring $\mathbb{K}[[x,y]]$. Since the ring of power series is a unique factorization domain there exists an irreducible power series $f \in I$. If $I \neq (f)$ then there exists a power series $g \in I$ such that f does not divide g and hence the power series f, g are coprime. By the Elimination Lemma we get $x^d, y^{d'} \in (f,g) \subset I$ which implies $x, y \in I$ i.e. $I = (x,y)$ and we are done. □

From the description of prime ideals it follows that the Krull dimension of $\mathbb{K}[[x,y]]$ is equal to 2.

D. Parameters of the ring $\mathbb{K}[[x,y]]$. Every ideal I of the ring $\mathbb{K}[[x,y]]$ is a $\mathbb{K}$-linear subspace of $\mathbb{K}[[x,y]]$ and its codimension $\operatorname{codim} I = \dim_{\mathbb{K}} \mathbb{K}[[x,y]]/I$ is defined. The powers of the maximal ideal $\mathcal{M}^k = (x^k, x^{k-1}y, \ldots, xy^{k-1}, y^k)$ have a finite codimension $\operatorname{codim} \mathcal{M}^k = \frac{1}{2}k(k+1)$. It is easy to see that $\operatorname{codim} I < \infty$ if and only if $I \supset \mathcal{M}^k$ for some $k \geqslant 0$ i.e. if I contains all monomials of degree big enough. A pair of power series f, g without constant term is a system of parameters (s.p.) of the ring $\mathbb{K}[[x,y]]$ if the ideal (f,g) has a finite codimension. This takes place if and only if x^d, $y^{d'} \in (f,g)$ for some $d, d' > 0$. Hence, from the Elimination Lemma it follows that a pair of power series f, g without constant term is a s.p. if and only if the series f, g are coprime.

Palamodov's Theorem *Let f, g be a s.p. of the ring $\mathbb{K}[[x,y]]$. Then $\mathbb{K}[[x,y]]$ is a finitely generated free module over $\mathbb{K}[[f,g]]$ whose rank is equal to the codimension of the ideal (f,g).*

Proof. Let m be the codimension of the ideal $I = (f,g)$ and let $e_1, \ldots, e_m$ be a sequence of power series such that the images of $e_1, \ldots, e_m$ under the natural epimorphism $\mathbb{K}[[x,y]] \to \mathbb{K}[[x,y]]/I$ form a $\mathbb{K}$-linear basis of $\mathbb{K}[[x,y]]/I$. For any $h \in \mathbb{K}[[x,y]]$ there exist constants $c_1, \ldots, c_m \in \mathbb{K}$ such that $h \equiv c_1e_1 + \cdots + c_me_m \pmod{I}$. We put $A_i^0(u,v) = c_i$ for $i = 1, \ldots, m$. We get then

$$h = \sum_{i=1}^{m} c_i e_i + h_1 f + h_2 g \text{ in } \mathbb{K}[[x, y]]$$

and

$$h_1 \equiv \sum_{i=1}^{m} c_{1i} e_i \mod (f, g),$$
$$h_2 \equiv \sum_{i=1}^{m} c_{2i} e_i \mod (f, g).$$

From the above relations we get:

$$h \equiv \sum_{i=1}^{m} c_i e_i + \sum_{i=1}^{m} (c_{1i} f) e_i + \sum_{i=1}^{m} (c_{2i} g) e_i \mod (f, g)^2.$$

Let $A_i^1(u, v) = c_i + c_{1i} u + c_{2i} v$; so we get

$$h \equiv \sum_{i=1}^{m} A_i^1(f, g) e_i \mod (f, g)^2.$$

In this way we define by induction the sequences of polynomials $A_i^k = A_i^k(u, v)$ $(i = 1, \dots, m, \ \ k = 0, 1, \dots, m)$ such that:

(1) $h \equiv \sum_{i=1}^{m} A_i^k(f, g) e_i \mod (f, g)^{k+1}$,

(2) A_i^k is a polynomial of degree $\leqslant k$; $A_i^{k+1} - A_i^k$ is a homogeneous form of degree $k + 1$.

Let us put $A_i = \sum_{k \geqslant 0} (A_i^{k+1} - A_i^k) + c_i$ for $i = 1, \dots, m$. It is easy to show that

$$h = \sum_{i=1}^{m} A_i(f, g) e_i.$$

It remains to check that the above representation is unique. It suffices to prove that

$$\sum_{i=1}^{m} A_i(f, g) e_i = 0 \quad \Rightarrow \quad A_i(u, v) = 0 \text{ in } \mathbb{K}[[u, v]] \text{ for } i = 1, \dots, m.$$

Let us suppose, to get a contradiction, that the set $I_0 = \{i : A_i(u, v) \neq 0\}$ is not empty. We get

$$\sum_{i \in I_0} A_i(0, 0) e_i \equiv 0 \mod (f, g)$$

hence $A_i(0, 0) = 0$ for $i \in I_0$. Dividing $A_i(u, v)$ by a sufficiently large power of u we may assume that $r = \inf\{\operatorname{ord} A_i(0, v)\} < \infty$. We get $A_i(u, v) = A_i(0, v) + u q_i(u, v) = v^r c_i(v) + u q_i(u, v)$, where not all $c_i(0)$ are equal zero.

So we have

$$\sum_{i=1}^{m} g^r c_i(g)e_i + \sum_{i=1}^{m} f q_i(f,g)e_i = 0$$

and

$$g^r \left(\sum_{i=1}^{m} c_i(g)e_i \right) \equiv 0 \mod (f).$$

The power series f, g are coprime because they form a s.p. Therefore from the last relation we obtain

$$\sum_{i=1}^{m} c_i(g)e_i \equiv 0 \mod (f)$$

and

$$\sum_{i=1}^{m} c_i(0)e_i \equiv 0 \mod (f,g)$$

so we get $c_i(0) = 0$ for all $i = 1, \ldots, m$, which is a contradiction.

An elementary treatment of parameters in power series ring in n variables is given in [7].

Acknowledgements: The author is very grateful to the anonymous referee for making many valuable suggestions.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 135 – 139

ABOUT CHOUIKHA'S ISOCHRONICITY CRITERION

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Abstract. Recently A.R.Chouikha gave a new characterization of isochronicity of center at the origin for the equation $x'' + g(x) = 0$, where g is a real smooth function defined in some neighborhood of $0 \in \mathbb{R}$. We present some new development of the subject. The present text is a short account of my paper "On Chouikha's isochronicity criterion", *arXiv:1201.6503*, where the proofs can be found. We correct the formulation of some results from the above paper.

Let us consider the second order differential equation

$$x'' + g(x) = 0 \tag{1}$$

where g is a real function defined in some neighborhood of $0 \in \mathbb{R}$ such that $g(0) = 0$, or equivalently the planar system

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= -g(x) \end{aligned}\right\} . \tag{2}$$

In what follows we shall exclusively concentrate on the system (2) with function g at least of class C^1.

As $g(0) = 0$, $0 \in \mathbb{R}^2$ is a singular point of the system (2). If in some neighborhood of a singular point all orbits of the system are closed and surround it, then the singular point is called a *center*.

A center is called *isochronous* if the periods of all orbits in some neighborhood of it are constant.

In future when speaking about isochronicity we always understand it with respect to $0 \in \mathbb{R}^2$ and the system (2).

The problem of characterization of isochronicity of the system (2) at $0 \in \mathbb{R}^2$ in term of function g is an old one.

2010 *Mathematics Subject Classification.* Primary 34C15, 34C25, 34C37.
Key words and phrases. Center, isochronicity, Urabe function.

To the best of our knowledge the first such characterization was done in 1937 by I.Kukles and N.Piskunov in [3], where even the case of continuous functions g is considered. The second one was described in 1962 by M.Urabe in [5] (see also [4]). Unfortunately these characterizations are not easy to handle and they are not really explicit.

We shall denote

$$G(x) = \int_0^x g(u)\,du. \tag{3}$$

Let us denote by X the continuous function defined in some neighborhood of $0 \in \mathbb{R}$ by

$$(X(x))^2 = 2\,G(x) \text{ and } xX(x) > 0 \; for \; x \neq 0. \tag{4}$$

Let us formulate now *Urabe Isochronicity Criterion.*

Theorem 1 ([5]). *Let g be a C^1 function defined in some neighborhood of $0 \in \mathbb{R}$. Let $g(0) = 0$ and $g'(0) = \lambda^2, \lambda > 0$. Then $0 \in \mathbb{R}^2$ is an isochronous center for the system (2) if and only if*

$$g(x) = \lambda \frac{X(x)}{1 + h(X(x))} \tag{5}$$

where the function X is defined by (4) and where h is a continuous odd function defined in some neighborhood of $0 \in \mathbb{R}$.

The function h is called *Urabe function* of the system (2).

Let us note that $\omega = \dfrac{2\pi}{\lambda}$ is the period of orbits of the above isochronous center.

Let us stress that from (3) and from assumptions on g in Urabe theorem it follows that $G(0) = 0$ and that in some punctured neighborhood of 0, $G(x) > 0$, G is of class C^2. Under our assumptions one proves that X is of class C^1. In fact, if $g \in C^k, k \geq 1$ (resp. g is real-analytic), then X is of class $C^k, X'(0) = \lambda > 0$ and h is of class C^{k-1} (resp. X and h are real-analytic).

From now on we shall always assume that $g \in C^1(]-\epsilon, \epsilon[))$ for some $\epsilon > 0$ and that

$$g'(0) = \lambda^2, \; \lambda > 0.$$

In September 2011 in a highly important paper [1], A.R.Chouikha published a completely new criterion of isochronicity ([1], Theorem B) which is much more direct and explicit that all previously known.

Theorem 2 ([1]). *Let $g \in C^1(]-\epsilon, \epsilon[)$ for some $\epsilon > 0$. Let $g(0) = 0$ and $g'(0) > 0$. If there exists δ, $0 < \delta \leq \epsilon$, such that for $|x| \leq \delta$ one has*

$$\frac{d}{dx}\left[\frac{G(x)}{g^2(x)}\right] = f(G(x)) \tag{6}$$

where f is a continuous functions defined on some interval $[0, \eta]$, *where* $\eta > 0$, *then* $0 \in \mathbb{R}^2$ *is an isochronous center for the system* (2).

If $g \in C^2(]-\epsilon, \epsilon[)$ *and* $0 \in \mathbb{R}^2$ *is an isochronous center for the system* (2), *then the condition* (6) *is satisfied.*

Consequently, if $g \in C^2(]-\epsilon, \epsilon[)$, *then* $0 \in \mathbb{R}^2$ *is an isochronous center for the system* (2) *if and only if the condition* (6) *is satisfied.*

We shall call the equation (6) the *Chouikha equation* and the function f is called *Chouikha function* of the system (2).

Let us pause now in the history of this theorem. In early February 2010, A.R. Chouikha communicated to me his *first* proof of his theorem valid only in real-analytic setting. Some time after he presented to me the *second* proof also valid only in real-analytic setting. The first proof was based on Urabe theorem, the second one on S.N. Chow and D. Wang [2] formula for the derivative of the first return map for the system (2). These proofs were not published at the time. At the beginning of July 2011, A.R. Chouikha and myself, simultaneously and independently obtained two different proofs of Chouikha theorem in smooth setting. Both proofs are the adaptation of the previous Chouikha's proofs in real-analytic setting. The Chouikha's proof published in [1] is the adaptation of his second proof. My proof is the adaptation of his first proof.

As a consequence of this last proof we obtain an unexpected closed relation between Urabe function h and Chouikha function f.

Theorem 3.

$$h(s) = \lambda \int_0^s f(\frac{q^2}{2}) dq, \tag{7}$$

where $g'(0) = \lambda^2$, $\lambda > 0$. *Thusf is real-analytic (resp. of class* C^∞*) if and only if* h *is real-analytic (resp. of class* C^∞*).*

From now on we shall suppose that $f \in C^1([0, \epsilon]), \epsilon > 0$, where in 0 and in ϵ one considers the one-sided first derivatives. As before $g \in C^1(]-\delta, \delta[)$, $\delta > 0$.

Theorem 4. *Let* $\epsilon > 0$ *and* $\lambda > 0$. *Let* $f \in C^1([0, \epsilon])$. *There exists* δ, $0 < \delta \leq \epsilon$ *and a unique function* $g \in C^1(]-\delta, \delta[)$, $g'(0) = \lambda^2$ *such that for every* $|x| < \delta$ *the Chouikha equation* (6)

$$\frac{d}{dx}\left[\frac{G(x)}{g^2(x)}\right] = f(G(x))$$

is satisfied.

Let us stress that if $f_1, f_2 \in C^1([0, \epsilon]), \epsilon > 0$, and $f_1 \neq f_2$ on every interval $[0, \eta]$, $0 < \eta \leq \epsilon$, then in any neighborhood of $0 \in \mathbb{R}$, $g_1 \neq g_2$, where g_1 and g_2 are the solutions of Chouikha equation that correspond to f_1 and to f_2 respectively.

Let us also note that if one supposes that $f \in C^k([0, \epsilon]), 1 \leq k \leq \infty$, or f is real-analytic, then the unique solution g of Chouikha equation is also of the same class. This gives a new light on the matter of Sec.4 of [1], proving the convergence of power series which appear there.

From now on we shall only consider the case of real-analytic or C^∞ functions g. Let us suppose that for function g, $0 \in \mathbb{R}^2$ is an isochronous center for the system (2).

In the real-analytic case there exists a natural bijective correspondence between the set of the couples of real-analytic functions f defined in some neighborhood of $0 \in \mathbb{R}$ and of real numbers $\lambda > 0$ with the set of the real-analytic functions g such that $0 \in \mathbb{R}^2$ is an isochronous center for the system (2). Indeed, to real-analytic function f defined in some neighborhood of $0 \in \mathbb{R}$ and to real number $\lambda > 0$ we associate the unique real-analytic function g such that $g(0) = 0,\ g'(0) = \lambda^2$ which is a solution of Chouikha equation, the existence of which is given by Theorem 4. Let us stress that the completely analogous statement is valid also in C^∞ framework.

As a consequence of Theorem 4 and of Theorem 3 we obtain a fact that seems to have been completely overlooked until now.

Theorem 5. *To every odd real-analytic (resp. of class C^∞) function h defined in some neighborhood of $0 \in \mathbb{R}$ and to every real number $\lambda > 0$ there corresponds a unique real-analytic (resp. of class C^∞) function g defined in some neighborhood of $0 \in \mathbb{R},\ g(0) = 0,\ g'(0) = \lambda^2$ such that $0 \in \mathbb{R}^2$ is an isochronous center for the system* (2) *and that h is its Urabe function.*

Let us denote by $Isochr(0, \omega)$ the germs of isochronous centers of the equation $x'' + g(x) = 0$ where g is a real-analytic function defined in some neighborhood of $0 \in \mathbb{R}$, $g(0) = 0$, $g'(0) > 0$. Let us denote by C_0^ω the germs of real-analytic functions defined in some neighborhood of $0 \in \mathbb{R}$. We can then state:

Theorem 6. *The Cartesian product $C_0^\omega \times \{x \in \mathbb{R}; x > 0\}$ and the set $Isochr(0, \omega)$ are in natural bijective correspondence. In other words the germs of real-analytic functions defined in some neighborhood of $0 \in \mathbb{R}$ and the strictly positive real numbers parametrize the germs of isochronous centers at 0 of equation $x'' + g(x) = 0$, with g a real-analytic function defined in some neighborhood of $0 \in \mathbb{R},\ g(0) = 0,\ g'(0) > 0$.*

Let us stress that the completely analogous statement to Theorem 6 is valid also in C^∞ framework.

Acknowledgment

I thank A.Raouf Chouikha (University Paris 13) for many years of helpful discussions on isochronous centers. I thank also Andrzej Maciejewski and Maria Przybylska (both from University of Zielona Gora, Poland) and Alain Albouy (Observatoire de Paris) for interesting discussions and important advices. Last but not least I thank Marie-Claude Werquin (University Paris 13) for linguistic corrections.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 141 – 153

JUMPS OF MILNOR NUMBERS IN FAMILIES OF NON-DEGENERATE AND NON-CONVENIENT SINGULARITIES

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Abstract. The non-degenerate jump of the Milnor number of an isolated singularity f_0 is the minimal non–zero difference between the Milnor numbers of f_0 and one of its non-degenerate deformations (f_s). In the paper the results by Bodin and the author (concerning the non-degenerate jump) are generalized to non-convenient singularities.

1. Introduction

Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be *an isolated singularity*, i.e. f_0 is the germ of a holomorphic function having an isolated critical point at 0. In the sequel a *singularity* means an isolated singularity.

A deformation of f_0 is a family $(f_s)_{s\in U}$ of isolated singularities (or smooth germs) analytically dependent on the parameter s in an open neighborhood U of $0 \in \mathbb{C}$. Let $\mu(f_s)$ denote the Milnor number of f_s. By the upper semi-continuity of $\mu(f_s)$ with respect to the Zariski topology [see [4], Prop. 2.57] the difference

$$\mu(f_0) - \mu(f_s), \qquad s \neq 0,$$

is non-negative and independent of $s \neq 0$ in a sufficiently small neighborhood of $0 \in \mathbb{C}$. We call it *the jump of Milnor numbers of the deformation* $(f_s)_{s\in U}$ and denote $\lambda((f_s))$.

The jump $\lambda(f_0)$ (or *the first jump*) is the minimum of non-zero jumps over all deformations (f_s) of f_0. Gusein-Zade proved in [3] that there exist singularities f_0 for which $\lambda(f_0) > 1$ and that for irreducible plane curve singularities it holds

2010 *Mathematics Subject Classification.* Primary 32S30, Secondary 14B07.

Key words and phrases. Deformation of singularity, Milnor number, Newton polygon, non-degenerate singularity.

$\lambda(f_0) = 1$. The paper concerns the non-degenerate jump of the Milnor number i.e. the case when deformations (f_s) consist of only non-degenerate singularities. First, we recall the needed notions.

Put $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let

$$f_0(x, y) = \sum_{(i,j)\in\mathbb{N}^2} a_{ij} x^i y^j \in \mathbb{C}\{x, y\}.$$

Put

$$\operatorname{supp}(f_0) := \{(i, j) \in \mathbb{N}^2 : a_{ij} \neq 0\}.$$

The *Newton diagram* of f_0 is the convex hull of

$$\bigcup_{(i,j)\in\operatorname{supp}(f_0)} \left((i, j) + \mathbb{R}^2_+\right), \quad \text{where} \quad \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \wedge y \geq 0\}.$$

We will denote it by $\Gamma_+(f_0)$. The boundary of the Newton diagram $\Gamma_+(f_0)$ is the union of two semilines and a finite set (may be empty) of compact, non-parallel segments. These segments constitute *the Newton polygon of* f_0, which we will denote by $\Gamma(f_0)$. They can be ordered in a natural way from the highest segment (closest to the vertical axes) to the lowest one. Often we will identify pairs $(i, j) \in \mathbb{N}^2$ with monomials $x^i y^j$. The singularity f_0 is *convenient*, if $\Gamma(f_0)$ has common points with OX and OY axes.

For a segment $\gamma \in \Gamma(f_0)$ we define

$$(f_0)_\gamma := \sum_{(i,j)\in\gamma} a_{ij} x^i y^j.$$

A singularity f_0 is *non-degenerate on* $\gamma \in \Gamma(f_0)$ (in the Kouchnirenko sense), if the system of equations

$$\frac{\partial (f_0)_\gamma}{\partial x}(x, y) = 0, \ \frac{\partial (f_0)_\gamma}{\partial y}(x, y) = 0$$

has no solutions in $\mathbb{C}^* \times \mathbb{C}^*$. We call a singularity f_0 *non-degenerate*, when f_0 is non-degenerate on every segment $\gamma \in \Gamma(f_0)$.

Let f_0 be a convenient singularity. By S we denote the area of the set bounded by OX and OY axes and the polygon $\Gamma(f_0)$. By a and b we denote the distances between the origin $(0, 0)$ and the common part of Newton polygon $\Gamma_+(f_0)$ with OX and OY axes, respectively.

We define the *Newton number* of f_0 by

$$\nu(f_0) := 2S - a - b + 1.$$

Let f_0 be a singularity. A deformation $(f_s)_{s\in U}$ of f_0 is called *non-degenerate* if f_s is non-degenerate for every $s \neq 0$ sufficielntly close to the origin. We will denote by $\mathcal{D}^{nd}(f_0)$ the set of all non-degenerate deformations of the singularity f_0. *The* *non-degenerate jump* $\lambda'(f_0)$ *of a singularity* f_0 is the minimum of non-zero jumps over all non-degenerate deformations (f_s) of f_0, i.e.

$$\lambda'(f_0) := \min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} \lambda((f_s)),$$

where by $\mathcal{D}_0^{nd}(f_0)$ we denote all the non-degenerate deformations (f_s) of f_0 for which $\lambda((f_s)) \neq 0$.

Now, we recall some results on the jump of convenient and non-degenerate singularities, which we will generalize to the non-convenient case. First, we define specific deformations of a convenient non-degenerate singularity f_0. Let $J(f_0)$ be the set of integer points (monomials) lying under the Newton polygon of f_0 except $(0,0)$. For any $(p,q) \in J(f_0)$ we define a deformation

$$f_s(x,y) = f_0(x,y) + sx^p y^q, \qquad s \in \mathbb{C},$$

and denote it by $(f_s^{(p,q)})$.

Theorem 1 (Bodin [1], Walewska [10])**.** *If* f_0 *is a non-degenerate and convenient singularity, then*

$$\lambda'(f_0) = \min_{(p,q)\in J_0(f_0)} \lambda((f_s^{(p,q)})),$$

where $J_0(f_0) \subset J(f_0)$ *is the set of points* $(p,q) \in J(f_0)$ *such that* $\lambda((f_s^{(p,q)})) \neq 0$.

Directly from the above theorem we have

Corollary 2. *If* f *and* $\tilde{f}$ *are two non-degenerate and convenient singularities, with the same Newton diagram, then* $\lambda'(f) = \lambda'(\tilde{f})$.

Using Theorem 1 Bodin gave the exact value of the non-degenerate jump of some singularities.

Theorem 3 (Bodin [1])**.** *Let* $f_0(x,y) = x^p - y^q$, *where* $p \geq q \geq 2$ *and let* d=GCD(p,q).

1. *If* $d < q$, *then* $\lambda'(f_0) = d$.
2. *If* $d = q$, *then* $\lambda'(f_0) = d - 1$.

In the first case the jump $\lambda'(f_0)$ *is realized by the deformation* $f_s^{(-b,q-a)}$, *where* $a, b \in \mathbb{Z}$ *are such that* $ap + bq = d$, *where* $0 < a < \dfrac{q}{d}$ *and* $b < 0$. *Moreover, the point* $(-b, q-a)$ *lies in an open triangle with vertices* $(0,q)$, $(0,0)$ *and* $(p,0)$.

In the second case the jump is realized by the deformation $f_s^{(p-1,0)}$.

Consider now a general case of a convenient and non-degenerate singularity f_0, whose Newton polygon consists of only one segment. Let $(p,0)$ and $(0,q)$ be the intersection points of the Newton polygon of f_0 with the axes OX and OY, respectively. From Corollary 2 and Theorem 3 we have the following

Theorem 4. *Let f_0 be a non-degenerate and convenient singularity, with the Newton polygon reduced to only one segment. Then this segment connects points $(p, 0)$ and $(0, q)$ for some $p, q \in \mathbb{N}$ such that $p, q \geq 2$. If $d := \mathrm{GCD}(p, q)$, then:*

1. *If $1 \leq d < \min(p, q)$, then $\lambda'(f_0) = d$,*
2. *If $d = \min(p, q)$, then $\lambda'(f_0) = d - 1$.*

Let f_0 be a non-degenerate and convenient singularity. Let

$$\Lambda'(f_0) = (\mu_0, \mu_1, \ldots, \mu_k)$$

be the strictly decreasing sequence of all possible Milnor numbers of all non-degenerate deformations (f_s) of f_0. In particular,

$$\mu_0 = \mu(f_0), \quad \mu_1 = \mu(f_0) - \lambda'(f_0), \quad \mu_k = 0.$$

From Theorem 4 we have a formula for μ_1 if f_0 is a singularity with one segment Newton polygon (in particular for irreducible f_0). The sequence $\Lambda'(f_0)$ may be strange. One can check that

1. for $f_0(x, y) = x^8 - y^5$, we have $\Lambda'(f_0) = (28, 27, \ldots, 0)$,
2. for $f_0(x, y) = x^8 - y^4$, we have $\Lambda'(f_0) = (21, 18, 17 \ldots, 0)$,
3. for $f_0(x, y) = x^7 - y^5$, we have $\Lambda'(f_0) = (24, 23, \ldots, 15, 13, 12, \ldots, 0)$.

Next theorem gives a formula for μ_2 for singularities with one segment Newton polygon.

Theorem 5 (Walewska [10])**.** *Let $f_0(x, y) = x^p - y^q$, $p \geq q \geq 2$, $p + q > 4$. Then $\mu_2 = \mu_1 - 1$, if μ_2 is defined.*

Consider now a general case of a singularity which Newton polygon consists of only one segment. From Corollary 2 and Theorem 5 we have the following

Theorem 6. *Let f_0 be a non-degenerate and convenient singularity whose Newton polygon consists of only one segment. If $\Lambda'(f_0) = (\mu_0, \mu_1, \ldots, \mu_k)$, $k \geq 2$, is the sequence of Milnor numbers associated to f_0, then $\mu_2 = \mu_1 - 1$.*

The main goal of this paper is to extend the above results to the case of non-convenient singularities.

2. Non-convenient singularities

A power series $f_0 \in \mathbb{C}\{x, y\}$ is *nearly convenient*, if the distance of the Newton diagram $\Gamma_+(f_0)$ to each axis of the coordinate system does not exceed 1. It is easy to notice that

Lemma 2.1. *If f_0 is a singularity, then f_0 is nearly convenient.*

Let f_0 be a singularity. Then f_0 is either convenient singularity or can be represented in one of the following forms

$$x\tilde{f}_1, \quad y\tilde{f}_2, \quad xy\tilde{f}_3, \tag{$\star$}$$

where $\tilde{f}_1$ and $\tilde{f}_2$ can be smooth germs or a convenient singularity and $\tilde{f}_3$ can be an invertible or a smooth germ or a convenient singularity. First, we consider the simplest cases when $\tilde{f}_i$ is not a convenient singularity.

Lemma 2.2. *Let f_0 be a singularity of one of the form listed in ($\star$). Assume that $\tilde{f}_i$ is not a convenient singularity. Then $\lambda'(f_0) = 1$ and $\mu_2 = \mu_1 - 1$, when μ_2 is defined.*

Proof. Consider the possible cases:

1. $f_0 = x\tilde{f}_1$, where $\tilde{f}_1$ is a smooth germ and $y \nmid f_0$. Then

a) if $\mathrm{ord}\tilde{f}_1(0, y) = 1$, then we easily check that $\mu(f_0) = 1$. This means that $\lambda'(f_0) = 1$ and μ_2 is undefined.

b) if $\mathrm{ord}\tilde{f}_1(0, y) =: k > 1$, then $\mu(f_0) = 2k-1$ and for the deformations $f_s(x, y) = f_0(x, y) + sy^{2k-1}$ and $\tilde{f}_s(x, y) = f_0(x, y) + sy^{2k-1} + sxy^{k-1}$ we have $\mu(f_s) = 2k - 2$ and $\mu(\tilde{f}_s) = 2k - 3$ for $s \neq 0$. Hence $\lambda'(f_0) = 1$ and $\mu_2 = \mu_1 - 1$.

2. $f_0 = y\tilde{f}_2$, where $\tilde{f}_2$ is a smooth germ and $x \nmid f_0$. We proceed similarly to case 1.

3. $f_0 = xy\tilde{f}_3$. Then

a) if $\tilde{f}_3$ is an invertible series, then we easily check that $\mu(f_0) = 1$. This means that $\lambda'(f_0) = 1$ and μ_2 is undefined.

b) if $\tilde{f}_3$ is a smooth germ then we proceed similarly to case 1. ∎

Let f_0 be a singularity. In the sequel we will assume that $\tilde{f}_1$, $\tilde{f}_2$, $\tilde{f}_3$ in ($\star$) are convenient singularities. Denote by (a_i, b_i), $i = 0, \ldots, k+1$ and γ_i, $i = 0, \ldots, k$, the consecutive vertices and segments of the Newton polygon $\Gamma(f_0)$, respectively. Let L_{γ_0} and L_{γ_k} be the lines that include the segments $\gamma_0 = \overline{(a_0, b_0), (a_1, b_1)}$ and $\gamma_k = \overline{(a_k, b_k), (a_{k+1}, b_{k+1})}$, respectively. It may happen that $L_{\gamma_0} = L_{\gamma_k}$.

Denote by $(r, 0)$ and $(0, t)$ the points of intersection of the lines L_{γ_k} and L_{γ_0} with the axes OX and OY, respectively. Of course, the coordinates r and t do not have to be integers.

If $a_0 = 0$, then the point (a_0, b_0) will be denoted by $(0, b)$. Similarly, if $b_{k+1} = 0$, then the point (a_{k+1}, b_{k+1}) will be denoted by $(a, 0)$. We will denote by $J(f_0)$ the set of all monomials $x^p y^q$, where $p + q \geq 1$, lying in the closed domain bounded by the axes OX, OY and by the set

$$\mathrm{conv}\left\{\{(r, 0), (0, t), \mathrm{supp}(f_0)\} + \mathbb{R}^2_+\right\}.$$

Note that for a convenient singularity the definition of the set $J(f_0)$ agrees with the one given in Section 1.

We associate to a singularity f_0 a convenient one f_0^{con} defined by

$$f_0^{\mathrm{con}} := \begin{cases} f_0, & \text{if } f_0 \text{ is a convenient singularity} \\ f_0 + x^m, & \text{if } f_0 \text{ is of the form } y\tilde{f}_1 \\ f_0 + y^n, & \text{if } f_0 \text{ is of the form } x\tilde{f}_2 \\ f_0 + x^m + y^n, & \text{if } f_0 \text{ is of the form } xy\tilde{f}_3 \end{cases}$$

where m and n are sufficiently large natural numbers.

It is easy to show that the Newton number of f_0^{con} does not depend on the choice of sufficiently large numbers m and n. So, we may define the Newton number of f_0 by

$$\nu(f_0) := \nu(f_0^{\mathrm{con}}).$$

We have the following formulas for the Newton number (see [7]).

Property 7. *Let f_0 be a singularity.*

1. *If f_0 is a convenient singularity* (see Fig. 1a)), *then $\nu(f_0) = 2S - a - b + 1$.*
2. *If f_0 can be written as $x\tilde{f}_1$, where $\tilde{f}_1$ is a convenient singularity* (see Fig. 1b)), *then $\nu(f_0) = 2S - a + b_0 + 1$.*
3. *If f_0 can be written as $y\tilde{f}_2$, where $\tilde{f}_2$ is a convenient singularity* (see Fig. 1c)), *then $\nu(f_0) = 2S + a_{k+1} - b + 1$.*
4. *If f_0 can be written as $xy\tilde{f}_3$, where $\tilde{f}_3$ is a convenient singularity* (see Fig. 1d)), *then $\nu(f_0) = 2S + a_{k+1} + b_0 - 1$.*

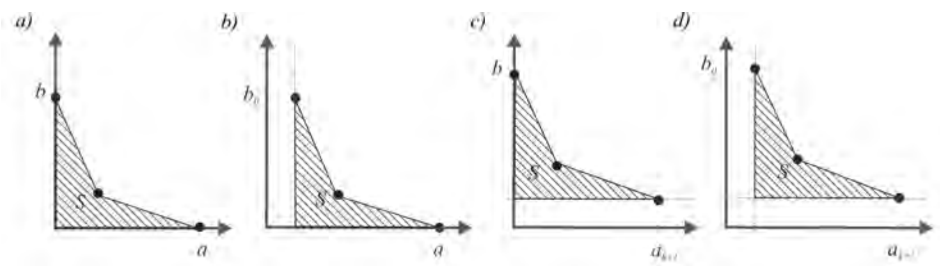

FIGURE 1. All possible variants of the Newton diagram of a nearly convenient singularity

From Kouchnirenko Theorem we have that if f_0 is a non-degenerate singularity, then $\mu(f_0) = \nu(f_0)$.

We prove that for any non-degenerate singularity f_0 there exists a deformation $(f_s^{(p,q)})$, where $(p,q) \in J(f_0)$, which realizes the jump $\lambda'(f_0)$.

Theorem 8. *If f_0 is non-degenerate, then*

$$\lambda'(f_0) = \min_{(p,q)\in J_0(f_0)} \lambda((f_s^{(p,q)})),$$

where $J_0(f_0) \subset J(f_0)$ is the set of points (p,q) such that $\lambda((f_s^{(p,q)})) \neq 0$.

Proof. Let f_0 be a non-degenerate singularity. Then f_0 can be represented in one of the forms

$$\tilde{f}_0, x\tilde{f}_1, y\tilde{f}_2, xy\tilde{f}_3,$$

where $x \nmid \tilde{f}_0$, $y \nmid \tilde{f}_0$, $y \nmid \tilde{f}_1$, $x \nmid \tilde{f}_2$. Note that it suffices to consider the cases when $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2, \tilde{f}_3$ are convenient singularities because the other cases are included in the Lemma 2.2. We will consider cases:

1. $f_0 = \tilde{f}_0$. This means that the singularity is convenient and we may directly apply Theorem 1.

2. Suppose that $f_0 = x\tilde{f}_1$, where $\tilde{f}_1$ is a non-degenerate and convenient singularity. Denote by (a_i, b_i), $i = 0, \ldots, k+1$, the consecutive vertices of the Newton polygon $\Gamma(f_0)$. We have to prove

$$\min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0) - \mu(f_s)) = \min_{(p,q)\in J_0(f_0)} \lambda((f_s^{(p,q)})).$$

The inequality „$\leq$" is obvious. We will prove the opposite inequality. For sufficiently large n we have

$$\min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0) - \mu(f_s)) = \min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0 + y^n) - \mu(f_s + y^n)).$$

Take any deformation $(f_s) \in \mathcal{D}_0^{nd}(f_0)$. Put $g_s := f_s + y^n$. Then g_s are convenient and $(g_s) \in \mathcal{D}_0^{nd}(f_0 + y^n)$ and $\mu(f_0 + y^n) - \mu(f_s + y^n) = \mu(f_0 + y^n) - \mu(g_s)$. We have

$$\begin{aligned}
\min_{(f_s)\in\mathcal{D}_0^{nd}(f_0)} (\mu(f_0 + y^n) - \mu(f_s + y^n)) &\geq \min_{(h_s)\in\mathcal{D}_0^{nd}(f_0+y^n)} (\mu(f_0 + y^n) - \mu(h_s)) \overset{Th.1}{=} \\
&= \min_{(p,q)\in J_0(f_0+y^n)} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)) = \\
&= \min_{(p,q)\in J_0(f_0)\cup J_0'} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)),
\end{aligned}$$

where J_0' is the set of points $(0,l)$, where $l \in (t,n]$, for which $\lambda((f_s^{(p,q)})) \neq 0$. We claim that $J_0' = \emptyset$. Suppose to the contrary that $J_0' \neq \emptyset$. So there exists a point $(p,q) \in J_0'$. Then $(p,q) = (0,l)$, for some $l \in (t,n]$. It is easy to check $\mu(f_0 + y^n) = \mu(f_0 + y^n + sy^l)$, which contradicts the assumption that $(f_s^{(0,l)}) \in \mathcal{D}_0^{nd}(f_0)$. So

$$\begin{aligned}
&\min_{(p,q)\in J_0(f_0)\cup J_0'} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)) = \\
&= \min_{(p,q)\in J_0(f_0)} (\mu(f_0 + y^n) - \mu(f_0 + y^n + sx^p y^q)) = \\
&= \min_{(p,q)\in J_0(f_0)} (\mu(f_0) - \mu(f_0 + sx^p y^q)).
\end{aligned}$$

3. In cases $f_0 = y\tilde{f}_2$ i $f_0 = xy\tilde{f}_3$ we proceed similarly to case 2. ■

3. The first jump of Milnor numbers

As for the non-degenerate and convenient singularities, we can give the exact value of the non-degenerate jump of some singularities. It happens that the Newton polygon of f_0 consists of only one segment. The following theorem extends Theorem 3 to the case of non-convenient singularities. It turns out that the formulas do not transfer automatically from convenient cases. There are new subcases.

Theorem 9. *Let $f_0(x,y) = x^i y^j (x^p - y^q)$, where $i, j \in \{0,1\}$, $p \geq q \geq 2$, $p+q \geq 5$ and let $d = \mathrm{GCD}(p,q)$.*

1. *If $d < q$, then $\lambda'(f_0) = d$.*
2. *If $d = q$ and $i = 0$ and $j = 1$, then $\lambda'(f_0) = \begin{cases} d, & \text{for} \quad q \neq p, \\ d-1, & \text{for} \quad q = p. \end{cases}$*
3. *If $d = q$ and $i = 1$ and $j = 1$, then $\lambda'(f_0) = d$.*
4. *If $d = q$ and $j = 0$, then $\lambda'(f_0) = d - 1$.*

Proof. Ad 1. Theorem 3, p. 1. implies that for the singularity $\tilde{f}_0(x,y) = x^p - y^q$ there exists a point P, which lies in the triangle with vertices $(0,q)$, $(0,0)$, $(p,0)$ and realizes the jump $\lambda'(\tilde{f}_0)$. According to the form of the singularity f_0 we consider the following cases.

a) $i = j = 0$. Then f_0 is a convenient singularity and from Theorem 3 we have $\lambda'(f_0) = d$.

b) $i = 1$ and $j = 0$. Translate the Newton diagram of $\tilde{f}_0$ together with the point P by the vector $[1,0]$. Using Property 7 p. 2. we easily check, that the point $P' := P + [1,0]$ realizes the jump equal to d.

Note that there exists no point P'' realizing a smaller jump than d. From Theorem 3, p. 1. we have that none of the points which lie on the axis OX realizes the jump smaller than d. We check, that for the points of the form $(0,k)$, where $k \in \mathbb{N}$ and $k \in (0,t)$ we have $\lambda((f_s^{(0,k)})) \geq d$. In fact, by assumption $p > q$ we have $|t - q| < 1$ (see Fig. 2). Moreover, Property 7, p. 2. implies that $\lambda((f_s^{(0,q)})) = q > d$ and $\lambda((f_s^{(0,q)})) < \lambda((f_s^{(0,k)}))$, where $k \in (0,q)$.

We check now that, for the points of the form $(1,m)$, where $m \in \mathbb{N}$ and $m \in (0,q)$ we get $\lambda((f_s^{(1,m)})) \geq d$. From Property 7, p. 2. $\lambda((f_s^{(1,q-1)})) = p + 1 > d$ and $\lambda((f_s^{(1,q-1)})) < \lambda((f_s^{(1,m)}))$, where $m \in (0, q-1)$ (see Fig. 2). This implies that $\lambda'(f_0) = d$ and this jump is realized by a point P'.

c) $i = 0$ and $j = 1$. Translate the Newton diagram of $\tilde{f}_0$ together with the point P by the vector $[0,1]$. From Property 7, p. 3. we have that the point $P' = P + [0,1]$ realizes the jump $\lambda'(f_0) = d$. Similarly to *b)* we easily check that, there exists no point which realizes the jump smaller than d.

d) $i = j = 1$. This follows from *b)* and *c)*.

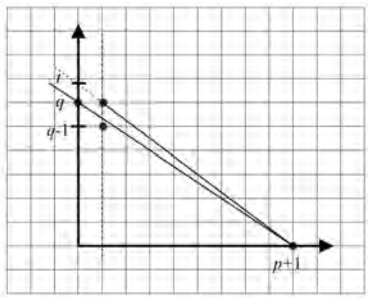

FIGURE 2. $f_0(x,y) = x(x^p - y^q)$

Ad 2. $d = q$, $i = 0$ and $j = 1$. In this case $r \in \mathbb{N}$ and $r = p + \dfrac{p}{q}$ (see Fig. 3). Consider the cases:

a) Let $q \neq p$. Note that $\lambda((f_s^{(r-1,0)})) = d$. It is sufficient to check that there exists no point realizing the jump smaller than d.

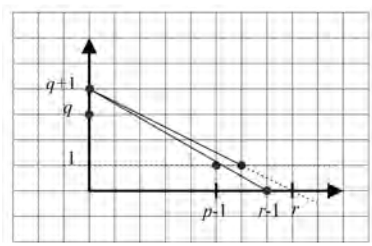

FIGURE 3. $f_0(x,y) = y(x^p - y^q)$

From Property 7, p. 3. $\lambda((f_s^{(p-1,1)})) = q+1 > d$ and $\lambda((f_s^{(0,q)})) = p-1 > d$ (see Fig. 3). Moreover $\lambda((f_s^{(k,0)})) > \lambda((f_s^{(r-1,0)}))$, if $k \in (0, r-1)$ and $\lambda((f_s^{(m,1)})) > \lambda((f_s^{(p-1,1)}))$, if $m \in (0, p-1)$ (see Fig. 3).

Moreover, Theorem 3, p. 2. implies that for the singularity $\tilde{f}_0(x,y) = x^p - y^q$ every point P which lies inside the triangle with vertices $(0,q)$, $(0,0)$, $(p,0)$ realizes the jump bigger or equal to d. If we translate the Newton diagram of $\tilde{f}_0$ by the vector $[0,1]$, then from Property 7, p. 3. we get, that every point P' lying inside the triangle with vertices $(0,q+1)$, $(0,1)$, $(p,1)$ realizes the jump bigger than d. So $\lambda'(f_0) = d$.

b) If $p = q$, then $\lambda((f_s^{(0,q)})) = d-1$. In this case $r = q+1$. Similarly to *a*) we check that there exists no point which realizes the jump smaller than $d-1$.

Ad 3. $d = q$, $i = 1$ and $j = 1$. Consider similarly to case 2.

Ad 4. Consider the cases:

a) $d = q$, $i = 0$ and $j = 0$. Then from Theorem 3 we have $\lambda'(f_0) = d - 1$.

b) $d = q$, $i = 1$ and $j = 0$. Note that $\lambda((f_s^{(p,0)})) = d-1$. It is sufficient to check that there exists no point realizing the jump better than $d - 1$. In fact, the assumption $p \geq q$ implies that $|t - q| \leq 1$ (see Fig. 4).

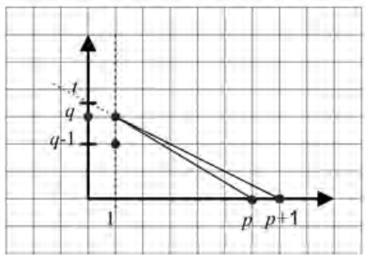

FIGURE 4. $f_0(x, y) = x(x^p - y^q)$

We have $\lambda((f_s^{(0,q)})) = q > d - 1$ and $\lambda((f_s^{(1,q-1)})) = p + 1 > d - 1$ (see Fig. 4). Property 7, p. 2. implies that $\lambda((f_s^{(0,k)})) > \lambda((f_s^{(0,q)}))$ for $k \in (0, q)$ and $\lambda((f_s^{(1,m)})) > \lambda((f_s^{(1,q-1)}))$ for $m \in (0, q - 1)$. Moreover, for singularity $\tilde{f}_0(x, y) = x^p - y^q$ each point P lying inside the triangle with vertices $(0, q)$, $(0, 0)$, $(p, 0)$ realizes the jump bigger than $d-1$. Hence and from Property 7, p. 2. we have that if we translate $\tilde{f}_0$ by the vector $[1, 0]$ then we get that each point P' lying inside the triangle with vertices $(1, q)$, $(1, 0)$, $(p+1, 0)$ realizes the jump bigger than $d-1$. Hence $\lambda'(f_0) = d - 1$. ■

From Lemma 2.2, Corollary 2 and Theorem 9 we have the following

Theorem 10. *Let f_0 be a non-degenerate singularity, with the Newton polygon reduced to at most one segment. Then $f_0(x, y) = x^i y^j \tilde{f}_0$, where $i, j \in \{0, 1\}$ and $\tilde{f}_0 \in \mathbb{C}\{x, y\}$ is a convenient power series. If $\tilde{f}_0$ smooth or invertible then $\lambda'(f_0) = 1$. If $\tilde{f}_0$ is a convenient singularity, which Newton polygon $\Gamma(\tilde{f}_0)$ has vertices at points $(p, 0)$ and $(0, q)$, $d := \mathrm{GCD}(p, q)$ and $p \geq q$, then*

1. *If $d < q$, then $\lambda'(f_0) = d$.*
2. *If $d = q$, $i = 0$ and $j = 1$, then* $\lambda'(f_0) = \begin{cases} d, & \text{for} \quad q < p, \\ d - 1, & \text{for} \quad q = p. \end{cases}$
3. *If $d = q$, $i = 1$ and $j = 1$, then $\lambda'(f_0) = d$.*
4. *If $d = q$ and $j = 0$, then $\lambda'(f_0) = d - 1$.*

4. THE SECOND JUMP OF MILNOR NUMBERS

Let f_0 be a non-degenerate singularity. Just as in the Introduction, we can consider the strictly decreasing sequence $(\mu_0, \mu_1, \ldots, \mu_k)$ of all possible Milnor numbers of all non-degenerate deformations (f_s) of f_0. In this case, we have results similar to the ones in the convenient case.

Theorem 11. *Let f_0 be a singularity of the form $f_0(x,y) = x^i y^j (x^p - y^q)$, $i, j \in \{0,1\}$, $p \geq q$. Then $\mu_2 = \mu_1 - 1$, if μ_2 is defined.*

Proof. For $i = 0$, $j = 0$ the assertion follows from Theorem 5. Note that if $x^p - y^q$ is not a singularity (i.e. $q = 1$) then the assertion follows from Lemma 2.2. If $x^p - y^q$ is a singularity we consider the case $i = 1$ or $j = 1$.

I. $q \nmid p$. Let us consider the subcases:

1. $i = 1$, $j = 0$. In this case we can repeat the argument of the proof of Theorem 5, p. 2. in [10] translating the whole configurations by the vector $[1, 0]$. Hence we get $\mu_2 = \mu_1 - 1$.

2. $i = 0$, $j = 1$. It suffices to consider only the case $q = 2$ because in the remaining cases we may repeat the argument from the proof of Theorem 5, p. 2 in [10]. Let $q = 2$. The fact $q \nmid p$ implies $\dfrac{3(p-1)}{2} \in \mathbb{N}$.

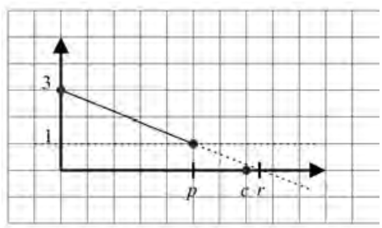

FIGURE 5. $f_0(x,y) = y(x^p - y^2)$

Moreover, for the point $(c, 0) := (\frac{3(p-1)}{2} + 1, 0)$ (see Fig. 5) we have $\lambda(f_s^{(c,0)}) = 1$. Of course $\mathrm{GCD}(c, 3) = 1$ hence from Theorem 3, p. 1. there exists a point lying inside the triangle with vertices $(0,3)$, $(0,0)$, $(c,0)$ realizing the jump $2 = \lambda'(f_0)+1$. Hence $\mu_2 = \mu_1 - 1$.

3. $i = 1$, $j = 1$. It follows from 2.

II. $q \mid p$. Let us consider the subcases:

1. $i = 1$, $j = 0$. We have:

(i) $p = q = 2$. Then $f_0(x,y) = x(x^2 - y^2)$. It is easy to check that the point $(2,0)$ realizes the jump equal to 1, while the deformation $f_s(x,y) = f_0(x,y) + sx^2 + sy^3$ realizes the jump equal to $2 = \lambda'(f_0) + 1$. Hence $\mu_2 = \mu_1 - 1$.

(ii) $p + q > 4$, $q \geq 2$. We repeat the argument from the proof of the Theorem 5, p. 1. in [10]. Hence and from Property 7 we have the assertion.

2. $i = 0$, $j = 1$. We have:

a) $q \neq p$. From Theorem 9 we have $\lambda'(f_0) = d$ and the deformation $f_s^{(r-1,0)}$ realizes this jump, where $r \in \mathbb{N}$, $r = p + \dfrac{p}{q}$ (see Fig. 6). Note that $\mathrm{GCD}(r-1, q+1) = 1$. From Theorem 3, p. 1., there exists a point (α, β) lying inside the triangle with vertices $(0, q+1)$, $(0,0)$, $(r-1, 0)$ realizing the jump equal to 1 for $f(x,y) = x^{r-1} - y^{q+1}$.

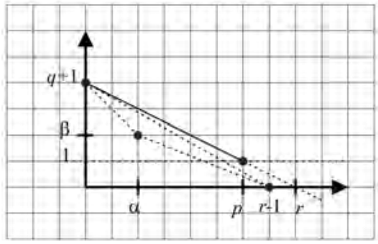

FIGURE 6. $f_0(x,y) = y(x^p - y^q)$

Therefore, the deformation $f_s(x,y) = f_0(x,y) + sx^{r-1} + sx^{\alpha}y^{\beta}$ realizes the jump $d+1 = \lambda'(f_0)+1$.

b) $q = p$. Let us consider the subcases:

i) $p = q = 2$. Then $f_0(x,y) = y(x^2 - y^2)$. It is easy to check that the point $(0,2)$ realizes the jump equal to 1, while the deformation $f_s(x,y) = f_0(x,y) + sx^3 + sy^2$ realizes the jump $2 = \lambda'(f_0)+1$. Hence $\mu_2 = \mu_1 - 1$.

ii) $p = q > 2$. From Theorem 9 $\lambda'(f_0) = d-1$ and this jump is realized by the point $(0,q)$. Note that $\mathrm{GCD}(q, q-1) = 1$. From Theorem 3, p. 1. there exists a point (α, β) lying inside the triangle with vertices $(0, q-1)$, $(0,0)$ and $(q,0)$ realizing the jump equal to 1 for $f(x,y) = x^q - y^{q-1}$.

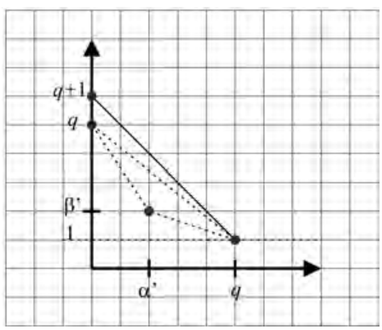

FIGURE 7. $f_0(x,y) = y(x^q - y^q)$

If we translate the diagram of $f(x,y) = x^q - y^{q-1}$ (with the point (α,β)) by the vector $[0,1]$ (see Fig. 7) we get the singularity $\tilde{f}(x,y) = x^q y - y^q$ and the point (α', β') such that the deformation $f_s(x,y) = f_0(x,y) + sy^q + sx^{\alpha'}y^{\beta'}$ realizes the jump $(d-1)+1 = d = \lambda'(f_0)+1$.

3. $i = j = 1$. Similarly to 2. ■

From Lemma 2.2, Corollary 2 and Theorem 11 we have the following

Theorem 12. *Let f_0 be a non-degenerate singularity with the Newton polygon reduced to at most one segment. If $\Lambda'(f_0) = (\mu_0, \mu_1, \dots, \mu_k)$, $k \geq 2$, is the sequence of Milnor numbers associated to f_0, then*

$$\mu_2 = \mu_1 - 1,$$

provided μ_2 is defined.

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Analytic and Algebraic Geometry
Łódź University Press 2013, 155 – 202

MULTIPLE ZETA VALUES AND THE WKB METHOD

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Abstract. The multiple zeta values $\zeta(d_1,\ldots,d_r)$ are natural generalizations of the values $\zeta(d)$ of the Riemann zeta functions at integers d. They have many applications, e.g. in knot theory and in quantum physics. It turns out that some generating functions for the multiple zeta values, like $f_d(x) = 1-\zeta(d)x^d+\zeta(d,d)x^{2d}-\ldots$, are related with hypergeometric equations. More precisely, $f_d(x)$ is the value at $t=1$ of some hypergeometric series ${}_dF_{d-1}(t)=1-x^dt+\ldots$, a solution to a hypergeometric equation of degree d with parameter x. Our idea is to represent $f_d(x)$ as some connection coefficient between certain standard bases of solutions near $t=0$ and near $t=1$. Moreover, we assume that $|x|$ is large. For large complex x the above basic solutions are represented in terms of so-called WKB solutions. The series which define the WKB solutions are divergent and are subject to so-called Stokes phenomenon. Anyway it is possible to treat them rigorously. In the paper we review our results about application of the WKB method to the generating functions $f_d(x)$, focusing on the cases $d=2$ and $d=3$.

1. Introduction

We study the following hypergeometric equations

$$(1-t)\partial(t\partial)^{d-1}g+x^dg=0, \tag{1.1}$$

where $\partial=\partial_t=\partial/\partial t$, with one solution in form of the hypergeometric series (see [BE1])[1]

2010 *Mathematics Subject Classification.* Primary 05C38, 15A15; Secondary 05A15, 15A18.
Key words and phrases. Multiple zeta values, hypergeometric differential equations, WKB expansion.
Supported by Polish OPUS Grant No 2012/05/B/ST1/03195 and by Polish-French PHC POLONIUM 2013 PROJECT No 28217 SG.

[1]Recall the standard formula ${}_pF_q(\alpha_1,\ldots\alpha_p;\beta_1,\ldots,\beta_q;t)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n n!}t^n$ where $(\alpha)_n=\alpha(\alpha+1)\ldots(\alpha+n-1)$ is the known Pochhammer symbol. Eq. (1.1) can be found in [Zud1] and [Zo2]

$$
\begin{aligned}
\varphi_1(t;x) &= {}_dF_{d-1}(-\varsigma^0 x,\ldots,-\varsigma^{d-1}x;1,\ldots,1;t) \\
&= 1-x^d t+(-x^d)\left(1-x^d\right)t^2/(2!)^d+\ldots
\end{aligned} \tag{1.2}
$$

here

$$
\varsigma = e^{2\pi i/d} \tag{1.3}
$$

is the primitive root of unity of degree d (other solutions $\varphi_2,\ldots,\varphi_d$ are given in Section 3.1). For $d=1$ we have the simple (and unique solution) $\varphi_1=(1-t)^x$, so this case is not interesting.

But when the degree of the equation is greater, $d\geq 2$, then something interesting happens. It turns out that the solution (1.2) evaluated at $t=1$ is a generating function for so-called **multiple zeta values** (MZV's, see [Zag1])[2]

$$
\zeta(d_1,\ldots,d_k)=\sum_{0<n_1<\ldots<n_k}\frac{1}{n_1^{d_1}\ldots n_k^{d_k}},\quad d_j\geq 1,\quad d_k\geq 2. \tag{1.4}
$$

Namely,

$$
\varphi_1(1;x)=f_d(x) \tag{1.5}
$$

where f_d is the following **generating function**:

$$
f_d(x)=1-\zeta(d)x^d+\zeta(d,d)x^{2d}-\ldots \tag{1.6}
$$

(see [Zo2] and Section 3 below).

It is easy to show the formula

$$
f_d(x)=\prod_{n=1}^{\infty}\left(1-\left(\frac{x}{n}\right)^d\right) \tag{1.7}
$$

which implies, in particular, that

$$
f_2(x)=\frac{\sin\pi x}{\pi x}. \tag{1.8}
$$

But for odd degrees we do not have similar formulas. Since the R. Apery's work [Ap] we know that the number $\zeta(3)$ is irrational, but it is not known whether it is algebraic or not. Due to formula (1.8) below we assume that:

$$
d=2 \text{ or } d>2 \text{ is odd.} \tag{1.9}
$$

The idea of this paper and of [Zo2, ZZ1, ZZ2, ZZ3] is to express the solution (1.2) in suitable basis $(\theta_1,\ldots,\theta_d)$ of solutions near $t=1$;

$$
\varphi_1=A_1(x)\theta_1+\ldots+A_d(x)\theta_d.
$$

The basis near $t=1$ is such that $\theta_j|_{t=1}=0$ for $j=1,\ldots,d-1$ and $\theta_d|_{t=1}$ is a known nonzero number. Therefore it is enough to find the coefficient $A_d(x)$ before θ_d. The coefficients $A_j(x)$ are analytic functions in $x\in\mathbb{C}\setminus 0$, with only possible

[2] In some sources the sum in Eq. (1.4) is denoted $\zeta(d_k,\ldots d_1)$.

singularities at $x=0$ and at $x=\infty$ (see Sections 3). So there appears an idea to consider behavior of the solutions when the parameter x becomes large.

For large $|x|$ there exist some special solutions of the form

$$g \sim x^{\gamma} e^{xS(t)} \left\{ \chi_0(t) + \chi_1(t) x^{-1} + \dots \right\},$$

known as the WKB solutions. Here the 'action' $S(t)$ and the amplitudes $\chi_j(t)$ satisfy some ODEs which are easy to integrate. There exist basic WKB solutions $g^{\sigma}(t;x) \sim \exp(\sigma x S_d(t))$ with $S_d(t) = \int_0^t \tau^{1/d-1}(1-\tau)^{-1/d} d\tau$ and $\sigma = \varsigma^{j+1/2}$ ($j=0,\dots,d-1$) to Eq. (1.1) (see Section 4). One would like to represent the solutions φ_1 and θ_j in the WKB basis. To this aim one could use some integral representations of the solutions φ_1 and θ_j and then to evaluate the corresponding integrals, which are of oscillatory type, using the stationary phase formula (see [Fed, He]).

This approach is tempting but it encounters serious obstacles. One of them is the question of uniqueness of the series defining the WKB solutions. The functions $\chi_j(t)$ satisfy an infinite series of ODEs and an infinite number of constants of integration of these equations has to be determined. In Definition 1 (in Section 4.1) we define so-called testing WKB solutions $g^{\sigma}_{\mathrm{test}}$ by choosing some arbitrary procedure of fixing the integration constants. But it is not the right choice. In Section 4.2 we define so-called normal WKB solutions $g^{\sigma}_{\mathrm{norm}}$ which are more natural, because they are obtained via some normalization procedure (i.e. a diagonalization) of a corresponding linear first order differential system and this procedure is unique.

But the main difficulty arises from the fact that the series defining the WKB solutions are divergent. It turns out that one can define analytic WKB solutions by applying an analytic version of the normalization procedure (see Section 4.3), but the domains of definition of the latter solutions are quite small: for $0<t<1$ the parameter x lies in a sector in $\mathbb{C}$ with vertex at $x=\infty$. Moreover, the analytic normalization requires solving some integral equation and the solutions obtained are not unique.

In Section 5 we develop a new approach in the asymptotic analysis of linear differential equations like Eq. (1.1). For t near 0 we approximate Eq. (1.1) with so-called Bessel type equation $\partial_y \left(y\partial_y\right)^{d-1} G + G = 0$ for $G(y)$ where $y = x^3 t$ (see Eq. (5.3)). Similarly, for $s = 1-t$ close to 0 we have an approximation by another Bessel type equation (Eq. (5.5)) for $H(z)$, where $z = x^d s^{d-1}$. These Bessel type equations have only two singular points: regular at $y=0$ (respectively at $z=0$) and irregular at $y=\infty$ (respectively at $z=\infty$). In Theorem 1 we prove that the hypergeometric equation (1.1) for $g(t;x)$ near $t=0$ is analytically equivalent with the corresponding Bessel type equation for $G(y)$ and that the corresponding equation for $h(s;x) = g(1-s;x)$ near $s=0$ is analytically equivalent with the Bessel type equation for $H(z)$. The Bessel type equations admit uniquely defined WKB type solutions $G^{\sigma}(y) \sim e^{d\sigma y^{1/d}}$ for $y\to\infty$ and $H^{\sigma} \sim e^{(d/(1-d))\sigma z^{1/d}}$ for $z\to\infty$. In Section 5.3 we define so-called principal WKB solutions $g^{\sigma}_{\mathrm{princ}}$ and $h^{\sigma}_{\mathrm{princ}}$ as images of the WKB solutions G^{σ} and H^{σ} using the above analytic equivalences.

To represent the solution $\varphi_1(t;x)$ (defined by the hypergeometric series (1.2)) in the basis $\left(g^{\sigma}_{\mathrm{princ}}\right)$ one expresses this hypergeometric function via a contour integral (in Section 6.1). This is an oscillatory type integral (or a mountain pass integral). It is evaluated asymptotically as $x \to \infty$ using well known stationary phase formula (or the mountain pass formula).

For the degree $d = 2$ one can write down suitable integral representations for the basic solutions $\theta_1(s;x)$ and $\theta_2(s;x)$ near $s = 1 - t = 0$. The corresponding stationary phase formula allows to represent θ_j in the basis $\left(h^{\sigma}_{\mathrm{princ}}\right)$. Because the relation between the bases $\left(g^{\sigma}_{\mathrm{princ}}\right)$ and $\left(h^{\sigma}_{\mathrm{princ}}\right)$ is given by a diagonal matrix (at least formally) it is possible to give new proofs of the formula (1.8). We give two proofs, one in Section 6.3 and another one in Section 7.2.1.

However, here we must underline that the existence of the integral formulas for $\theta_{1,2}$ in the case $d = 2$ follows from the formula $\theta_j(s) = -s\partial_s\varphi_j(s)$, which is a consequence of so-called self-duality for the MZV's $\zeta(2,\dots,2)$ (see Eqs. (2.8)–(2.9) and Lemma 3 below).

In the case of odd $d > 2$ there are no integral formulas for the basic solutions θ_j, $j = 1,\dots,d$. But we can find such formulas for corresponding solutions $\Theta_j(z)$ (to the Bessel type equation) which approximate the solutions θ_j. Evaluating these integrals, using the mountain pass formula for large $|z|$, one finds expansions of the functions Θ_j in the basis (H^{σ}). Next, one uses the equivalence of the hypergeometric and the Bessel equations near $s = 0$ to expand θ_j in the principal WKB basis $\left(h^{\sigma}_{\mathrm{princ}}\right)$. We do it for the case $d = 3$.

The WKB solutions G^{σ} (respectively H^{σ}) are subject to so-called Stokes phenomenon. It relies upon the property that the formal solutions G^{σ} are asymptotic expansions of some genuine analytic solutions G^{σ}_j, defined in some sectors $\mathcal{S}_j$, but in intersection of two adjacent sectors the relation between the corresponding bases is given by so-called Stokes matrix (which is not identical). This explains the divergence of the series defining G^{σ} and is responsible for the unpleasant fact that the coefficients in the expansion of the function $\Phi_1(y)$ (approximating φ_1) given by the stationary phase formula are not exact. More precisely, only the dominating terms $\mathrm{const}\cdot e^{d\sigma y^{1/d}}$, as $|y| \to \infty$ and $\arg y$ is fixed, are correct. Other terms are determined by an analysis leading to computation of the Stokes matrices. The same is true for the WKB solutions H^{σ} and representations of $\Theta_j(z)$ in terms of (H^{σ}) for $|z| \to \infty$ and fixed $\arg z$. This is done in Section 7.1.

In Section 7.2 we apply the above theory to get a representation

$$A_d(x) = \sum a_{\sigma} \cdot F^{\sigma}(x)$$

for the connection coefficient before θ_d in the representation of φ_1 in the basis (θ_j). Here $F^{\sigma}(x)$ are functions of WKB type. For $d = 2$ we prove that the functions F^{σ} are single valued, i.e. the corresponding Stokes operators are trivial.

For $d = 3$ we have

$$F^\sigma = \pm x^{-3/2} e^{2\pi\sigma x/\sqrt{3}} \omega^\sigma (x^{-1/2})$$

which are subject to a nontrivial Stokes phenomenon. Moreover, their monodromy, as x makes a turn around ∞, is nontrivial (due to the factor $x^{-3/2}$). This implies that the function $A_3(x)$ is a solution of a meromorphic sixth order linear equation with irregular singularity at $x = \infty$ (Theorem 2).

Since the function $A_3(x)$ is entire (and holomorphic at $x = 0$) it is quite plausible that the equation satisfied by F^σ's has regular singularity at $x = 0$. Then this equation should take the following form

$$\begin{gathered} f^{(VI)} + c_1 x^{-1} f^{(V)} + c_2 x^{-2} f^{(IV)} + \left(c_3 + c_4 x^{-3}\right) f^{(III)} + \left(c_5 x^{-1} + c_6 x^{-4}\right) f^{(II)} \\ + \left(c_7 x^{-2} + c_8 x^{-5}\right) f^{(I)} + \left(c_9 + c_{10} x^{-3} + c_{11} x^6\right) f = 0 \end{gathered}$$

where $c_3 = 2\left(2\pi/\sqrt{3}\right)^3$, $c_9 = \left(2\pi\sqrt{3}\right)^6$ and other coefficients c_j are computable (most probably are expressed in an algebraic way via π and $\sqrt{3}$). But then the coefficients $b_k = (-1)^k \zeta(3, \ldots, 3)$ in the expansion $f_3 = \sum b_k x^{3k}$ should satisfy a recurrent relation, hence all the zeta values $\zeta(3, \ldots., 3)$ are expressed via $\zeta(3)$ and $\zeta(3)$ would satisfy an algebraic equation with coefficients depending on the c_j's. We plan to calculate the coefficients c_j in a separate paper.

Sections 2 of the paper is devoted to presentation of some basic facts about MZV's and about their relations with hypergeometric series.

2. MZV's, polylogarithms and hypergeometric series

The Multiple Zeta Values (MZV's) $\zeta(d_1, \ldots, d_k)$ are defined in Eq. (1.4). Any such quantity has its **weight** $d = d_1 + \ldots + d_k$, **depth** equal k and **height** $h = \sharp\{i : d_i > 1\}$.

They form a graded algebra, where the grading is defined by the weight. Indeed, we can rewrite the product of two infinite sums

$$\left(\sum_{n_1 < \ldots < n_k}\right)\left(\sum_{m_1 < \ldots < m_l}\right)$$

in the product $\zeta(d_1, \ldots, d_k)\zeta(e_1, \ldots, e_l)$ as a finite sum corresponding to different orderings of the index set $\{n_1, \ldots, n_k, m_1, \ldots, m_l\}$. The corresponding identity is sometimes called the **first shuffle product**. For example, we have

$$\zeta(2)\zeta(2) = 2\zeta(2,2) + \zeta(4) \tag{2.1}$$

which implies $\zeta(4) = \pi^4/90$. It was Euler who used this sort of shuffle relations to prove that $\zeta(2k) = \pi^{2k} \times$(rational number).

Important is the problem of calculation of the dimension D_d of the space $\mathfrak{Z}_d$ (over the field $\mathbb{Q}$) generated by the MZV's of weight d. There exists a conjecture (see [Zag1]) that these dimensions satisfy the recursion $D_d = D_{d-2} + D_{d-3}$ (with $D_0 = 1$ and $D_d = 0$ for $d < 0$). This is equivalent to the property

$$\sum D_d t^d = \frac{1}{1 - t^2 - t^3}.$$

M. Hoffman [Hof] conjectured that the algebra of MZV's is generated by special values of the form $\zeta(d_1, \ldots, d_k)$ with $d_j \in \{2, 3\}$. This conjecture was recently proved by F. Brown [Bro]; in the proof some explicit relations between the values $\zeta(2, \ldots, 2)$, $\zeta(2r + 1)$ and $\zeta(2, \ldots, 2, 3, 2, \ldots, 2)$ (proved by D. Zagier [Zag2]) are used.

There exists the following **Kontsevich–Drinfeld formula** ([KoZa]) for the MZV's. Let

$$\omega_0(t) = dt/t, \quad \omega_1(t) = dt/(1 - t) \tag{2.2}$$

be two 1−forms. For given $d_1, \ldots, d_k$ we define the d−form

$$\begin{aligned} \Omega_{d_1,\ldots,d_k} &= \omega_0(t_{d_1+\ldots+d_k}) \ldots \omega_0(t_{d_1+\ldots+d_{k-1}+2})\omega_1(t_{d_1+\ldots+d_{k-1}+1}) \\ &\quad \ldots \omega_0(t_{d_1}) \ldots \omega_0(t_2)\omega_1(t_1); \end{aligned} \tag{2.3}$$

there are k forms ω_1 with arguments $t_1, t_{d_1+1}, \ldots, t_{d_1+\ldots+d_{k-1}+1}$. Next, we integrate it over the simplex $\{0 \leq t_1 \leq \ldots \leq t_d \leq 1\}$:

$$\zeta(d_1, \ldots, d_k) = \int_{0 \leq t_1 \leq \ldots \leq t_d \leq 1} \Omega_{d_1,\ldots,d_k}. \tag{2.4}$$

For example, we have[3]

$$\int_{0 \leq t_1 \leq t_2 \leq 1} \frac{dt_2}{t_2} \frac{dt_1}{1 - t_1} = \sum_{n \geq 1} \frac{1}{n} \int_0^1 t_2^{n-1} dt_2 = \sum \frac{1}{n^2} = \zeta(2). \tag{2.5}$$

The latter formula is generalized to the **generalized polylogarithms**

$$\begin{aligned} \mathrm{Li}_{d_1,\ldots,d_k}(t) &= \textstyle\sum_{0<n_1<n_2<\ldots<n_k} t^{n_k}/n_1^{d_1} \ldots n_k^{d_k} \\ &= \textstyle\int_{0 \leq t_1 \leq \ldots \leq t_d \leq t} \Omega_{d_1,\ldots,d_k}. \end{aligned} \tag{2.6}$$

It implies another shuffle multiplication. The product

$$\left(\int_{t_1 \leq \ldots \leq t_d \leq t}\right)\left(\int_{s_1 \leq \ldots \leq s_e \leq t}\right)$$

[3]Such integrals appear as coefficients in some knot invariants and in evaluation of some Feynmann integrals in quantum physics.

of integrals is represented as a finite sum of integrals according to the ordering of the variables set $\{t_1, \ldots, t_d, s_1, \ldots, s_d\}$. For example, we have

$$
\begin{aligned}
(2.7) \quad \mathrm{Li}_2(t)\mathrm{Li}_1(t) &= \left(\int_{0\le t_1\le t_2\le t} \frac{dt_2 dt_1}{t_2(1-t_1)}\right)\left(\int_0^t \frac{dt_3}{1-t_3}\right) \\
&= \left(2\int_{0\le t_1\le t_3\le t_2\le t} + \int_{0\le t_1\le t_2\le t_3\le t}\right)\frac{dt_2 dt_3 dt_1}{t_2(1-t_3)(1-t_1)} \\
&= 2\mathrm{Li}_{1,2}(t) + \mathrm{Li}_{2,1}(t).
\end{aligned}
$$

The **second shuffle formula** leads to an interesting shuffle algebra (see [MPH, Zud1]), but there is no place to describe its details.

The Drinfeld–Kontsevich formula (2.4) leads to the following **MZV duality**. Namely, we put $s_1 = 1-t_d, \ldots, s_d = 1-t_1$; thus $\omega_{\varepsilon_j}(t_j) = \omega_{1-\varepsilon_j}(1-s_{d-j+1})$ and we get

$$(2.8) \quad \zeta(1,\ldots 1, m_1+2, \ldots, 1, \ldots, 1, m_r+2) = \zeta(1, \ldots 1, n_r+2, \ldots, 1, \ldots, 1, n_1+2)$$

where the sequences of 1's have lengths n_j in the left-hand side and m_{r-j+1} in the right hand side. We observe that the quantities

$$(2.9) \qquad \zeta(2,\ldots,2) \quad \text{and} \quad \zeta(1,3,\ldots,1,3)$$

are invariant with respect to the MZV duality. We have also the formula

$$(2.10) \qquad \zeta(3) = \zeta(1,2)$$

which is proved in many ways in the literature.

There exist interesting generating functions which imply series of relations between MZV's. One of them is following (see [BBB]):

$$(2.11) \quad \sum_{m,n\ge 0} x^{m+1}y^{n+1}\zeta(m+2,1,\ldots,1) = 1 - \exp\left\{\sum_{k\ge 2}\frac{x^k+y^k-(x+y)^k}{k}\zeta(k)\right\}$$

where the sequence of 1's has length n.

Some of the generating series are expressed via hypergeometric functions. In the next example we put

$$G(d,k,h) = \sum \zeta(d_1,\ldots,d_k),$$

where in the sum the weight $d = d_1 + \ldots + d_k$, the depth k and the height $h = \sharp\{i : d_i > 1\}$ are fixed and $d_k \ge 2$. Let also α and β satisfy

$$\alpha+\beta = x+y, \quad \alpha\beta = z.$$

Then we have the following identity for

$$\Phi(x,y,z) = \sum G(d,k,h)x^{d-k-h}y^{k-h}z^{h-1}$$

(see [OhZa]):

$$\Phi = \frac{1}{xy-z}\{1 - {}_2F_1(\alpha - x, \beta - x; 1-x; 1)\} \tag{2.12}$$

$$= \frac{1}{xy-z}\left\{1 - \exp\left(\sum_{n\geq 2}\frac{x^n+y^n-\alpha^n-\beta^n}{n}\zeta(n)\right)\right\}. \tag{2.13}$$

This result was generalized in [AOW] and [Li]. Specializing Eq. (2.13) to $xy = z$ one obtains the formula

$$\sum_{d,k,h} G(d,k,h)x^{d-k-1}y^{k-1} = \sum \zeta(d)x^{d-k-1}y^{k-1}. \tag{2.14}$$

In particular,

$$\sum_{d_1+\ldots+d_k=d} \zeta(d_1,\ldots,d_k) = \zeta(d) \tag{2.15}$$

where the depth k is fixed. For $k = 2$ the latter identity is known as the **Euler formula**.

We note also the following Borwein formula for the generating function $f_{1,3}(x) = 1 - \zeta(1,3)x^4 + \zeta(1,3,1,3)x^8 - \ldots$:

$$f_{1,3}(x) = f_4\left(x/\sqrt{2}\right) \tag{2.16}$$

which follows from a corresponding identity for generating functions for polylogarithms (see [KoZa], [BBBL]). This formula was conjectured by D. Zagier in [Zag1].

It was conjectured in [BBB] and proved in [Zhao] that

$$\zeta(3,\ldots,3) = 8^k \cdot \zeta(1,\bar{2},\ldots,1,\bar{2}) \tag{2.17}$$

where

$$\zeta(1,\bar{2},\ldots,1,\bar{2}) = \sum_{0<m_1<n_1<\ldots<m_k<n_k} \frac{(-1)^{n_1+\ldots+n_k}}{m_1n_1^2\ldots m_kn_k^2} \tag{2.18}$$

is so-called **alternating Euler sum**. The generating function for the latter values

$$f_{1,\bar{2},\ldots,1,\bar{2}}(x) = \sum \zeta(1,\bar{2},\ldots,1,\bar{2}) \cdot (-x^3)^k \tag{2.19}$$

is related with the following sixth order equation:

$$(1-t)\partial(1-t)\partial t\partial(1+t)\partial(1+t)\partial_t t\partial_t g - x^6 g = 0.$$

Namely, this equation has two solutions analytic near $t = 0$ and of the form $\varphi_1 = 1+ O(x^6)$ and $\varphi_2 = \sum_{0<m<n}\frac{(-t)^n}{mn^2} + O(x^6)$. Then $f_{1,\bar{2},\ldots,1,\bar{2}}(x) = \varphi_1(1;x) - x^3\varphi_2(1;x)$. The Zhao's result implies that $f_{1,\bar{2},\ldots,1,\bar{2}}(x) = f_3(x/2) = \prod\left(1 - \left(\frac{x}{2n}\right)^3\right)$.

Some hypergeometric series are also used in irrationality proofs of some zeta values. Here we refer the reader to the exemplary papers [CFR, Zud2, Hut].

We finish this section by noticing that some third order linear differential equations, similar to Eq. (1.1) for $d = 3$ were considered by F. Beukers with C. Peters in [BePe] and by S.-T. Yau with B. Lian in [LYau]. In [BePe] the equation

$$(t^4 - 34t^3 + t^2)\partial^3 z + (6t^3 - 153t^2 + 3t)\partial^2 z + (7t^2 - 112t + 1)\partial z + (t - 5)z = 0,$$

which is directly related with the recurrence used by R. Apéry in his proof of irrationality of $\zeta(3)$ (see [Ap], [vPo]), turns out to be a Picard–Fuchs equation for periods of some K3 surface. In [LYau] the authors consider equations of the form

$$\left((t\partial)^3 - t\left(\sum_{i=1}^{3} r_i\,(t\partial)^i\right)\right) z = 0;$$

they are Picard–Fuchs equations for a one-parameter deformations of K3 surfaces and are used in the mirror symmetry property for K3 surfaces. However the choice of parameters r_j used in [LYau] is different than in Eq. $(1.1)_{d=3}$.

3. Two bases of solutions

3.1. **Basic solutions near $t = 0$.** Recall that we consider Eq. (1.1). The hypergeometric function (1.2) is one of the basic solutions. We may represent it as a series in powers of x^d with coefficients depending on t. Also other solutions can be written in the form $g = \phi(t;x) = \phi_0(t) - \phi_1(t)x^d + \phi_2(t)x^{2d} - \ldots$, where the coefficient functions satisfy the series of equations: $(t\partial)^d\,\phi_0 = 0$ and $(t\partial)^d\,\phi_k = \frac{t}{1-t}\phi_{k-1}$ for $k \geq 1$. The first equation has d independent solutions which we can choose in the following form:

$$\varphi_{1,0}(t) = 1, \quad \varphi_{2,0} = \ln\left(x^d t\right), \ldots, \varphi_{d,0} = \frac{1}{(d-1)!}\ln^{d-1}\left(x^d t\right) \tag{3.1}$$

(this special choice is justified in Section 5). The other equations are solved as follows:

$$\phi_k(t) = \int_{0<t_d\ldots<t_1<t} \frac{dt_1}{t_1}\cdots\frac{dt_{d-1}}{t_{d-1}}\frac{dt_d}{1-t_d}\phi_{k-1}(t_d). \tag{3.2}$$

It is easy to see that the coefficients ϕ_k decrease very fast with k (like $1/k!$), so the obtained solutions are analytic functions in $x^d \in \mathbb{C}\setminus 0$ with known singularities at $x = 0$.

The above implies that the basic solutions to Eq. (1.1) are of the form

$$\varphi_j(t;x) = \varphi_{j,0}(t) - \varphi_{j,1}(t)x^d + \varphi_{j,2}(t)x^{2d} - \ldots, \qquad j = 1,\ldots,d, \tag{3.3}$$

with $\varphi_{j,k}$ given by the integral recurrence (3.2). They can be rewritten as follows:

$$\begin{aligned}
&\varphi_1 = 1 + O(t),\\
&\varphi_2 = \varphi_1 \ln\left(x^d t\right) + \psi_2,\\
&\varphi_3 = \tfrac{1}{2!}\varphi_1 \ln^2(x^d t) + \psi_2\ln(x^d t) + \psi_3,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\varphi_{d-1} = \tfrac{1}{(d-1)!}\varphi_1 \ln^{d-1}\left(x^d t\right) + \ldots + \psi_{d-1}\ln(x^d t) + \psi_d
\end{aligned} \tag{3.4}$$

where $\varphi_1, \psi_2, \ldots, \psi_d$ are analytic in t near $t = 0$. (The above form of the basic solutions can be explained by the defining equation $\lambda^d = 0$ for the leading exponents in the solutions $\phi = t^\lambda + \ldots$.)

Of course, for us the principal is the first of these solutions. Using the Drinfeld–Kontsevich formula (2.6) we find

$$\begin{aligned}\varphi_{1,2}(t) &= \int_{0<t_d\ldots<t_1<t} \frac{dt_1}{t_1}\cdots\frac{dt_{d-1}}{t_{d-1}}\frac{dt_d}{1-t_d} \\ &= \sum_{n=1}^{\infty}\int_{0<t_d\ldots<t_1<t} \frac{dt_1}{t_1}\cdots\frac{dt_{d-1}}{t_{d-1}}t_d^{n-1}dt_d = \sum\frac{t^n}{n^d} = \mathrm{Li}_d(t),\end{aligned}$$

i.e. a polylogarithm. Other coefficient functions $\varphi_{1,k}$ are also expressed via polylogarithms and we have

$$\varphi_1 = 1 - \mathrm{Li}_d(t)x^d + \mathrm{Li}_{d,d}(t)x^{2d} - \ldots,$$

which implies formula (1.5).[4]

Remark 1. *Other solutions $\varphi_2, \ldots, \varphi_d$ also admit expressions in terms of hypergeometric series. For example, in the case $d = 2$ we can take the following perturbation of Eq. (1.1): $t\left\{(1-t)\partial_t t\partial_t g + x^2 g\right\} - \mu^2 g = 0$ with small parameter μ (see* [ZZ1]*). It has the solutions η_μ and $\eta_{-\mu}$, where $\eta_\mu = \frac{\Gamma(1+x+\mu)}{\Gamma(1+x-\mu)\Gamma(1+2\mu)}\cdot t^\mu\cdot F(\mu+x,\mu-x;1+2\mu;t)$, and therefore*

$$\widehat{\varphi}_2 = \lim_{\mu\to 0}\left(\eta_\mu - \eta_{-\mu}\right)/2\mu$$

is a solution to Eq. $(1.1)_{d=2}$ with the logarithmic term (arising from $t^\mu \approx 1+\mu\ln t$).

Since $\frac{\Gamma(1+x+\mu)}{\Gamma(1+x-\mu)\Gamma(1+2\mu)} \approx 1 + 2\mu(\Psi(1+x)-\Psi(1))$, where Ψ denotes the Euler Psi function and $\Psi(1) = -\gamma$ is the Euler–Mascheroni constant, it follows that $\widehat{\varphi}_2 = \varphi_2 + 2(\Psi(1+x)+\gamma-\ln x)\cdot\varphi_1$ and the analytic part of the solution φ_2 equals $\psi_2 = \frac{\partial}{\partial\mu}F(\mu+x,\mu-x;1+2\mu;t)|_{\mu=0}$.

Moreover, from the expansions $\Psi(1+x) = -\gamma + \zeta(2)x - \zeta(3)x^2 + \zeta(4)x^3 - \ldots$ (see [BE1, Eq. 1.17(5)]*) and $\frac{\pi}{\tan\pi x} = \frac{1}{x} - 2\zeta(2)x - 2\zeta(4)x^3 - \ldots$ (compare* [BE1, Eq. 1.20(3)] *we get $\widehat{\varphi}_2(1;x) = -\frac{\cos\pi x}{x} + \frac{1}{x}f_2(x)$. It implies that the function*

$$\check{\varphi}_2 = \widehat{\varphi}_2 - x^{-1}\cdot\varphi_1$$

is a solution to Eq. (1.1), independent with φ_1 and such that

$$\check{\varphi}_2(1;x) = -\frac{\cos\pi x}{x}.$$

[4]Also other series ψ_j appearing in the formulas for φ_j are generating functions for some polylogarithms. For instance, in [ZZ1] it is proved that in the case $d = 2$ we have $\varphi_{2,k} = \mathrm{Li}_{2,\ldots,2}(t)\ln(x^2t) - 2\sum_{j-1}^{k}\mathrm{Li}_{2,\ldots,3,\ldots,2}(t)$, where only one index in Li equals 3. After a simple resummation one finds $\varphi_2(1;x) = 2f_2(x)\ln x + 2x^2f_2(x)\left\{\zeta(3)+\zeta(5)x^2+\zeta(7)x^4+\ldots\right\}$. However we should not regard the latter identity as something important.

Also the below solutions θ_j are expressed via the polylogarithms and $\ln s$.

In the case of higher order equations ($d > 2$) the perturbation relies on adding a differential operator of lower order with $d-1$ small parameters.

3.2. **Basic solutions near $t = 1$.** With the variable $s = 1 - t$ Eq. (1.1) takes the form

$$s\partial_s(1-s)\partial_s \dots (1-s)\partial_s g + (-1)^d x^d g = 0. \tag{3.5}$$

Analogously as in Section 3.1 we consider solutions of the form $g(1-s) = \theta_j(s;x)$ such that

$$\begin{aligned} \theta_j &= (-x^{d/(d-1)})^j \left\{\theta_{j,0}(s) + \theta_{j,1}(s)x^d + \dots\right\}, \quad (j = 1, \dots, d-1), \\ \theta_d &= \theta_{d,0}(s) + \theta_{d,1}(s)x^d + \dots \end{aligned} \tag{3.6}$$

where

$$\theta_{j,0} = \frac{1}{j!}\ln^j(1-s) = \mathrm{Li}_{1,\dots,1}(s), \quad (j = 1, \dots, d-1), \quad \theta_{d,0} = 1-d+\theta_{d-1,0}\ln x^d \tag{3.7}$$

and

$$\theta_{j,k}(s) = \int_{0<s_d \dots <s_1<s} \frac{ds_1}{1-s_1} \dots \frac{ds_2}{1-s_{d-1}} \frac{ds_d}{s_d} \theta_{j,k-1}. \tag{3.8}$$

It is clear that these solutions are analytic in $x \in \mathbb{C} \diagdown 0$ with known singularities at the origin.

Their behavior near $s = 0$ is following:

$$\begin{aligned} \theta_j(s;x) &= \tfrac{1}{j!}\left(x^{d/(d-1)}s\right)^j + O(s^d) \quad (j = 1, \dots, d-1), \\ \theta_d(s;x) &= \theta_{d-1}\ln\left(x^d s^{d-1}\right) + (1-d) + O(s). \end{aligned} \tag{3.9}$$

(compare [ZZ1, ZZ3]).

3.3. **Some relations between the two bases.** Firstly, we underline the following property which follows directly from independence of the two systems $\varphi = (\varphi_1, \dots, \varphi_d)^\top$ and $\theta = (\theta_1, \dots, \theta_d)^\top$ of solutions (see [ZZ3]).

Lemma 1. *The matrix $M = M(x)$ defined by $\theta = M\varphi$ is an analytic function of $x \in \mathbb{C} \diagdown 0$ with regular singularity at $x = 0$.*

Also the following obvious statement is important in this paper.

Lemma 2. *Let*

$$\varphi_1(t;x) = A_1(x) \cdot \theta_1(1-t;x) + \dots + A_d(x) \cdot \theta_d(1-t;x)$$

be the representation of $\varphi_1(t;x)$ near $t = 1$ in the basis θ (with the connection coefficients A_j). Then the generating function (1.6) is expressed via the last connection coefficient,

$$f_d(x) = (1-d) \cdot A_d(x).$$

In the case of standard hypergeometric equation of second order we have the following property which is proved by direct checking.

Lemma 3. *Let $d = 2$. Then, if $\varphi(t; x)$ is a solution to Eq. (1.1), then $\theta(s; x) = -s\partial_s \varphi(s; x)$ is a solution to Eq. (3.5). In particular, we have*

$$\theta_{1,2}(s; x) = -s\partial_s \varphi_{1,2}(s; x).$$

This lemma will be used below in explanation of the formula (1.8) for $f_2(x)$. On the other side, it has simple explanation in terms of the MZV duality relations.

Together with Eq. (1.1) one can consider the following equation:

$$[(1 - t)\partial_t]^{d-1} t\partial_t g + x^d g = 0. \tag{3.10}$$

It has one solution of the form

$$\phi_1(t; x) = 1 - \mathrm{Li}_{1,\ldots,1,2}(t)x^d + \mathrm{Li}_{1,\ldots,1,2,1,\ldots,,2}(t)x^{2d} - \ldots$$

(where each sequence of 1's is of length $d - 1$) and hence $\phi_1(1; x) = f_{1,\ldots,1,2}(x) = 1-\zeta(1, \ldots, 1, 2)x^d+\ldots$ is a generating function for multiple zeta values $\zeta(1, \ldots, 1, 2 \ldots 1, \ldots, 1, 2)$. But the MZV duality (see Eq. (2.8)) implies that the latter numbers equal $\zeta(d, \ldots, d)$. Therefore

$$\phi(1; x) = f_d(x)$$

is the generating function for $\zeta(d, \ldots, d)$ from Eq. (1.6). Of course, for $d = 2$ it is nothing new, because the values $\zeta(2, \ldots, 2)$ are fixed under the duality transformation.

There exists another relation between Eqs. (1.1) and (3.10). Namely,

if $\varphi(t; x)$ is a solution to Eq. (1.1) near $t = 0$ then for $s = 1 - t \approx 0$ the function $\vartheta(s; x) = (s\partial_s)^{d-1} \varphi(s; -x)$ is a solution to Eq. (3.10) near $t = 1$ but for the parameter x replaced with $-x$, i.e. to the equation

$$(s\partial_s)^{d-1} (1 - s)\partial_s g + (-x)^d g = 0.$$

4. WKB solutions

Theoretically Eq. (1.1) for large parameter x can be solved using the WKB method. This means that one represents a solution as a finite sum of terms of the form

$$x^\gamma e^{xS(t)} \left\{\chi_0(t) + \chi_1(t)x^{-1} + \ldots\right\}. \tag{4.1}$$

In general the series in the above formula are divergent, but this divergence can be somehow controlled. Below we present three approaches to the WKB solutions to Eq. (1.1): formal, via normal forms and using the stationary phase formula (in Section 6).

The name of the method comes from the names of its authors G. Wentzel [Wen], H. Kramers [Kr] and L. Brillouin [Bri]. Originally it was used to solve approximately the Schrödinger equation [Sch], but here we use it to the hypergeometric equation.

4.1. **Testing WKB solutions.** These are solution of the form

$$g(t;x) = x^{\gamma} e^{xS(t)} \chi(t;x^{-1}), \tag{4.2}$$

where χ is a power series in x^{-1}. Substituting it into equation (1.1) we get

$$x^d \left\{(1-t)t^{d-1}\left(\dot{S}\right)^d + 1\right\}\chi + x^{d-1}\frac{1-t}{t}\mathcal{P}_1\chi + \ldots + \frac{1-t}{t}\mathcal{P}_d\chi = 0, \tag{4.3}$$

where $\dot{S} = dS/dt$ and $\mathcal{P}_j$ are some differential operators and the first of them is following:

$$\mathcal{P}_1\chi = d\cdot\left(t\dot{S}\right)^{d-2}\cdot\left\{t\partial S\cdot t\partial\chi + \frac{d-1}{2}(t\partial)^2 S\cdot\chi\right\}. \tag{4.4}$$

It follows that the '*action*' $S(t)$, the solution to the '*Hamilton–Jacobi equation*'

$$(1-t)t^{d-1}\left(\dot{S}\right)^d + 1 = 0, \tag{4.5}$$

equals

$$S = \sigma S_d(t) := \sigma\int_0^t \frac{d\tau}{\tau^{(d-1)/d}(1-\tau)^{1/d}}, \quad \sigma = \varsigma^{j+1/2}, \quad j = 0,\ldots,d-1, \tag{4.6}$$

where ς is the root of unity from Eq. (1.3). These d possibilities correspond to d solutions, which can be expanded as follows

$$g^{\sigma}_{\text{test}}(t;x) = (\sigma x)^{\gamma}\, e^{\sigma x S_d(t)}\left\{\chi_0(t) - \frac{\chi_1(t)}{\sigma x} + \frac{\chi_2(t)}{(\sigma x)^2}\cdots\right\}, \quad \gamma = -\frac{d-1}{2}. \tag{4.7}$$

The functions χ_j satisfy the '*transport equations*'

$$\mathcal{P}_1\chi_0 = 0, \quad \mathcal{P}_1\chi_1 = \mathcal{P}_2\chi_0, \ldots$$

where in definition of $\mathcal{P}_j$ we use $S = S_d$. The first transport equation is easy: we have $\chi_0 =$const$\cdot\left(t\dot{S}_d\right)^{(1-d)/2}$. We choose it in the form

$$\chi_0(t) = \left(\frac{1-t}{t}\right)^{(d-1)/2d}. \tag{4.8}$$

To solve the other equations one introduces the new variable

$$u = \left(\frac{t}{1-t}\right)^{1/4} \text{ for } d = 2 \text{ and } u = \left(\frac{t}{1-t}\right)^{1/d} \text{ for odd } d \geq 3; \tag{4.9}$$

thus $\chi_0(t) = u^{-1}$ $(d = 2)$ or $\chi_0(t) = u^{(1-d)/2}$ (odd $d \geq 3$). The following result was proved in[ZZ1] for $d = 2$ and in [ZZ3] for $d = 3$ but it holds in general case.

Lemma 4. *The functions $\chi_j(t)$, $j>1$, can be chosen as Laurent polynomials in u, such that the term with u^{-1} (respectively $u^{(1-d)/2}$) is absent.*[5]

For example, when $d=2$ we have

$$\chi_{k+1}(t)=(T\chi_k)(u)=\frac{1}{8u}\int^u \frac{1}{v}\partial_u\left(v(1+v^4)\,\partial_u\chi_k\right)dv.$$

This gives

$$\chi_1=-\left(u^{-3}+3u\right)/16,\quad \chi_2=3(3u^{-5}-5u^3)/8^3. \tag{4.10}$$

A general algebraic formula can be obtained using the functions $\omega_k(u)=(2k-1)\,u^{-2k-1}+(-1)^{k+1}(2k+1)\cdot u^{2k-1}$, $k=1,2,\ldots$, which satisfy the recurrent relations: $T\omega_1=-\frac{3\cdot 1}{8\cdot 4}\omega_2$, $T\omega_k=-\frac{4k^2-1}{8}\left\{\frac{\omega_{k+1}}{k+1}-\frac{\omega_{k-1}}{k-1}\right\}$. It follows that $\chi_k(t)=a_{k,k}\omega_k(u)+a_{k,k-2}\omega_{k-2}(u)+\ldots$, for some coefficients $a_{k,l}$ which are calculated inductively. The latter coefficients grow very fast with k; for instance, we have $a_{k,k}=(2k-1)\,(-1/8)^{k-1}\,((2k-3)!!)^2/(2k-2)!!$.

Definition 1. *The formal expressions*

$$g^\sigma_{\text{test}}(t;x)\sim\frac{e^{\sigma x S_d(t)}}{(\sigma x)^{(d-1)/2}}\cdot\left(\frac{1-t}{t}\right)^{(d-1)/2d},$$

$\sigma=\varsigma^{j+1/2}$, $j=0,\ldots,d-1$, defined in equation (4.7) with the coefficients $\chi_j(t)$ defined as above (without u^{-1} or $u^{(1-d)/2}$ for $j>1$) are called the ***testing WKB solutions associated with*** *$t=0$.*

We introduce also another system of ***testing WKB solutions associated with*** *$s=1-t=0$:*

$$\begin{aligned} h^\sigma_{\text{test}}(s;x) &= \xi_\sigma(\sigma x)^{d/2}e^{-\sigma x S_d(1)}\cdot g^\sigma_{\text{test}}(1-s;x) \\ &\sim \sqrt{-\sigma x}\cdot e^{-\sigma x(S_d(1)-S_d(1-s))}\cdot\left(\frac{s}{1-s}\right)^{(d-1)/2d}, \end{aligned} \tag{4.11}$$

where $\xi_\sigma\in\mathbb{S}^1$.

Above we agree that for $0<t<1$ and $\arg x=0$ we take:[6]

$$g^\pm\sim\frac{\exp}{\sqrt{\pm ix}}=e^{\mp i\pi/4}\frac{\exp}{\sqrt{x}},\quad h^\pm\sim\sqrt{\frac{x}{\pm i}}\exp=e^{\mp i\pi/4}\sqrt{x}\exp$$

---

[5] The general solution to the system of transport equations contains infinitely many constants, to each particular solution $\chi_j(t)$ we can add $c_j\chi_0(t)$ for a constant c_j. It the case of Schrödinger equation one avoids analogous problem of arbitrary constants of integration by assuming that the wave functions (representing bound states of a quantum system) vanish at infinity; that restriction leads to so-called Born–Sommerfeld quantization condition (see [Sch]).

[6] In [ZZ1] the notations g_0^+ and g_0^- for g^i_{test} and g^{-i}_{test}, $i=e^{i\pi/2}$, are used. In [ZZ3] one uses the notations $g_0^-, g_0^\epsilon, g_0^{\bar\epsilon}$ for g^σ_{test}, $\sigma=-1,\epsilon=e^{i\pi/3},\bar\epsilon$. Also for h^σ_{test} analogous notations are used.

for $d = 2$ and

$$g^\sigma \sim \frac{\exp}{\sigma x}, \quad h^- \sim \sqrt{x}\exp, \quad h^\epsilon = \bar{\epsilon}\sqrt{x}\exp, \quad h^{\bar{\epsilon}} \sim \epsilon\sqrt{x}\exp,$$

$(\sigma = -1, \epsilon, \bar{\epsilon})$ for $d = 3$.

4.2. **Formal reduction to normal form.** Here we present an alternative way to derive WKB type solutions to equations with a parameter like Eq. (1.1). The obtained basic WKB solutions g^σ_{norm} differ from the testing WKB solutions g^σ_{test} from Definition 1 by factors which depends on x. There are reasons to regard the new solutions are more natural than the testing solution.

In the presentation we describe only the simplest case $d = 2$. Here we will use the notations $g^\pm$ (see Note 6).

Putting

$$g_1 = g, \quad g_2 = \dot{g}/x \tag{4.12}$$

we rewrite Eq. (1.1) in form of the following first order system

$$\frac{d}{dt}\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = A(t;x)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix},$$

where

$$A = xA_1(t) + A_0(t), \quad A_1 = \begin{pmatrix} 0 & 1 \\ 1/t(t-1) & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 \\ 0 & -1/t \end{pmatrix}.$$

The normal form of such system is a diagonal (or independent) system obtained by means of a formal linear change which depends on t.

The first step is the diagonalization of the matrix $A_1(t)$ with the eigenvalues

$$\lambda_1^\pm(t) = \pm i/\sqrt{t(1-t)} = \pm i \cdot \dot{S}_2(t). \tag{4.13}$$

We put

$$X^+ = \lambda_1^+(t)g_1 + g_2, \quad X^- = \lambda_1^-(t)g_1 + g_2 \tag{4.14}$$

and we get

$$\begin{aligned} \dot{X}^+ &= \lambda_1^+(t)xX^+ - \tfrac{1}{4}\left(\tfrac{3}{t} - \tfrac{1}{1-t}\right)X^+ - \tfrac{1}{4}\left(\tfrac{1}{t} + \tfrac{1}{1-t}\right)X^-, \\ \dot{X}^- &= \lambda_1^-(t)xX^- - \tfrac{1}{4}\left(\tfrac{1}{t} + \tfrac{1}{1-t}\right)X^+ - \tfrac{1}{4}\left(\tfrac{3}{t} - \tfrac{1}{1-t}\right)X^-. \end{aligned} \tag{4.15}$$

The general theory says that such system can be diagonalized by means of an infinite series of 'shearing' transformations. Let us apply some initial changes, in order to compare the obtained (partial) normal form with the results of the previous and next subsections. We put

$$X^+ = X_1^+ + \left(\frac{b_1}{x} + \frac{b_2}{x^2} + \ldots\right)X_1^-, \quad X^- = \left(\frac{c_1}{x} + \frac{c_2}{x^2} + \ldots\right)X_1^+ + X_1^-, \tag{4.16}$$

where b_j, c_j depend on t, and we expect to obtain the following separated system

$$\dot{X}_1^+ = \lambda^+(t;x)X_1^+, \quad \dot{X}_1^- = \lambda_1^-(t;x)X_1^-, \tag{4.17}$$

$$\lambda^{\pm}(t;x)=\lambda_1^{\pm}(t)x+\lambda_0^{\pm}(t)+\lambda_{-1}^{\pm}(t)x^{-1}+\dots$$

The resulted system of equations onto b_j, c_j, $\lambda_j^{\pm}$ is easily solved; moreover, in algebraic way. Using the variable $u=(t/(1-t))^{1/4}$ (see Eq. (4.9)) we get $b_1=-c_1=-i/8\,(t(1-t))^{1/2}=-i(1+u^4)/8u^2$, $b_2=c_2=(1-2t)\,/32t(1-t)=\left(1-u^8\right)/32u^4$ and $\lambda_0^{\pm}=\mp\frac{1}{4}\left(\frac{3}{t}-\frac{1}{1-t}\right)$, $\lambda_{-1}^{\pm}=\mp i/32\,(t(1-t))^{3/2}=\mp i(1+u^4)^3/32u^6$, $\lambda_{-2}^{\pm}=(2t-1)/128t^2(1-t)^2=(u^4-1)(1+u^4)^4/128u^8$.

General solutions to the system (4.17) are of the form

$$\begin{aligned} X_1^{+}&=K_{+}\tfrac{e^{ixS(t)}}{t^{3/4}(1-t)^{1/4}}\exp\left\{\tfrac{-i}{16x}\left(u^2-\tfrac{1}{u^2}\right)-\tfrac{1}{512x^2}\left(u^4+2+\tfrac{1}{u^4}\right)+\dots\right\},\\ X_1^{-}&=K_{-}\tfrac{e^{-ixS(t)}}{t^{3/4}(1-t)^{1/4}}\exp\left\{\tfrac{i}{16x}\left(u^2-\tfrac{1}{u^2}\right)-\tfrac{1}{512x^2}\left(u^4+2+\tfrac{1}{u^4}\right)+\dots\right\},\end{aligned} \tag{4.18}$$

with arbitrary constants $K_{\pm}$ (which may depend on x). Substituting this to Eq. (4.16) and then to $g=\frac{1}{2\lambda}\left(X^{+}-X^{-}\right)$ (see Eq. (4.14)) one finds a general solution to Eq. (1.1) in the form

$$g=K_{+}g_{\text{norm}}^{+}(t;x)+K_{-}g_{\text{norm}}^{-}(t;x),$$

where

$$g_{\text{norm}}^{\pm}(t;x)=\left(1+(5/256)x^{-2}+\dots\right)\cdot g_{\text{test}}^{\pm}(t;x) \tag{4.19}$$

and $g_{\text{test}}^{\pm}$ are the testing WKB solutions (see Definition 1 and Eq. (4.7)).

For general degree $d\geq 2$ we have $g_1=g, g_2=\partial g/x,\dots,g_d=\partial^{d-1}g/x^{d-1}$ in an analogue of Eqs. (4.12), $\lambda_1^{\sigma}=\sigma\dot{S}_d(t)$, $\sigma=\varsigma^{j+1/2}$, $j=0,\dots,d-1$, in Eq. (4.13) and we finally obtain the diagonal system

$$\dot{X}_1^{\sigma}=\lambda^{\sigma}(t;x)X_1^{\sigma},\quad \lambda^{\sigma}=\lambda_1^{\sigma}(t)x+\lambda_0^{\sigma}(t)+\lambda_{-1}^{\sigma}(t)x^{-1}+\dots, \tag{4.20}$$

with solutions $X_1^{\sigma}=K_{\sigma}\cdot\exp\int_0^t\lambda^{\sigma}(\tau;x)d\tau$, which imply the formula

$$g=\sum_{\sigma}K_{\sigma}\cdot g_{\text{norm}}^{\sigma}(t;x) \tag{4.21}$$

for a general (formal) solution to the hypergeometric equation (1.1).

Definition 2. *The solutions g^{σ} are called the* ***normal WKB solutions associated with the point*** $t=0$*. Corresponding* ***normal WKB solutions associated with the point*** $s=1-t=0$ *are* $h_{\text{norm}}^{\sigma}(s;x)=\xi_d\,(\sigma x)^{d/2}\,e^{-\sigma x S_d(1)}g^{\sigma}(1-s;x)$ *(where ξ_d is the same as in Definition 1).*

The normal WKB solutions are also defined uniquely, because the reduction to the normal form is unique and essentially algebraic. They seem to be more important than the testing WKB solutions g_{test}^{σ}, because we can show that they are represented by analytic functions in some sectorial domains (due to some Birkhoff's theorem discussed below).

Note also that the normal form system (4.20) is more natural than the WKB solutions g_{norm}^{σ}, because the latter involve the initial condition $S_d(0)=0$.

Remark 2. *The relation between* $g^{\sigma}_{\mathrm{norm}}$ *and* $g^{\sigma}_{\mathrm{test}}$ *is of the form*

$$g^{\sigma}_{\mathrm{norm}}(t;x) = C^{\sigma}_{\mathrm{norm}}(x^{-1}) \cdot g^{\sigma}_{\mathrm{test}}(t;x),$$

where $C^{\sigma}_{\mathrm{norm}}(x^{-1}) = 1 + O(x^{-1})$ *are formal series. It seems that all the series* $C^{\sigma}_{\mathrm{norm}}(x^{-1})$ *are the same for any index* σ *and depend on* x^{-d}. *This is proved for* $d = 2$ *in* [ZZ1]. *Also from Eq. (4.19) it follows that these series are nontrivial.*

4.3. **Analytic normalization.** We have seen that the process (which is standard) of successive reduction of Eq. (4.15) to the normal (diagonal) form is essentially algebraic. It is also unique. Unfortunately, it is divergent.

The problem of analytic interpretation of the WKB method is highly nontrivial. There exist known results about WKB functions which are analytic in some rather special domains and have the same asymptotic expansions as the formal WKB series. But those analytic functions undergo dramatic changes when the domains are changed; this is the famous Stokes phenomenon studied in Section 7.

Additional complication arises from the dependence of two variables: x (which is large) and t (which is bounded). In a traditional approach, used mostly by the physicists [He, BNR], the parameter x is real and the variable t may vary in some complex domain. In that domain there exist so-called Stokes lines which separate domains of uniqueness of the WKB functions. Several Stokes lines meet at so-called turning points, which are the ramification points of the derivative $\dot{S}(t) = dS/dt$ of the 'action' (like $\dot{S}(t) = \sqrt{q(t)}$ for the Schrödinger equation $\ddot{\psi} = -x^2 q(t)\psi$). In our situation, the fact that $\dot{S}(t)$ is infinite at $t = 0$ and $t = 1$ causes additional complication.

Since our principal aim is to study analytic properties of the connection coefficient $A_d(x)$ in Lemma 2, we should rather consider complex x, while t can stay real. When one allows $\arg x$ to vary the Stokes lines also should vary in a controllable way (see [DePh]). But this controlling is rather troublesome and we prefer to use our own method.

One ingredient of this method is exemplified in Theorem 1 below (we refer the reader to our original work [ZZ2]). It allows to treat analytically WKB functions in two domains in $\mathbb{C} \times \mathbb{C} = \{(t,x)\} : \mathcal{U}_0 \times \mathcal{V}_\infty$ and $\mathcal{U}_1 \times \mathcal{V}_\infty$, where $\mathcal{U}_{0,1}$ are neighborhoods of $t = 0, 1$ and $\mathcal{V}_\infty = (\mathbb{C}, \infty)$. In these domains we are able to control perfectly the Stokes lines and their $x-$dependence (see Section 7).

Another ingredient (realized in this section) is an analogue of a theorem due to G. D. Birkhoff [Bir] about WKB functions analytic in domains like $\mathcal{W} \times \mathcal{S}$ where $\mathcal{W}$ is a neighborhood of the 'interior' of the segment $[0, 1]$ in the $t-$plane and $\mathcal{S}$ is a sector in the $x-$plane. The above domains have non-empty suitable intersections which allows to provide an analytic realization of formal WKB type series for solutions of differential equations and of the connection coefficient $A_d(x)$.

The reduction (4.16) is divergent (as a power series in x^{-1}) and the WKB solutions $g^{\pm}$ are only formal solutions. G. Birkhoff [Bir] was the first who proved that such a system can be diagonalized analytically in some sectorial domains. Below we present a scheme of the Birkhoff's proof in the case $d=2$.

We apply a change

$$X^{+}=X_1^{+}+V^{12}(t)X_1^{-},\quad X^{-}=V^{21}(t)X_1^{+}+X_1^{-} \tag{4.22}$$

which should transform system (4.15), i.e.

$$\frac{d}{dt}\begin{pmatrix}X^{+}\\ X^{-}\end{pmatrix}=\begin{pmatrix}B^{11} & B^{12}\\ B^{21} & B^{22}\end{pmatrix}\begin{pmatrix}X^{+}\\ X^{-}\end{pmatrix},$$

to the diagonal form

$$\dot{X}_1^{+}=D_{+}(t)X_1^{+},\quad \dot{X}_1^{-}=D_{-}(t)X_1^{-}. \tag{4.23}$$

We get $D_{+}=B^{11}+B^{12}V^{21}$, $D_{-}=B^{21}V^{12}+B^{22}$ and two independent Riccati equations

$$\begin{aligned}\dot{V}^{12} &= B^{11}V^{12}-V^{12}B^{22}+B^{12}-V^{12}B^{21}V^{12},\\ \dot{V}^{21} &= B^{22}V^{21}-V^{21}B^{11}+B^{21}-V^{21}B^{12}V^{21}.\end{aligned}$$

The latter differential equations are rewritten in form of the following integral equations:

$$V^{12}(t)=\int_{\Gamma_1(t)}e^{P(t)-P(\tau)}\left\{B^{12}(\tau)-V^{12}(\tau)B^{21}(\tau)V^{12}(\tau)\right\}d\tau, \tag{4.24}$$

$$V^{21}(t)=\int_{\Gamma_2(t)}e^{P(\tau)-P(t)}\left\{B^{21}(\tau)-V^{21}(\tau)B^{12}(\tau)V^{21}(\tau)\right\}d\tau, \tag{4.25}$$

$P(t)=\int_0^t(B^{11}(\iota)-B^{22}(\iota))d\iota=2ixS_2(t)+\dots$. Here $\Gamma_1(t)$ and $\Gamma_2(t)$ are some well chosen paths in the $\tau-$plane.

One would like to treat Eqs. (4.24)–(4.25) as fixed point equations in suitable functional spaces. For this the nonlinear operators defined by the right-hand sides should be contracting, at least bounded (see [Was, Zo3]).

The crucial element in the proof of the latter property is the possibility to estimate the factors $e^{\pm(P(t)-P(\tau))}\approx\exp\left\{\pm 2ix(S_2(t)-S_2(\tau))\right\}$. Thus, if $t\in(0,1)$ is real, then for $\operatorname{Im}x>0$ we take the integration paths as segments $\Gamma_1=[0,t]$ and $\Gamma_2=[1,t]$; when $\operatorname{Im}x<0$ we take $\Gamma_1=[1,t]$ and $\Gamma_2=[0,t]$.

But the entries $B^{ij}(t)$ of the matrix B have poles at $t=0$ and $t=1$. Moreover, we want to extend the range of $\arg x$ and to allow complex values of t. We choose three small constants $\alpha>0$, $\beta>0$ and $0<\tau_0<<\beta$ and define the following domains: $\mathcal{W}=\{t=t_1+it_2:\ \beta<t_1<1-\beta,\quad |t_2|<\beta t_1(1-t_1)\}\subset\mathbb{C}$ (a neighborhood of the open segment $(\beta,1-\beta)\subset\mathbb{R}$) and $\mathcal{D}_u,\mathcal{D}_d\subset\mathbb{C}^2$ ('up' and 'down') by the conditions

$$\begin{aligned}\operatorname{Im}xS_2(t),\operatorname{Im}x(S_2(1)-S_2(t)) &> -\alpha,\ t\in\mathcal{W}\quad(\text{for }\mathcal{D}_u),\\ \operatorname{Im}xS_2(t),\operatorname{Im}x\,(S_2(1)-S_2(t)) &< \alpha,\ t\in\mathcal{W}\quad(\text{for }\mathcal{D}_d).\end{aligned}$$

If $(t, x) \in \mathcal{D}_u$ then the contour Γ_1 begins at $\tau = \tau_0$ and ends at $\tau = t$ and the path Γ_2 begins at $\tau = 1 - \tau_0$ and ends at $\tau = t$ and with $\operatorname{Im} x(S(t) - S(\tau)) < 0$. For $(t, x) \in \mathcal{D}_d$ the choice of the contours is opposite.

Solving the integral equations in the domains $\mathcal{D}_u$ and $\mathcal{D}_u$ one obtains analytic solutions $g_u^{\pm}(t; x)$ and $g_d^{\pm}(t; x)$ respectively. They have the same formal asymptotic expansions as the principal WKB solutions $g^{\pm}(t; x)$.

We note the conjugation symmetry of the above construction:

$$\overline{g_u^+(t; x)} = g_d^-(\bar{t}; \bar{x}), \quad \overline{g_u^-(t; x)} = g_d^+(\bar{t}; \bar{x}).$$

In the case of general degree $d \geq 2$ the corresponding system of Riccati type equations consists of $d(d-1)$ equations for the off-diagonal entries $V^{\sigma\rho}(t)$ of the matrix $V(t)$ (with 1's on the diagonal) such that $X = V X_1$. The corresponding integral equations take the form

$$V^{\sigma\rho}(t) = \int_{\Gamma^{\sigma\rho}} e^{(\sigma-\rho)x(S_d(t)-S_d(\tau))} F^{\sigma\rho}(\tau, V(\tau)) d\tau. \tag{4.26}$$

Here there are $2d$ domains $\mathcal{D}_{1,2}, \mathcal{D}_{2,3}, \ldots, \mathcal{D}_{2d,1}$ being neighborhoods of the sectorial sets $[\beta, 1-\beta] \times \overline{S_{k,k+1}}$, where $\overline{S_{k,k+1}}$, $k = 1, \ldots 2d$ (and $2d + 1 = 1$), are closed sectors defined by division of a neighborhood of $x = \infty$ by the lines $\arg x = j\pi/d$, $j = 0, \ldots, d-1$. One obtains solutions $g^{\sigma}_{k,k+1}(t; x)$ analytic in the domains $\mathcal{D}_{k,k+1}$. From the construction they satisfy the following symmetry properties:

$$g^{\sigma}_{k+2,k+3}(t, \varsigma x) = g^{\varsigma\sigma}_{k,k+1}(t; x), \tag{4.27}$$

$$\overline{g^{\sigma}_{k,k+1}(t; x)} = g^{\bar{\sigma}}_{2d-k+1,2d-k+2}(\bar{t}; \bar{x}), \tag{4.28}$$

$\varsigma = e^{2\pi i/d}$.

Let us summarize the results of this subsection in the following

Proposition 1. *For $d > 2$ there exist $2d$ systems of solutions $(g^{\sigma}_{k,k+1})$, $k = 1, \ldots, 2d$, analytic in the domains $\mathcal{D}_{k,k+1}$ (defined above) whose formal expansions are the same as for the normal WKB solutions $g^{\sigma}_{\mathrm{norm}}$ from Definition 2. They satisfy relations (4.27) and (4.28).*

For $d = 2$ there exist two such systems $(g^{\sigma}_u) = \left(g^{\sigma}_{1,2}\right)$ and $(g^{\sigma}_d) = \left(g^{\sigma}_{2,1}\right)$ analytic in the domains $\mathcal{D}_u = \mathcal{D}_{1,2}$ and $\mathcal{D}_d = \mathcal{D}_{2,1}$.

5. Bessel approximations

5.1. **Bessel type equations and their basic solutions.** Consider series (1.2) when $x \to \infty$ and

$$y = x^d t$$

is finite. Then we get

$$\varphi_1(t;x) \approx \Phi_1(y) := \sum_{n=0}^{\infty} \frac{(-y)^n}{(n!)^d} =_0 F_{d-1}(1,\ldots,1;-y), \tag{5.1}$$

i.e. a confluent hypergeometric function. For $d=2$ the function Φ_1 is expressed via a Bessel function:[7]

$$\Phi_1(y)|_{d=2} = J_0(2\sqrt{y}). \tag{5.2}$$

The function Φ_1 satisfies a special confluent hypergeometric equation, which we call the **Bessel type equation**:

$$\partial_y (y\partial_y)^{d-1} G + G = 0. \tag{5.3}$$

The other independent solutions to Eq. (5.3) are

$$\begin{array}{l} \Phi_2(t) = \Phi_1(y)\ln y + \Psi_2(y), \\ \Phi_3(t) = \frac{1}{2!}\Phi_1 \ln^2 y + \Psi_2 \ln y + \Psi_3(y), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Phi_d(y) = \frac{1}{(d-1)!}\Phi_1 \ln^{d-1} y + \frac{1}{(d-2)!}\Psi_2 \ln^{d-2} y + \ldots + \Psi_d(y) \end{array} \tag{5.4}$$

(where Ψ_j are some entire functions), they approximate the solutions φ_j.

Of course, Eq. (5.3) is obtained from Eq. (1.1) by the change $t = y/x^d$, $\partial_t = x^d\partial_y$ and taking limit as $x \to \infty$. We shall do analogous change with Eq. (3.5) by taking x large and

$$z = x^d s^{d-1}$$

finite. The obtained **Bessel type equation** is following:

$$(1-d)^d \cdot z^{\frac{1}{d-1}} \left(z^{\frac{d-2}{d-1}} \partial_z \right)^d H + H = 0. \tag{5.5}$$

It has basic solutions of the form

$$\begin{array}{l} \Theta_j(z) = \frac{1}{j!} z^{j/(d-1)} F_j(z) = \frac{1}{j!} z^{j/(d-1)} \cdot (1 + O(z)), \quad (j = 1,\ldots,d-1), \\ \Theta_d(z) = \Theta_{d-1}(z)\ln z + \Xi_d(z), \end{array} \tag{5.6}$$

where $F_j(z)$ are some concrete confluent hypergeometric series and Ξ_d is an entire function.

For $d=2$ we have

$$\Theta_1|_{d=2} = \sqrt{z} J_1(2\sqrt{z}) \tag{5.7}$$

---

[7] Recall that the Bessel function with index μ equals $J_\mu(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(\mu+n-1)n!} \left(\frac{w}{2}\right)^{2n+\mu}$.

and for $d = 3$ we have

$$\Theta_1|_{d=3} = \sqrt{z}\left(1 + \sum_{n=1}^{\infty} \frac{z^n}{(2n+1)!(2n-1)!!}\right) = \sqrt{z} \cdot {}_0F_2\left(\alpha, \beta; \frac{z}{8}\right), \tag{5.8}$$

$$\Theta_2|_{d=3} = 2\sum_{n=1}^{\infty} \frac{z^n}{(2n)!(2n-2)!!} = z \cdot {}_0F_2\left(\gamma, \delta; \frac{z}{8}\right), \tag{5.9}$$

where $\alpha = \delta = n + 1/2$, $\beta = n - 1/2$, $\gamma = n + 1$.

5.2. **Formal and analytic WKB solutions.** The Bessel type equation (5.3) has irregular singular point at $y = \infty$ and equation (5.5) has irregular singular point at $z = \infty$. Any linear meromorphic differential equation with an irregular singular point has uniquely defined (up to a multiplicative constants) formal solution which we call the WKB solutions.

For Eq. (5.3) the **WKB solutions** are of the form

$$G^\sigma(y) = \left(\sigma y^{1/d}\right)^\gamma e^{d\sigma y^{1/d}} \left\{1 - \frac{a_1}{\sigma y^{1/d}} + \frac{a_2}{\left(\sigma y^{1/d}\right)^2} - \ldots\right\}, \quad \gamma = -\frac{d-1}{2}, \tag{5.10}$$

and the **WKB solutions** for Eq. (5.5) are following:

$$H^\sigma(z) = \sqrt{-\sigma z^{1/d}} e^{(d/(1-d)\sigma z^{1/d}} \left\{1 + \frac{b_1}{\sigma z^{1/d}} + \frac{b_2}{\left(\sigma z^{1/d}\right)^2} + \ldots\right\}, \tag{5.11}$$

where $\sigma = \varsigma^{j+1/2}$, $j = 0, \ldots, d-1$, (as usual), the choice of the square root $\sqrt{-\sigma z^{1/d}}$ is defined in Definition 1 and the coefficients are computed recursively.

The dependence of the above functions on the roots $y^{1/d}$ and $z^{1/d}$ is not useful in calculations. Often we will use the variables

$$v = y^{1/d}, \quad w = z^{1/d} \tag{5.12}$$

and denote corresponding WKB solutions as

$$\widetilde{G}^\sigma(v) = -G^\sigma(v^3), \quad \widetilde{H}^\sigma(w) = H^\sigma(w^d). \tag{5.13}$$

They satisfy the following Bessel type equations:

$$(v\partial_v)^d \widetilde{G} + d^d \cdot v^d \widetilde{G} = 0, \tag{5.14}$$

$$(1/d - 1)^d \cdot w^{\frac{d}{d-1}} \left(w^{\frac{-1}{d-1}} \partial_w\right)^d \widetilde{H} + d^d \cdot \widetilde{H} = 0. \tag{5.15}$$

Like in Section 4.2 we can transform each of the Eqs. (5.14)–(5.15) to a corresponding linear system which is next diagonalized using shearing transformations. The obtained diagonal system has basic solutions which must equal the WKB solutions from Eqs. (5.13). This formal reduction of the Bessel type equations to the normal form is in complete agreement with the analogous reduction of the hypergeometric equation.

But when we want to obtain analytic normal forms, then one encounters some differences with what is done in Section 4.3. For example, in the case of Eq. (5.14) one arrives to an analogue of Eq. (4.26), i.e.

$$V^{\sigma\rho}(v)=\int_{\Gamma^{\sigma\rho}} e^{d(\sigma-\rho)(v-\tau)}F^{\sigma\rho}(\tau,V(\tau))d\tau,$$

but now the paths $\Gamma^{\sigma\rho}=\Gamma^{\sigma\rho}(v)$ of integration are chosen rather differently.

Consider sectors $\mathcal{S}_1,\ldots,\ \mathcal{S}_{2d}$ with angles $2\pi/d-\delta$ ($\delta>0$ small) and with the bisectrices $\arg v=0,\pi/d,\ldots,(d-1)\pi/d$. These bisectrices $\mathcal{R}_j$ correspond to the situations when $\operatorname{Im}(\sigma-\rho)\,v=0$ (for some σ and ρ) and are called the **rays of division** associated with the pair (σ,ρ).

With given unordered pair $\{\sigma,\rho\}$ two rays of division $\mathcal{R}_j$ and $\mathcal{R}_{j+d}$ are associated (here $j+d$ is taken mod $2d$). Consider larger sectors $\mathcal{S}_{j-[d/2]}\cup\ldots\cup\mathcal{S}_j\cup\ldots\cup\mathcal{S}_{j+[d/2]}$ and $\mathcal{S}_{j+d-[d/2]}\cup\ldots\cup\mathcal{S}_{j+d}\cup\ldots\cup\mathcal{S}_{j+d+[d/2]}$ with the above rays as their bisectrices; they cover a neighborhood of $v=\infty$. For $v\in\ldots\cup\mathcal{S}_j\cup\ldots$ (respectively $v\in\ldots\cup\mathcal{S}_{j+3}\cup\ldots$) the path $\Gamma^{\sigma\rho}(v)$ runs parallel to the ray $\mathcal{R}_j$ from $\tau=\infty$ to $\tau=v$. Due to the fact that the factors $e^{d(\sigma-\rho)\tau}$ in the corresponding integral equations are bounded for $\tau\in\Gamma^{\sigma\rho}(v)$ the solutions to the integral equations exist and are analytic in the sectors $\mathcal{S}_k$.

We denote the analytic solutions in the sectors $\mathcal{S}_j$ obtained above by

$$\widetilde{G}^{\sigma}_j(v),\ v\in\mathcal{S}_j,\quad j=1,\ldots,6. \tag{5.16}$$

They are formally equivalent to the formal WKB solutions form Eqs. (5.10)–(5.13). (But for $d=2$ we have only two sectors $\mathcal{S}_1=\mathcal{S}_r$ (right)and $\mathcal{S}_2=\mathcal{S}_l$ (left) with bisectrices $\mathcal{R}_1=\{\arg v=0\}$ and $\mathcal{R}_2=\{\arg v=\pi\}$ and angles $2\pi-\delta$ and two sets of solutions $\widetilde{G}^{\pm}_{r,l}(v)$.

Analogously we obtain systems of analytic solutions to Eq. (5.15):

$$\widetilde{H}^{\sigma}_j(w),\quad w\in\mathcal{S}_j,\quad j=1,\ldots,2d. \tag{5.17}$$

Remark 3. *Functions (5.16) and (5.17) were constructed by solving corresponding integral equations. But there exist explicit integral formulas for analytic WKB solutions to Bessel type equations (and to general hypergeometric confluent equations) due to A. Duval and C. Mitschi* [DuMi] *(see also* [ZZ3]*). For example, for $d=3$ the following **Mellin–Barnes integral***

$$G^{-}_{DM}(y)=\frac{1}{2\pi i}\int_\gamma \Gamma^3(-\tau)y^\tau d\tau,$$

where γ is a path from $\tau=-i\infty$ to $\tau=+i\infty$ which leaves the poles $\tau=1,2,\ldots$ of the Gamma function from the right, defines a solution to the Bessel type equation (5.3) for $d=3$. (The function G^{-}_{DM} is a particular case of the so-called Meijer G–functions, [Me] *and* [BE1]*). It turns out that $G^{-}_{DM}(y)$ is analytic in the sector $\left\{-\pi-\varepsilon<\arg y^{1/3}<\pi+\varepsilon\right\}$ and has the form $G^{-}_{DM}=e^{-3y^{1/3}}y^{-1/3}\Omega_0(y^{-1/3})$ (like G^-).*

Moreover other WKB solutions can be taken in the form

$$G^{\epsilon}_{DM}(y) = e^{3\epsilon y^{1/3}} y^{-1/3}\Omega_0(\bar{\epsilon} y^{-1/3}), \quad G^{\bar{\epsilon}}_{DM}(y) = e^{3\bar{\epsilon} y^{1/3}} y^{-1/6}\Omega_0(\epsilon y^{-1/3})$$

(where the notations $-, \epsilon, \bar{\epsilon}$ *are like in Note 6). The new WKB solutions* H^{-}_{DM}, H^{ϵ}_{DM}, $H^{\bar{\epsilon}}_{DM}$ *to the Bessel type equation (3.7) are defined similarly, via the following Mellin–Barnes integral:*

$$\begin{aligned} H^{-}_{DM}(z) &= \frac{1}{2\pi i}\int_{\gamma} \Gamma(1-\tau)\Gamma(1/2-\tau)\Gamma(-\tau)(-z/8)^{\tau} d\tau \\ &= e^{\frac{3}{2}z^{1/3}} z^{1/6}\Omega_1(z^{-1/3}). \end{aligned}$$

Also for other degrees $d \neq 3$ *Duval and Mitschi define WKB solutions* G^{σ}_{DM} *and* H^{σ}_{DM} *analytic in suitable sectors about infinity.*

Finally, we note that analyticity of the WKB solutions in sectors can be proved in still another way, using the fact that the formal WKB solutions are defined via Gevrey type series, by applying corresponding Borel and Laplace transforms. We refer the reader to the books of W. Balser [Bal] *and J.-P. Ramis* [Ram].

5.3. **Equivalences of hypergeometric equation and its Bessel approximations.** Importance of the above approximations can be seen from the following result, which is a special case of a more general theorem proved in [ZZ2, Theorem 2]. Let $\Phi = (\Phi_1, \dots, \Phi_d)$, $\Theta = (\Theta_1, \dots, \Theta_d)$ denote the bases (5.1)–(5.4) and (5.6) and φ, θ be corresponding bases from Section 3.

Theorem 1. *There exist matrix-valued functions* $\mathcal{H}_0(t) = I + O(t)$ *and* $\mathcal{H}_1(s) = I + O(s)$, *defined in a neighborhood of* $t = 0$ *and* $s = 1 - t = 0$ *in* $\mathbb{C}$ *and analytic there, such that*

$$\varphi\mathcal{H}_0 = \Phi, \quad \theta\mathcal{H}_1 = \Theta.$$

Proof. Let

$$\mathcal{F}_0 = \begin{bmatrix} \varphi_1 & \dots & \varphi_d \\ \dots & \dots & \dots \\ \partial_t^{d-1}\varphi_1 & \dots & \partial_t^{d-1}\varphi_d \end{bmatrix}, \quad \mathcal{G}_0 = \begin{bmatrix} \Phi_1 & \dots & \Phi_d \\ \dots & \dots & \dots \\ \partial_t^{d-1}\Phi_1 & \dots & \partial_t^{d-1}\Phi_d \end{bmatrix}$$

be the fundamental matrices associated with the bases φ (see Eq. (3.4)) and Φ and $\partial_t\Phi_j = x^d\partial_y\Phi_j$ means differentiation with respect to the time t. Then we have

$$\mathcal{H}_0(t; x) = \mathcal{F}_0^{-1}\mathcal{G}_0.$$

Analogously the fundamental matrices $\mathcal{F}_1$ and $\mathcal{G}_1$ associated with the fundamental systems θ and Θ define the matrix-valued function

$$\mathcal{H}_1(s; x) = \mathcal{F}_1^{-1}\mathcal{G}_1.$$

It is clear from Section 3 that the matrices $\mathcal{F}_0(t, x)$ and $\mathcal{G}_0(t, x)$ are analytic in (t, x) for $t \in (\mathbb{C}\setminus 0, 0)$ and $x \in \mathbb{C}\setminus 0$. It was observed in [ZZ2] that the matrices $\mathcal{F}_0$

and $\mathcal{G}_0$ have the same monodromy properties as t turns around 0 and as x turns around 0 (or around ∞) and have the same singularities at $t = 0$ and at $x = 0$. Moreover, from the analysis in Sections 6 and 7 it follows that these matrices have almost the same asymptotic as $x \to \infty$, i.e. in sectorial domains. Therefore the matrix valued function $\mathcal{H}_0$ is single valued in the both variables and is bounded at possible singularities: $t = 0$, $x = 0$ and $x = \infty$. It follows that it is analytic in $t \in (\mathbb{C}, 0)$ and constant in $x \in \mathbb{C}$.

The same arguments prove that $\mathcal{H}_1(s; x)$ is holomorphic in $s \in (\mathbb{C}, 0)$ and constant in $x \in \mathbb{C}$. ∎

Theorem 2 from [ZZ2] is a generalization of a theorem of W. Wasow from [Was] about reduction of equations of the form $d^2x/dt^2 = \left\{\lambda^2 ta(t) + \lambda b(t, 1/\lambda)\right\} x$, $a(0) = 1$ (with analytic germs a and b and large λ) to the *Airy equation* $\partial_T^2 y = Ty$, $T = t\lambda^{2/3}$, which is also of the Bessel type. In [ZZ2] a slightly weaker result was proved; namely, it was stated that $\mathcal{H}_0(t, x)$ is analytic in $t \in (\mathbb{C}, 0)$ and $x^{-1} \in (\mathbb{C}, 0)$.

Definition 3. *The functions* $g^\sigma_{\mathrm{princ}} = G^\sigma \mathcal{H}_0^{-1}$ *are called the* ***principal WKB solutions near*** $t = 0$ *to hypergeometric equations (1.1) and the functions* $h^\sigma_{\mathrm{princ}} = H^\sigma \mathcal{H}_1^{-1}$ *are called the* ***principal WKB solutions near*** $s = 1 - t = 0$ *to the same equation.*

Remark 4. *Since the WKB solutions* G^σ *to Eq. (5.3) and* H^σ *to Eq. (5.5) are formal the principal WKB solutions* $g^\sigma_{\mathrm{princ}}$ *and* $h^\sigma_{\mathrm{princ}}$ *are also only formal. Their relations with the formal and normal WKB solutions from Definition 1 and Definition 2 are of the form*

$$g^\sigma_{\mathrm{princ}} = K^\sigma_{\mathrm{princ}}(x^{-1}) \cdot g^\sigma_{\mathrm{test}}, \quad h^\sigma_{\mathrm{princ}} = L^\sigma_{\mathrm{princ}}(x^{-1/(d-1)}) \cdot h^\sigma_{\mathrm{test}} \tag{5.18}$$

for some series $K^\sigma_{\mathrm{princ}}(x^{-1}) = 1 + O(x^{-1})$ *and* $L^\sigma_{\mathrm{princ}}(x^{-1/(d-1)}) = 1 + O(x^{-1/(d-1)})$. *Here* $L^\sigma_{\mathrm{princ}}$ *is a series in powers of* $x^{-1/(d-1)}$ *because the hypergeometric equation (1.1) is a perturbation of the Bessel type equation (5.5) and in the perturbation we encounter powers of* $s = z^{1/(d-1)} x^{-d/(d-1)}$*; in fact we solve it by solving a system of equations in variations (see* [ZZ3]*).*

Therefore

$$g^\sigma_{\mathrm{princ}}(1 - s) = \xi_d^{-1} \frac{K^\sigma_{\mathrm{princ}}}{L^\sigma_{\mathrm{princ}}} (\sigma x)^{-d/2} e^{\sigma x S_d(1)} \cdot h^\sigma_{\mathrm{princ}}(s). \tag{5.19}$$

We have not calculated the series $K^\sigma_{\mathrm{princ}}(x^{-1})$ *and* $L^\sigma_{\mathrm{princ}}(x^{-1})$*, but there is no reason to expect that they are equal. But Eq. (4.19) above and Lemma 5 below suggest that probably* $K^\sigma_{\mathrm{princ}}(x^{-1}) = L^\sigma_{\mathrm{princ}}(x^{-1}) = C_{\mathrm{norm}}(x^{-2}) = 1 + (5/256)x^{-2} + \ldots$ *for* $d = 2$.

On the other hand, if we choose analytic versions (i.e. in some sectors) of the formal WKB solutions to Eqs. (5.3) and (5.5), like in Section 5.2, then by applying *the operators* $\mathcal{H}_0^{-1}$ *and* $\mathcal{H}_1^{-1}$ *to them we obtain analytic principal WKB solutions in corresponding domains.*

Moreover, the domain of definition of $\mathcal{H}_0(t)$ *is not limited to a small neighborhood of* $t=0$. $\mathcal{H}_0$ *is analytic in a disc* $\{|t|<1-\varepsilon_0\}$ *for small* ε_0. *Similarly* $\mathcal{H}_1(s)$ *is analytic in* $\{|s|<1-\varepsilon_0\}$. *These two domains have quite big intersection.*

Finally, because there exist analytic (in sectors) versions G_j^σ *and* H_j^σ *of the formal WKB functions, application of* $\mathcal{H}_0^{-1}$ *and* $\mathcal{H}_1^{-1}$ *to them gives corresponding analytic principal WKB solution to the hypergeometric equation.*

Definition 4. *We introduce the following WKB type formal functions*

$$F^\sigma(x)=\frac{g^\sigma_{\text{princ}}(1-s;x)}{h^\sigma_{\text{princ}}(s;x)}=\xi_d^{-1}\,(\sigma x)^{-d/2}\,e^{\sigma x S_d(1)}\omega^\sigma(x^{-1/(d-1)}).$$

Here $\omega^\sigma(x^{-1/(d-1)})=K^\sigma_{\text{princ}}(1/x)/L^\sigma_{\text{princ}}(1/x^{1/(d-1)})$ *and*

$$S_2(1)=\pi \text{ and } S_3(1)=2\pi/\sqrt{3}.$$

We have

$$F^\pm \;=\; \frac{1}{x}e^{\pm ix\pi}\omega^\pm(1/x)\,, \tag{5.20}$$

$$F^\sigma \;=\; \pm\frac{e^{-2x\sigma\pi/\sqrt{3}}}{x^{3/2}}\omega^\sigma\left(x^{-1/2}\right), \tag{5.21}$$

for $d=2$ and $d=3$ respectively; in Eq. (5.21) $\pm=+$ for $\sigma=\epsilon,\bar\epsilon$ and $=-$ for $\sigma=-1$.

In the case $d=3$ the series $\omega^\sigma(x^{-1/2})$ are not single valued. We can write instead

$$x^{-3/2}\omega^\sigma_\pm=\pm\sqrt{x}\cdot x^{-2}\omega^\sigma_0(x^{-1})+x^{-2}\omega^\sigma_1(x^{-1}).$$

Then we have six WKB type functions

$$F^\sigma_\pm=x^{-3/2}e^{2\sigma x\pi/\sqrt{3}}\omega^\sigma_\pm. \tag{5.22}$$

In the case of odd $d>3$ there are $d(d-1)$ similar WKB functions.

6. Integral representations and stationary phase formula

6.1. **Integral formulas.** Some of the series defining solutions of hypergeometric and Bessel type equations have integral representations. We begin with the standard representation of the Bessel functions:

$$\begin{aligned} J_n(w) \;&=\; \tfrac{1}{2\pi i}\oint\limits_{|u|=1}\exp\left(\frac{w}{2}\left(u-1/u\right)\right)\frac{du}{u^{n+1}} \\ &=\; \tfrac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(iw\sin\alpha\right)e^{-in\alpha}d\alpha. \end{aligned} \tag{6.1}$$

This formula was obtained by Bessel and can be found in the literature (see [BE2, GM]). Let us recall its simple proof whose argumentation can be used in more general situations. The series $\sum_{m=0}^{\infty}(-1)^m(w^2/4)^{m+n/2}/(m+n)!m!$ which defines $J_n(w)$ admits the following residue representation:

$$\operatorname{res}_{u=0}\frac{1}{u^{n+1}}\left(\sum\frac{(wu/2)^m}{m!}\right)\left(\sum\frac{(-w/2u)^m}{m!}\right).$$

Next we use the Cauchy formula.

For a non-integer index μ we have the following *Schläfli representation:*

$$\begin{aligned}J_\mu(w) = & \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(i(w\sin\alpha-\mu\alpha)\right)d\alpha \\ & -\frac{\sin\pi\mu}{\pi}\int_0^\infty \exp\left(-w\sinh\beta-\mu\beta\right)d\beta.\end{aligned} \tag{6.2}$$

This follows from some generalization of the residuum formula for J_n with integer n. We have $J_\mu(w)=\frac{1}{2\pi i}\int_C \exp\left(\frac{1}{2}w\left(u-1/u\right)\right)u^{-\mu-1}du$ where C is a contour which begins and ends at $u=-\infty$ and surrounds $u=0$ in positive direction. Next the contour C is deformed to two half-lines along $(-\infty,-1)$ (parametrized by $-e^\beta$) and the circle $|u|=1$. For more details we refer reader to [BE2, Eq. 7.3(9)]. (In the original Schläfli formula the first integral in Eq. (6.2) is replaced with $\frac{1}{\pi}\int_0^\pi\cos\left(w\sin\alpha-\mu\alpha\right)d\alpha$.)

Now we are ready to present a multidimensional contour integrals. We have

$$\Phi_1=\left(\frac{1}{2\pi i}\right)^{d-1}\underset{|Q_0|=\ldots=|Q_{d-2}|=1}{\int\cdots\int}\exp\left\{-y^{1/d}\sum_{j=0}^{d-1}\varsigma^j P_j\right\}\prod_{j=0}^{d-2}\frac{dQ_j}{Q_j} \tag{6.3}$$

for the generalized Bessel function (5.1). Here and below $\varsigma=e^{2\pi i/d}$ and

$$\begin{aligned}&P_0=Q_0, P_1=Q_1Q_0^{-1/(d-1)},\ldots,P_{d-2}=Q_{d-2}Q_{d-3}^{-1/2}\ldots Q_0^{-1/(d-1)},\\ &P_{d-1}=Q_{d-2}^{-1}Q_{d-3}^{-1/2}\ldots Q_0^{-1/(d-1)};\end{aligned} \tag{6.4}$$

thus $\prod P_j=1$.

For the hypergeometric function (1.2) we get the following formula:

$$\varphi_1=\left(\frac{1}{2\pi i}\right)^{d}\underset{|Q_0|=\ldots=|Q_{d-2}|=1}{\int\cdots\int}\left\{\prod_{j=0}^{d-1}\left(1-t^{1/d}P_j\right)^{\varsigma^j}\right\}^x\prod_{j=0}^{d-1}\frac{dQ_j}{Q_j}. \tag{6.5}$$

In the proof one uses the expansions

$$(1-z)^{-a}=\sum\frac{\Gamma(a+n)}{\Gamma(a)n!}z^n$$

and

$${}_dF_{d-1}(a_1,\ldots,a_d;1,\ldots,1;t)=\sum\frac{\Gamma(a_1+n)}{\Gamma(a_1)n!}\cdots\frac{\Gamma(a_d+n)}{\Gamma(a_d)n!}t^n.$$

Using the Schläfli formula (6.2) we can prove the formula (with the Euler–Mascheroni constant γ)

$$
(6.6)\qquad \begin{array}{rcl} (\Phi_2+2\gamma\Phi_1)\,|_{d=2} &=& \frac{1}{i\pi}\int_{-\pi}^{\pi}\alpha\exp\left(2i\sqrt{y}\sin\alpha\right)d\alpha \\ && -2\int_0^{\infty}\exp\left(-2\sqrt{y}\sinh\beta\right)d\beta \end{array}
$$

for another solution $\lim_{\nu\to 0}\frac{1}{\nu}\left\{J_\nu(2\sqrt{y})-J_{-\nu}(2\sqrt{y})\right\}$ to the Bessel type equation (5.3) for $= 2$.

The Schläfli formula admits a generalization to the case of hypergeometric integrals (see [ZZ1]). It allows to prove the following formula for the solution $\widehat{\varphi}_2$ (for $d=2$) from Remark 1:

$$
(6.7)\qquad \begin{array}{rcl} \widehat{\varphi}_2|_{d=2} &=& \frac{1}{2\pi i}\int_{|v|=1}\left(\frac{1-\sqrt{t}v}{1-\sqrt{t}/v}\right)^x\ln\left(\frac{1-\sqrt{t}v}{v^2(1-\sqrt{t}/v)}\right)\frac{dv}{v} \\ && -\int_1^{1/\sqrt{t}}\left(\frac{1-\sqrt{t}v}{1-\sqrt{t}/v}\right)^x\left\{\frac{\sin\pi x}{\pi}\ln\left(\frac{1-\sqrt{t}v}{v^2(1-\sqrt{t}/v)}\right)+3\cos\pi x\right\}\frac{dv}{v}. \end{array}
$$

Unfortunately, we do not have integral formulas for the basic solutions θ_j to the hypergeometric equation near $1-t=0$ for odd $d>2$. (For $d=2$ we can use the duality formula from Lemma 3.) The reason for this is that the recurrence relations for the coefficients in the series defining θ_j are of length greater than two.

Fortunately, we can find such formulas for the solutions Θ_j to the Bessel type equation (5.5).

In the case $d=2$ the duality relation implies

$$
\Theta_j(z)|_{d=2}=-z\partial_z\Phi_j(z),\quad j=1,2,
$$

and, in particular,

$$
\Theta_1(z)|_{d=2}=\sqrt{z}J_1(2\sqrt{z}).
$$

For $d=3$ we have the following formulas (for the proofs see [ZZ3]):

$$
(6.8)\qquad \begin{array}{rcl} \Theta_1|_{d=3} &=& -\frac{z^{1/6}}{8\pi} \\ && \cdot\int_{C'}\frac{d\tau}{(1-\tau)^{3/2}}\int_{-\pi}^{\pi}d\alpha\sinh\left(z^{1/3}e^{i\alpha/2}\right)\exp\left(\frac{1}{2}z^{1/3}e^{-i\alpha}\tau\right)e^{-i\alpha/2}, \end{array}
$$

$$
(6.9)\qquad \Theta_2|_{d=3}=\frac{z^{1/3}}{2\pi}\int_{-\pi}^{\pi}\cosh\left(z^{1/3}e^{i\alpha/2}\right)\exp\left(\frac{1}{2}z^{1/3}e^{-i\alpha}\right)e^{-i\alpha}d\alpha.
$$

In Eq. (6.8) C' is a contour which begins and ends at $\tau=0$ and surrounds $\tau=1$ in positive direction. (The third solution $\Theta_3|_{d=3}$ to the Bessel like equation (5.5) can be found by taking the perturbation $8\left\{z^2\partial_z\sqrt{z}\partial_z\sqrt{z}\partial_z-\nu(\nu-1/2)(\nu-1)\right\}H-zH=0$ and passing to the limit as $\nu\to 0$ with suitable combination of the basic solutions.)

6.2. **The stationary phase formula.** Recall (see [He]) that the stationary phase formula concerns integrals of the type

$$I(\lambda) = \int e^{\lambda\phi(\alpha)}\chi(\alpha)d^k\alpha \tag{6.10}$$

over a $k-$dimensional manifold when $|\lambda| \to \infty$. Assuming that the 'phase' $\phi(\alpha)$ has finitely many critical points $\alpha_1, \dots, \alpha_n$, which are Morsean, one has the following asymptotic **stationary phase formula**:

$$I(\lambda) \sim \sum_i \chi(\alpha_i) \frac{1}{\sqrt{\det(-D^2\phi(\alpha_i))}} e^{\lambda\phi(\alpha_i)} \left(\frac{2\pi}{\lambda}\right)^{k/2}. \tag{6.11}$$

Usually, in applications, the large parameter λ is imaginary and the phase ϕ is a real function; then the integral in Eq. (6.10) is called the **oscillating integral**. Otherwise the name **mountain pass integral** is sometimes used; with such case we deal in this paper. In the case of real x and t the integrals (6.3), $(6.5)_{d=2}$, (6.6) and (6.7) are oscillating integrals and for $d > 2$ we deal with mountain pass integrals.

We want to apply formula (6.11) to the above integrals with large $|y|$ or $|z|$. However here the large parameter λ is not purely imaginary and the phase ϕ is not a real function. So we shall assume that λ lies in some sector S (in the complex plane) with vertex at ∞. Then the sum in Eq. (6.11) becomes restricted to those critical points α_i for which the function

$$z \to \exp\left\{\lambda D^2\phi(\alpha_i)(z, z)\right\}$$

is integrable, i.e. the eigenvalues μ_j of the Hessian $D^2\phi(\alpha_i)$ satisfy

$$\mathrm{Re}(\lambda\mu_j) \leq 0.$$

We shall also deal with integrals of the type

$$J(\lambda) = \int_{\beta_0}^{\beta_1} e^{\lambda\varphi(\beta)}\chi(\beta)d\beta, \tag{6.12}$$

where the 'phase' function φ is noncritical. Assume that

$$\varphi' < 0, \quad \chi(\beta) = (\beta - \beta_0)^{\sigma-1}(D + l.o.t.), \tag{6.13}$$

where the function $\chi_1(\beta) = D + l.o.t.$ is analytic near β_0. In this case, for large λ, with $\mathrm{Re}\,\lambda \geq 0$, and $\mathrm{Re}\,\sigma > 0$ we have

$$J(\lambda) \sim D \cdot \Gamma(\sigma) \cdot \exp\{\lambda\varphi(\beta_0)\} \cdot (-\lambda\varphi'(\beta_0))^{-\sigma} \tag{6.14}$$

(see [ZZ3, Lemma 3.7]). Moreover, this formula holds also when $\mathrm{Re}\,\sigma < 0$ and is not integer, but the integral in Eq. (6.12) is replaced by $(1 - e^{-2\pi i\sigma})^{-1}$ times an integral along a contour which surrounds the point β_0 in negative direction.

The aim of this subsection is to derive initial terms of the asymptotic expansions of the functions expressed via the above contour integrals.

Let us consider firstly the simplest case of the oscillating integral $\Phi_1(y)|_{d=2} = \frac{1}{2\pi}\int \exp\left(2i\sqrt{y}\sin\alpha\right)d\alpha$. The phase function $\phi(\alpha) = 2i\sin\alpha$ has two critical points $\alpha_1 = \frac{\pi}{2}$ with $\phi(\alpha_1) = 2i$, $\phi''(\alpha_1) = -2i$ and $\alpha_2 = -\frac{\pi}{2}$ with $\phi(\alpha_2) = -2i$, $\phi''(\alpha_2) = 2i$. Therefore we obtain the following (well known) asymptotic formula for $y \to \infty$:

$$\Phi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}y^{1/4}}\left(e^{i(2\sqrt{y}-\pi/4)} + e^{-i(2\sqrt{y}-\pi/4)}\right). \tag{6.15}$$

In the right-hand side of Eq. (6.6) the second integral can be ignored, because it decreases like $y^{-1/2}$ (without any exponent). The first integral in that formula is an oscillating integrals and standard application of Eq. (6.11) gives (for $y \to \infty$)

$$\left(\Phi_2 + 2\gamma\Phi_1\right)|_{d=2} \sim \frac{\sqrt{\pi}}{2iy^{1/4}}\left(e^{i(2\sqrt{y}-\pi/4)} - e^{-i(2\sqrt{y}-\pi/4)}\right). \tag{6.16}$$

In the case of the oscillating integral $(6.3)_{d\geq 3}$ the phase equals

$$\phi(Q) = \sum \varsigma^j P_j.$$

Its critical points are calculated using a Lagrange multiplier κ corresponding to the restriction $\prod P_j = 1$. One finds $P_j = \kappa\varsigma^{-j}$, where $\kappa^d = -1$. This gives d points $P^{(k)}$, $k = 0, \ldots, d-1$, $P_j^{(k)} = \varsigma^{k-j+1/2}$, and to $d!$ critical points $Q^{(l)}$ (when we take into account choices of the roots $Q_0^{1/(d-1)}, \ldots, Q_{d-2}^{1/2}$. Next, one substitutes $P_j = P_j^{(k)}e^{ip_j}$ and $Q_j = Q_j^{(l)}e^{iq_j}$, where p_j and q_i satisfy definite linear relations (see Eqs. (6.4)). The Taylor expansion of the phase at $Q^{(l)}$ takes the form $\phi(q) = \phi(Q^{(l)}) + \frac{1}{2}\sum a_{mn}^{(l)}q_m q_n + \ldots$ and the corresponding contribution in the stationary phase formula takes the form

$$(2\pi)^{(1-d)/2}\left(\det \mathcal{A}^{(l)}\right)^{-1/2} \cdot e^{-y^{1/d}\phi(Q^{(l)})} \cdot y^{(1-d)/2d}, \quad \mathcal{A}^{(l)} = \left(a_{mn}^{(l)}\right)$$

In the case $d = 3$ we obtain, as $y \to \infty$,

$$\Phi_1|_{d=3} \sim \frac{1}{\pi\sqrt{3}y^{1/3}}\left(\frac{e^{3\epsilon y^{1/3}}}{\epsilon} + \frac{e^{3\bar{\epsilon}y^{1/3}}}{\bar{\epsilon}} + \frac{e^{-3y^{1/3}}}{-1}\right), \quad \epsilon = e^{i\pi/3}. \tag{6.17}$$

(We have not finished calculations for $d > 3$.)

For the integral (6.5) the phase

$$\phi(Q) = \sum \varsigma^j \ln(1 - t^{1/d}P_j)$$

also has $d!$ critical points.

For $d = 2$ the critical points in Eq. $(6.5)_{d=2}$ are $Q^{\pm} = \sqrt{t} \pm i\sqrt{s}$, $s = 1-t$, and $\phi(Q^{\pm}e^{iq}) = \pm iS_2(t) \mp iu^2q^2$, $u = \sqrt[4]{t/s}$. Therefore the leading term of the oscillatory integral corresponding to the critical point $\alpha_{\pm}$ equals

$$e^{\pm ixS(t)}\frac{1}{2\pi}\int \exp(\mp ixu^2q^2)dq \sim \frac{1}{2u\sqrt{\pm i\pi x}}e^{\pm ixS_2(t)}.$$

We obtain

$$\varphi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}}\left\{\frac{e^{ixS_2(t)}}{u\sqrt{ix}} + \frac{e^{-ixS_2(t)}}{u\sqrt{-ix}}\right\}. \tag{6.18}$$

For $d=3$ the critical points are $Q^{\sigma,\pm}$, $\sigma=-1,\epsilon,\bar{\epsilon}$, such that

$$Q_1^{\sigma,\pm} = \frac{1}{t^{1/3}-\bar{\sigma}s^{1/3}},\quad Q_2^{\sigma,\pm} = \pm\sqrt{\frac{u+\bar{\epsilon}\bar{\sigma}}{u+\epsilon\bar{\sigma}}},\quad u=\left(\frac{t}{s}\right)^{1/3},\quad s=1-t.$$

Here the absolute values of $Q_j^{\sigma,\pm}$ are different from 1, so it is rather a mountain pass integral than an oscillating integral. We deform the initial integration contour, the torus $\mathbb{T}_0=\{Q_1=e^{i\alpha},\ Q_2=e^{i\beta}:0\le\alpha,\beta\le 2\pi\}$, to another contour $\mathbb{T}_1$ such that it passes through the critical points and near these points we can write $Q_1=Q_1^{\sigma,\pm}e^{iq_1}$, $Q_2=Q_2^{\sigma,\pm}e^{iq_2}$ (see [ZZ3] for details).

One has $\phi(Q^{\sigma,\pm})=\sigma S_3(t)$ and the corresponding matrix $\mathcal{A}^\sigma$ defining the quadratic terms equals

$$-\sigma u\begin{pmatrix}\frac{3}{4}(2-\sigma u) & i\frac{\sqrt{3}}{2}\sigma u\\ i\frac{\sqrt{3}}{2}\sigma u & 2+\sigma u\end{pmatrix},$$

with the determinant $3\left(\sigma u\right)^2$.

The leading part of the hypergeometric function $(6.3)_{d=3}$ arising from a neighborhood of the point $Q^{\sigma,\pm}$ for large $|x|$ equals $e^{\sigma x S_3(t)}$ times

$$\left(\frac{1}{2\pi}\right)^2\int\int e^{-x(\mathcal{A}q,q)/2}d^2q = \frac{1}{2\pi\sqrt{3}}\times\left\{\left(\frac{1-t}{t}\right)^{1/3}\frac{1}{\sigma x}\right\}.$$

It agrees, up to a constant, with the first term in the testing WKB solution $g^\sigma_{\text{test}}(t;x)$ given in Definition 1. We get the following formal expansion as $x\to\infty$:

$$\varphi_1|_{d=3} \sim \frac{1}{2\pi\sqrt{3}}\left\{\frac{e^{-xS_3(t)}}{-ux} + \frac{e^{\epsilon xS_3(t)}}{\epsilon ux} + \frac{e^{\bar{\epsilon}xS_3(t)}}{\bar{\epsilon}ux}\right\}. \tag{6.19}$$

Let us present the corresponding stationary phase expansions for the functions $\Theta_j(z)|_{d=2,3}$. For $d=2$ we have the following expansions, as $z\to\infty$,

$$\begin{aligned}\Theta_1|_{d=2} &\sim \tfrac{-1}{2\sqrt{\pi}}\left\{\sqrt{\tfrac{z^{1/2}}{i}}e^{-2i\sqrt{z}} + \sqrt{\tfrac{z^{1/2}}{-i}}e^{2i\sqrt{z}}\right\},\\ (\Theta_2+2\gamma\Theta_1)|_{d=2} &\sim \tfrac{\sqrt{\pi}}{2i}\left\{\sqrt{\tfrac{z^{1/2}}{i}}e^{-2i\sqrt{z}} - \sqrt{\tfrac{z^{1/2}}{-i}}e^{2i\sqrt{z}}\right\}.\end{aligned} \tag{6.20}$$

In [ZZ3] it was found that the integrals (6.8) and (6.9) have the following expansions:

$$
\begin{aligned}
\Theta_1|_{d=3} &\sim \sqrt{1/3}\cdot\left\{z^{1/6}e^{\frac{3}{2}z^{1/3}} - \epsilon z^{1/6}e^{-\frac{3}{2}\bar\epsilon z^{1/3}} - \bar\epsilon z^{1/6}e^{-\frac{3}{2}\epsilon z^{1/3}}\right\},\\
\Theta_2|_{d=3} &\sim \sqrt{2/3\pi}\left\{z^{1/6}e^{\frac{3}{2}z^{1/3}} + \epsilon z^{1/6}e^{-\frac{3}{2}\bar\epsilon z^{1/3}} + \bar\epsilon z^{1/6}e^{-\frac{3}{2}\epsilon z^{1/3}}\right\},\\
\Theta_3|_{d=3} &\sim -2i\sqrt{2\pi/3}z^{1/6}\left\{\epsilon e^{-\frac{3}{2}\bar\epsilon z^{1/3}} - \bar\epsilon e^{-\frac{3}{2}\epsilon z^{1/3}}\right\} +\\
&\qquad \sqrt{6/\pi}\ln 2\cdot z^{1/6}\left\{e^{\frac{3}{2}z^{1/3}} + \epsilon e^{-\frac{3}{2}\bar\epsilon z^{1/3}} + \bar\epsilon e^{-\frac{3}{2}\epsilon z^{1/3}}\right\}.
\end{aligned}
\tag{6.21}
$$

Remark 5. *The formulas (6.15)–(6.21) cannot be treated rigorously and the reason for this is not the fact that the corresponding series are divergent. In fact, only one or two leading terms are correct when* $\arg y$ *or* $\arg x$ *or* $\arg z$ *is fixed. This is related with the Stokes phenomenon discussed in detail in Section 7. Also there the correct coefficients in the expansions (6.15)–(6.21) are computed.*

6.3. **Applications.**

6.3.1. *Expansion in the principal WKB solutions.* The first application is the correct WKB expansion of the analytic solution φ_1 to our hypergeometric equation.

Proposition 2. *(a) For* $d = 2$ *and* $0 < t < 1$, $x > 0$ *we have*

$$\varphi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}}\left\{g^+_{\text{princ}} + g^-_{\text{princ}}\right\}.$$

(b) For $d = 3$ *and* $0 < t < 1$, $x > 0$ *we have*

$$\varphi_1|_{d=3} \sim \frac{1}{2\pi\sqrt{3}}\left\{g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} - 2g^-_{\text{princ}}\right\}.$$

Here g^σ_{princ} *are the principal WKB solutions from Definition 3. Of course, these expansions are subject to the limitation from Remark 5.*

This follows from Definition 3 and the fact that the solution $\Phi_1(y)|_{d=2,3}$ has the same representation as in Proposition 2 with g^σ replaced with G^σ. In the point (b) the coefficient before g^-_{princ} is different than in Eq. (6.19); but by Remark 5 this coefficient is not determined in that formula. It is calculated in Section 7.

We can formulate a result like Proposition 2 but with respect to the basic solutions θ_j. The formulas (6.20) for $d = 2$ and (6.21) (for $d = 3$) give representation of the solutions Θ_j to a Bessel type equations in the WKB bases H^σ. By Theorem 1 the same relations connect the solutions θ_j and h^σ_{princ}. But for us important is the coefficient before θ_d in the representation of the WKB solutions h^σ_{princ} in the basis θ. We have the following result (where F^σ are defined in Definition 4).

Proposition 3. *(a) If $d=2$ and $0<t<1$, $x>0$ then we have*

$$h^+_{\rm princ} = -h^-_{\rm princ} = \frac{-1}{\sqrt{\pi}}\cdot\theta_2 \mod \theta_1.$$

This implies that

$$\varphi_1 = \frac{i}{2\pi}\left\{F^+ - F^-\right\}\cdot\theta_2 \mod \theta_1.$$

(b) If $d=3$ and $0<t<1$, $x>0$ then we have

$$h^-_{\rm princ} = 0\cdot\theta_3,\quad h^\epsilon_{\rm princ} = -h^{\bar\epsilon}_{\rm princ} = \frac{-i}{4}\sqrt{\frac{3}{2\pi}}\cdot\theta_3 \mod(\theta_1,\theta_2).$$

This implies that

$$\varphi_1 = \frac{i}{(2\pi)^{3/2}}\left\{F^{\bar\epsilon} - F^\epsilon\right\}\cdot\theta_3 \mod(\theta_1,\theta_2).$$

In other sectors the relations are different than in item (b), but always we have something like $h^\sigma_{\rm princ} = {\rm const}\cdot\frac{i}{4}\sqrt{\frac{3}{2\pi}}\cdot\theta_3$, where the constant is either 0 or 1 or -1 (see the next section).

6.3.2. *Gaussian type integrals for $d=2$.* In the case $d=2$ in [ZZ1] we continued further the stationary phase expansion. We have $Q=Q^\pm e^{iq}$ (as above). We put $q=A/(u\sqrt{x_\pm})$, $x_\pm=\pm ix$, and we expand $ix\Delta_\pm\phi := ix(\phi-\phi_\pm)$ in powers of $x_\pm^{-1/2}$. We get

$$ix\Delta_\pm\phi = \pm ix_\pm \ln(1\mp iu^2\left(e^{iA/u\sqrt{x_\pm}}-1\right))\mp ix_\pm\ln\left(1\mp iu^2\left(e^{-iA/u\sqrt{x_\pm}}-1\right)\right).$$

The $x^0_\pm$–term of this expression equals $-A^2$ and other terms, denoted by $\Omega(A)$, can be grouped as follows:

$$x_\pm u^2\left[\sum_{m\geq 0,n\geq 2} c_{m,n}u^{4m}\left(\frac{A^2}{u^2x_\pm}\right)^n\right]+(\pm i\sqrt{x_\pm}u^3A)\left[\sum_{m\geq 0,n\geq 1} d_{m,n}u^{4m}\left(\frac{A^2}{u^2x_\pm}\right)^n\right]$$

for some real coefficients $c_{m,n}$ and $d_{m,n}$ (which do not depend on the sign $\pm$). We get an integral of the form $\frac{1}{2\pi u\sqrt{x_\pm}}\int e^{-A^2}\times e^\Omega dA$, where $e^{\Omega(A)}$ is expanded in powers of A and integrated. By analogy with the Gaussian integrals we can assume that

$$\langle A^n\rangle := \frac{1}{\sqrt{\pi}}\int e^{-A^2}A^n dA = (n-1)!!\cdot\left(\frac{1}{2}\right)^{n/2}$$

if n is even and zero otherwise. Our computations lead to the following properties of the basic solutions to the hypergeometric equation.

Lemma 5. *(a) We have*

$$\varphi_1|_{d=2} \sim \frac{1}{2\sqrt{\pi}} K_{\text{princ}}(x^{-2}) \left(g_{\text{test}}^{+} + g_{\text{test}}^{-}\right),$$

where $K_{\text{princ}}(x^{-2})$ is a formal series with real coefficients such that $K_{\text{princ}}(x^{-2}) = 1 + \frac{5}{256}x^{-2} + \ldots \not\equiv 1$ (compare Eq. (4.19)).

(b) We have

$$\widehat{\varphi}_2 \sim \frac{\sqrt{\pi}}{2i} \left\{D_{+}(x^{-1})g_{\text{test}}^{+} - D_{-}(x^{-1})g_{\text{test}}^{-}\right\},$$

where $\widehat{\varphi}_2$ is defined in Remark 1 and $D_{\pm}(x^{-1})$ are formal series satisfying

$$D_{+}(x^{-1}) + D_{-}(x^{-1}) = 2K_{\text{princ}}(x^{-2}).$$

First proof of formula (1.8). By Remark 1, Proposition 2 and Lemma 5 we have

$$\theta_1(s) = -\frac{K_{\text{princ}}}{2\sqrt{\pi}} \left\{s\partial_s g_{\text{test}}^{+} + s\partial_s g_{\text{test}}^{-}\right\}$$

and a second solution can be taken in the form

$$\widehat{\theta}_2(s) = -s\partial_s \hat{\varphi}_2 \sim -\frac{\sqrt{\pi}}{2i} \left\{D_{+} s\partial_s g_{\text{test}}^{+} - D_{-} s\partial_s g_{\text{test}}^{-}\right\}.$$

Since $\widehat{\varphi}_2 = \varphi_2 + \text{const}\cdot\varphi_1$, also $\widehat{\theta}_2 = \theta_2 + \text{const}\cdot\theta_1$, and hence Eq. (2.9) gives $\widehat{\theta}_2(0) = \theta_2(0) = -1$.

For the WKB functions $g_{\text{test}}^{\pm}$ we find the identity (see [ZZ1])

$$s\partial_s g_{\text{test}}^{\pm}(s) = xe^{\pm i\pi x} g_{\text{test}}^{\mp}(t) = \mp i h_{\text{test}}^{\mp}(s), \quad t = 1 - s,$$

where $\pi = S_2(1)$. This, together with the results of the previous, yields the following:

$$\begin{aligned} \theta_1(s) &\sim -x\frac{K_{\text{princ}}}{2\sqrt{\pi}}\{e^{i\pi x} g_{\text{test}}^{-}(t) + e^{-i\pi x} g_{\text{test}}^{+}(t)\}, \\ \widehat{\theta}_2(s) &\sim x\frac{\sqrt{\pi}}{2i}\{-D_{+}e^{i\pi x} g_{\text{test}}^{-}(t) + D_{-}e^{-i\pi x} g_{\text{test}}^{+}(t)\}. \end{aligned} \tag{6.22}$$

It implies that the formula

$$\varphi_1(t) = -\frac{2K_{\text{princ}}}{D_{+} + D_{-}} \frac{\sin \pi x}{\pi x} \cdot \widehat{\theta}_2(s) \mod \theta_1,$$

This and the equalities $\hat{\theta}_2(0) = -1$, $D_{+} + D_{-} = 2K_{\text{princ}}$ (see Lemma 5(b)) imply the formula $f_2(x) = -A_2(x) = \sin \pi x / \pi x$.

Finally, we note that Eq. (6.22) implies the equality $K_{\text{princ}}^{\pm} = L_{\text{princ}}^{\pm}$ and hence $F^{\pm} = e^{\pm ix}/x$ (see Definition 4). Then the formula $\varphi_1 = -\frac{\sin \pi x}{\pi x} \cdot \theta_2 \mod \theta_1$ follows also from Proposition 3 (but it needs the analysis from Section 7). ∎

7. The Stokes phenomenon

The Stokes phenomenon is related with 'jumps' of constants in the asymptotic expansions of solutions of linear meromorphic differential equations near irregular critical point. Here we define the Stokes operators as acting on the basic WKB solutions. For precise informations about Stokes operators (in the case of a linear equation near an irregular singularity) we refer the reader to [Was], [Zo3] and to [ZZ2], where the Stokes phenomenon for the genuine WKB solutions of equations with large parameter is discussed.

The Stokes phenomenon [St] is related with normalization of a linear system $\dot{z} = A(t)z$ in a neighborhood of an irregular singular point, say at $t = 0$. The neighborhood of $t = 0$ is divided into sectors $\mathcal{S}_j$, such that there exist changes $z = \mathcal{B}_j(t)y$ holomorphic with respect to $t \in \mathcal{S}_j$ which lead to a diagonal system $\dot{y} = \mathrm{diag}(d_1(t), \ldots, d_n(t))y$. But the matrix-valued functions $\mathcal{B}_j$ are different in different sectors. The difference between $\mathcal{B}_j$ and $\mathcal{B}_{j+1}$ is measured via so-called Stokes matrices (see [Zo3]).

In the context of WKB solutions, e.g. for $t \in (0,1)$ and large parameter x, usually the Stokes matrices are related with solutions near one of the endpoints of the time interval, $t = 0$ or $t = 1$ (see [He]). One would like to define analogues of the Stokes operators for the WKB solutions, but when the time $t \in (0,1)$ is real and the large parameter x varies in some sectors near $x = \infty$, i.e. in $(\mathbb{C}, \infty)$. However, a rather detailed analysis performed in [ZZ2] demonstrates that it is not possible to do this in uniform way with respect to t. Moreover, calculations of the Stokes operators associated with the third order hypergeometric equation $(1.1)_{d=3}$ demonstrate that the Stokes operators at the two endpoints of the interval $(0,1)$ are essentially different.

When studying the Stokes phenomenon in [He] and [Fed] greater attention is focused on analytic properties of the WKB solutions with respect to the time t, while the parameter $x \approx +\infty$ is usually real. The so called Stokes lines are drawn in the complex $t-$plane near the 'turning points' points $t = 0$ and $t = 1$. In this section we focus our attention on the parameter x, which will vary in whole sectors near infinity, and the time t will vary in a small neighborhood of the interval $(\beta, 1-\beta) \subset \mathbb{C}$ (like in Section 4.3).

Below we firstly calculate the Stokes operators for the Bessel type equations $(5.14)_{d=2,3}$ and $(5.15)_{d=2,3}$, i.e. in the WKB bases $\widetilde{G}^{\sigma}$ and $\widetilde{H}^{\sigma}$ in Eqs. (5.13). We use essentially two methods: one from the book of J. Heading [He] and using perturbation of the Bessel type equations to equations with regular singularities and then considering corresponding monodromy matrices. An alternative approach is to use results of the paper [DuMi] which imply that the principal Stokes matrix differs from the identity only at one place.

It is worth to underline the fact that the Heading's method is sufficient only in the case $d = 2$. In the case $d \geq 3$ it is insufficient.

Finally, in the second part of this section, we apply the results about the Bessel type equations to analysis of the Stokes phenomenon for the principal WKB solutions $g^{\sigma}_{\mathrm{princ}}$ and $h^{\sigma}_{\mathrm{princ}}$ the hypergeometric equation (1.1). We show that the connection coefficient $A_d(x)$ from Lemma 3.2 is a sum of WKB type the formal summands F^{σ}, they are subject to Stokes phenomenon which is trivial in the case $d=2$ and nontrivial in the case $d=3$.

7.1. **Stokes operators for the Bessel type equations.**

7.1.1. *The case* $d=2$. We begin with Eq. $(5.14)_{d=2}$. By a sectorial normalization theorem the solutions $\widetilde{G}^{\pm}(v)$ from Eq. $(5.13)_{d=2}$ represent asymptotic series for solutions $\widetilde{G}^{\pm}_{r,l}(v)$ which are analytic in some sectors about $v=\infty$ (in the complex $v-$plane).

There are two such sectors: $\mathcal{S}_r$ (right) and $\mathcal{S}_l$ (left) with vertex at ∞ of angle $2\pi-2\delta$ ($\delta>0$ and small) and with the rays $\arg v=0$ and $\arg v=\pi$ as their bisectrices. The latter rays are called the **rays of division**. Then the sectors $\mathcal{S}_u=\mathcal{S}_r\cap\mathcal{S}_l\cap\{\operatorname{Im} v>0\}$, and $\mathcal{S}_d=\mathcal{S}_r\cap\mathcal{S}_l\cap\{\operatorname{Im} v<0\}$ have angle $\pi-2\delta$. The sectors $\mathcal{S}_u$ and $\mathcal{S}_d$ are 'transitional' sectors; their bisectrices are called the **Stokes lines**. $\widetilde{G}^{\pm}_r$ and $\widetilde{G}^{\pm}_l$ are the corresponding solutions in the sectors $\mathcal{S}_r$ and $\mathcal{S}_l$ respectively obtained from the sectorial normalization theorem.

We note the following relations (where $f\prec h$ means that the function f is much smaller than the functions h) :

$$\widetilde{G}^{+}_{r,l}\prec\widetilde{G}^{-}_{r,l}\ \text{in}\ \mathcal{S}_u,\qquad \widetilde{G}^{-}_{r,l}\prec\widetilde{G}^{+}_{r,l}\ \text{in}\ \mathcal{S}_d. \tag{7.1}$$

The solutions $\widetilde{G}^{\pm}_r$ (respectively $\widetilde{G}^{\pm}_l$) are analytic in the adjacent sectors $\mathcal{S}_u$ (up) and $\mathcal{S}_d$ (down). Therefore they are expressed as linear linear combinations of the corresponding solutions $\widetilde{G}^{\pm}_l$ (respectively $\widetilde{G}^{\pm}_r$). The corresponding matrices C_u and C_d of changes between the basic solutions are called the **Stokes matrices**.

Each Stokes matrix is triangular with 1 on the diagonal. We have

$$C_u=\begin{pmatrix}1 & c_{12}\\ 0 & 1\end{pmatrix},\quad C_d=\begin{pmatrix}1 & 0\\ c_{21} & 1\end{pmatrix}. \tag{7.2}$$

This means that, after passing from the sector $\mathcal{S}_r$ to the sector $\mathcal{S}_l$, the basic solutions undergo the following changes:

$$\widetilde{G}^{+}_r = \widetilde{G}^{+}_l,\quad \widetilde{G}^{-}_r=\widetilde{G}^{-}_l+c_{12}\widetilde{G}^{+}_l\quad (\text{in}\quad \mathcal{S}_u), \tag{7.3}$$

$$\widetilde{G}^{+}_l = \widetilde{G}^{+}_r+c_{21}\widetilde{G}^{-}_l,\quad \widetilde{G}^{-}_l=\widetilde{G}^{-}_r\quad (\text{in}\quad \mathcal{S}_d). \tag{7.4}$$

The rule is that to a given solution one can add a solution with smaller asymptotic at infinity. We shall calculate the coefficients c_{12} and c_{21} using the method from [He], where Stokes matrices associated with the Bessel equation were computed (see also [Zo3]).

We note also the following *symmetry property*:

$$\widetilde{G}_l^+(e^{i\pi}v) = -\widetilde{G}_r^-(v), \quad \widetilde{G}_l^-(e^{i\pi}v) = \widetilde{G}_r^+(v), \quad v > 0. \tag{7.5}$$

Let $\widetilde{G}_r^+(v)$ on the ray $\arg v = 0$ (in the sector $\mathcal{S}_r$) be represented by the following combination of the basic solutions $\widetilde{\Phi}_1(v) = \Phi_1(v^2)$, $\widetilde{\Phi}_2(v) = \Phi_2(v^2) = \widetilde{\Phi}_1 \ln v^2 + \widetilde{\Psi}_2(v^2)$:

$$\widetilde{G}_r^+(v) = K_1\widetilde{\Phi}_1(v) + K_2\widetilde{\Phi}_2(v), \quad v > 0, \tag{7.6}$$

for some coefficients K_1 and K_2. After passing to the ray $\arg v = \pi$ (in $\mathcal{S}_l$) and the substitution $v \to -v$ (using Eqs. (7.5) and the logarithmic singularity of $\widetilde{\Phi}_2$) we get

$$-\widetilde{G}_r^-(v) = (K_1 + 2\pi i K_2)\widetilde{\Phi}_1(v) + K_2\widetilde{\Phi}_2(v), \quad v > 0. \tag{7.7}$$

Analogously, after passing to the ray $\arg x = 2\pi$ and using an analogue of the relations (7.5), we get

$$-\widetilde{G}_r^+(v) - c_{21}\widetilde{G}_r^-(v) = (K_1 + 4\pi i K_2)\widetilde{\Phi}_1(v) + K_2\widetilde{\Phi}_2(v), \quad v > 0. \tag{7.8}$$

Eqs. (7.6)–(7.8) imply the representation (on $\arg v = 0$)

$$\widetilde{\Phi}_1(v) = \frac{i}{2\pi K_2}(\widetilde{G}_r^+ + \widetilde{G}_r^-), \quad \widetilde{\Phi}_2(v) = \left(\frac{1}{K_2} - \frac{iK_1}{2\pi K_2^2}\right)\widetilde{G}_r^+ - \frac{iK_1}{2\pi K_2^2}\widetilde{G}_r^-,$$

and that

$$c_{21} = 2.$$

Moreover, the asymptotic formula (6.18) implies that $K_2 = i/\sqrt{\pi}$.

In the same way one proves that $c_{12} = -2$ and obtains the representation

$$\widetilde{\Phi}_1(v) = \frac{1}{2\sqrt{\pi}}(\widetilde{G}_l^- - \widetilde{G}_l^+), \quad \arg v = \pi.$$

Calculation of the Stokes matrices associated with the Bessel type equation $(5.15)_{d=2}$ runs practically in the same way as above. The formal WKB solutions

$$\widetilde{H}^\pm(w) = \sqrt{-w_\pm}e^{-2w_\pm}\left\{1 + \frac{b_1}{w_\pm} + \frac{b_2}{w_\pm^2} - \dots\right\}, \quad w_\pm = \pm iw.$$

satisfy the Bessel type equation $(5.15)_{d=2}$ with another pair of solutions

$$\widetilde{\Theta}_1(w) = w - \frac{1}{2}w^2 + \dots, \quad \widetilde{\Theta}_2(w) = \widetilde{\Theta}_1(w)\cdot\ln w + \widetilde{\Xi}_3(w) \tag{7.9}$$

(with analytic $\widetilde{\Theta}_1$ and $\widetilde{\Xi}_3$).

Now we have the same sectors $\mathcal{S}_{r,l}$, with analytic solutions $\widetilde{H}_{r,l}^\pm$, and $\mathcal{S}_{u,d}$ about $w = \infty$, but with domination relations different than in Eq. (7.1). Therefore the corresponding Stokes matrices take the following form

$$D_u = \begin{pmatrix} 1 & 0 \\ d_{21} & 1 \end{pmatrix}, \quad D_d = \begin{pmatrix} 1 & d_{12} \\ 0 & 1 \end{pmatrix}. \tag{7.10}$$

Anyway (using also Eqs. (6.20)) we arrive to the following result, where Eq. (7.17) is a consequence of the factor $\sqrt{-w_\pm}$ in definition of $\widetilde{H}^\pm$: we have $\widetilde{H}_l^\pm(e^{2\pi i}w) = -\widetilde{H}_l^\pm(w)$.

We summarize this in the following

Proposition 4. *(a) We have* $c_{12} = -2$ *and* $c_{21} = 2$ *in Eqs (7.2). Moreover,*

$$(7.11)\quad \widetilde{\Phi}_1(v) = \frac{1}{2\sqrt{\pi}}\left(\widetilde{G}_r^+ + \widetilde{G}_r^-\right), \quad \widetilde{\Phi}_2 = -i\sqrt{\pi}\cdot\widetilde{G}_r^+ \mod \widetilde{\Phi}_1, \ \arg v = 0;$$

$$(7.12)\quad \widetilde{\Phi}_1(v) = \frac{1}{2\sqrt{\pi}}\left(\widetilde{G}_l^- - \widetilde{G}_l^+\right), \quad \widetilde{\Phi}_2 = -i\sqrt{\pi}\cdot\widetilde{G}_l^+ \mod \widetilde{\Phi}_1, \ \arg v = \pi.$$

(b) We have $d_{12} = -2$ *and* $d_{21} = 2$ *in Eqs (7.10). Moreover,*

$$(7.13)\quad \widetilde{\Theta}_1 = \frac{1}{2\sqrt{\pi}}\left(\widetilde{H}_r + \widetilde{H}_r^-\right), \quad \widetilde{\Theta}_2 = -i\sqrt{\pi}\cdot\widetilde{H}_r^+ \mod \widetilde{\Theta}_1, \quad \arg w = 0;$$

$$(7.14)\quad \widetilde{\Theta}_1 = \frac{1}{2\sqrt{\pi}}\left(\widetilde{H}_l^- - \widetilde{H}_l^+\right), \quad \widetilde{\Theta}_2 = i\sqrt{\pi}\cdot\widetilde{H}_l^- \mod \widetilde{\Theta}_1, \quad \arg w = \pi.$$

In particular, we get

$$(7.15)\qquad \widetilde{H}_r^+(w) = -\widetilde{H}_r^- = \left(i/\sqrt{\pi}\right)\cdot\widetilde{\Theta}_2 \mod \widetilde{\Theta}_1, \qquad \arg w = 0;$$

$$(7.16)\qquad \widetilde{H}_l^+(w) = \widetilde{H}_l^- = \left(-i/\sqrt{\pi}\right)\cdot\widetilde{\Theta}_2 \mod \widetilde{\Theta}_1, \qquad \arg w = \pi;$$

$$(7.17)\qquad \widetilde{H}_l^+(w) = -\widetilde{H}_l^- = \left(i/\sqrt{\pi}\right)\cdot\widetilde{\Theta}_2 \mod \widetilde{\Theta}_1, \qquad \arg w = -\pi.$$

Above we give the representation of the function $\widetilde{\Phi}_1(v)$ for v on the two rays of division. But, in fact, these formulas hold true in the whole sectors $\mathcal{S}_{r,l}$ which contains the corresponding ray of division. The same remark applies in other expansions.

7.1.2. *The case* $d = 3$. Eq. $(5.14)_{d=3}$ has the following independent solutions

$$\widetilde{\Phi}_1(v) = \Phi_1(v^3), \quad \widetilde{\Phi}_2(v) = \widetilde{\Phi}_1 \ln v^3 + \widetilde{\Psi}_2(v), \quad \widetilde{\Phi}_3 = \frac{1}{2}\Phi_1 \ln^2(v^3) + \widetilde{\Psi}_2 \ln v^3 + \widetilde{\Psi}_3,$$

where $\widetilde{\Phi}_1$, $\widetilde{\Psi}_2$ and $\widetilde{\Psi}_3$ are entire functions and depend on v^3. We have also the system $\widetilde{G}_j^\sigma$ of WKB type solutions defined in the sectors $\mathcal{S}_j$ about $v = \infty$ (see Eq. (5.16) and Figure 1 below).

The **rays of division** $\mathcal{R}_j$ (or the *anti-Stokes lines*) are given by $\arg v = 0$, $\pi/3$, $2\pi/3$, π, $4\pi/3$, $5\pi/3$, i.e. they are the bisectrices of the sectors $\mathcal{S}_j$. Then the sectors $\mathcal{S}_{12} = \mathcal{S}_1 \cap \mathcal{S}_2$, $\mathcal{S}_{23}$, $\mathcal{S}_{34}$, $\mathcal{S}_{45}$, $\mathcal{S}_{56}$, $\mathcal{S}_{61}$ have angle $\pi/3 - \delta$ (see Figure 1); their bisectrices are known as the **Stokes lines**. The corresponding Stokes matrices C_{ji} are the matrices of changes between the basic solutions $\left\{\widetilde{G}_i^\sigma\right\}$ and $\left\{\widetilde{G}_j^\sigma\right\}$ in the sectors $\mathcal{S}_i$ and $\mathcal{S}_j$.

Each matrix C_{ji}, after suitable ordering of the basic solutions, becomes upper triangular with 1's on the diagonal. For example, in the sector $\mathcal{S}_{12}$ we have

$$\widetilde{G}_j^- \prec \widetilde{G}_j^\epsilon \prec \widetilde{G}_j^{\bar\epsilon}, \quad j = 1, 2.$$

The Stokes matrix associated with the sector $\mathcal{S}_{12}$ equals

$$C_{21} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}, \tag{7.18}$$

where the parameters a, b, c are to be determined.

Other Stokes matrices can be obtained from the matrix C_{21} using the fact that Eq. (5.16) is invariant with respect to:

— the rotation $v \to \epsilon^2 v$ (where $\epsilon = e^{i\pi/3}$),

— the complex conjugation $v \to \bar v$.

Formally the rotation $\epsilon^2 v$ is reflected in the cyclic permutation of solutions, $\widetilde{G}_{j+2}^\sigma(\epsilon^2 v) = \widetilde{G}_j^{\epsilon^2\sigma}(v)$. The double rotation results in the change $\widetilde{G}_{j+4}^\sigma(\epsilon^4 v) = \widetilde{G}_j^{\epsilon^4\sigma}(v)$. The complex conjugation induces the change $\widetilde{G}_j^\sigma(v) = \widetilde{G}_{7-j}^{\bar\sigma}(\bar v)$; but here also the orientation of the $v-$plane is reversed. Compare also Eqs. (4.27)–(4.28).

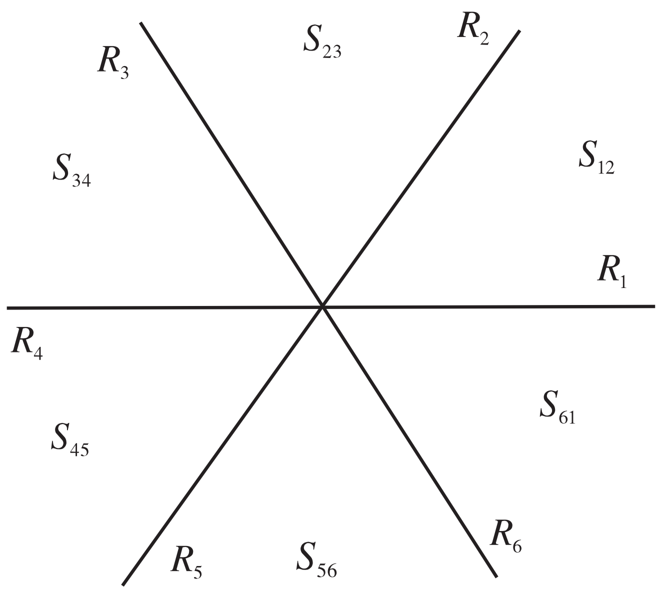

FIGURE 1. Rays of division

Therefore the Stokes matrices C_{43} and C_{65} are obtained from C_{21} by application of conjugation with suitable permutation matrices. The matrix C_{16} is obtained from C_{21} by: complex conjugation, taking the inverse and conjugation with the permutation (1) (23) . The matrices C_{32} and C_{54} are obtained from the matrix C_{16} by permutations.

In the calculation of the Stokes matrix C_{21} we follow the Heading method described in the previous section for the case $d = 2$. We represent the function $\widetilde{G}^{-}(v)$ in the ray $\mathcal{R}_1 = \{\arg v = 0\}$ in the basis $\left\{\widetilde{\Phi}_j\right\}$,

$$\widetilde{G}_1^{-} = K_1\widetilde{\Phi}_1 + K_2\widetilde{\Phi}_2 + K_3\widetilde{\Phi}_3$$

(with coefficients K_j), and we pass to the rays $\mathcal{R}_3$, $\mathcal{R}_5$ and $\mathcal{R}_1$, using actions of the matrices $C_{31} = C_{32}C_{21}$, $C_{53} = C_{54}C_{43}$ and $C_{15} = C_{16}C_{65}$ and substitutions $\epsilon^2 v$, $\epsilon^4 v$ and $\epsilon^6 v$ in the argument. We arrive at the following relation

$$b = 3 + \bar{a} + \bar{c}, \tag{7.19}$$

but the parameters a and c are not determined.

We repeat the same analysis, but starting from the ray $\mathcal{R}_6 = \{\arg v = -\pi/3\}$ and use the matrices $C_{26} = C_{21}C_{16}$, C_{42} and C_{64}. Again we get relation (7.19).

In order to calculate the constants a and c we use the known property (see [Gl] or [Zo1]) that Stokes operators are limits of monodromy operators of a perturbed equation which has regular singularities.

An obvious perturbation of Eq. (5.3) is the our initial hypergeometric equation, i.e. $(1 - yx^{-3})\partial_y y\partial_y y\partial_y G + G = 0$, and the corresponding perturbation of Eq. (5.14) is

$$\left(1 - (v/x)^3\right)\partial_v v\partial_v v\partial_v \widetilde{G} + 27v^2\widetilde{G} = 0. \tag{7.20}$$

Together with perturbation (7.20) we shall consider the following one:

$$\left(1 + (v/x)^3\right)\partial_v v\partial_v v\partial_v \widetilde{G} + 27v^2\widetilde{G} = 0, \tag{7.21}$$

i.e. with change of the sign before $(v/x)^3$.

Eq. (7.20) has three additional singular points $v_1 = x, \quad v_2 = \epsilon^2 x, \quad v_3 = \epsilon^{-2} x$ which tend to infinity as $x \to \infty$ and where we assume that x is real positive. The latter singular points lie in the division rays $\mathcal{R}_1$, $\mathcal{R}_3$ and $\mathcal{R}_5$ and the monodromy matrices M_1, M_2 and M_3 (in some basis of solutions) defined by prolongation of solutions along curves around these points (in the clockwise direction) should tend (as $x \to \infty$) to matrices equivalent to C_{26}^{-1}, C_{42}^{-1} and C_{64}^{-1} respectively.

On the other hand, each monodromy matrix M_j, $j = 1, 2, 3$, is equivalent to some monodromy matrix $\mathcal{M}_1$ related with the hypergeometric equation $(1.1)_{d=3}$ and corresponding to the singular point $t = 1$. Since the basic solutions of the latter equation near $s = 1 - t = 0$ are $s + \ldots$, $s^2 + \ldots$, and $(s^2 + \ldots)\ln x^3 s + \alpha + \ldots$ the corresponding monodromy matrix $\mathcal{M}_1$ has all eigenvalues equal to 1 and its Jordan decomposition consists of two cells; anyway, the characteristic polynomial is $P(\lambda) = \det(\mathcal{M}_1 - \lambda) = (1 - \lambda)^3$. Looking at the matrix C_{26} in [ZZ3] one finds that its characteristic polynomial is $\left(1-\lambda)(\lambda^2-\left(2-|c|^2\right)\lambda+1\right)$. It follows that $c=0$.

Equation (7.21) is related with the modified hypergeometric equation $(1+t)\partial t\partial t\partial g+x^3g=0$, where one checks that the basic solutions near $s=1+t=0$ are $s+\ldots$, $s^2+\ldots$ and $\left(s^2+\ldots\right)\ln s+\ldots$. Here also the corresponding monodromy matrix has eigenvalues 1 and two Jordan cells. On the other hand, the monodromy matrices related with the singular points $v=-x,\epsilon x,\bar{\epsilon}x$ of equation (7.21) tend to the matrices C_{53}^{-1}, C_{31}^{-1}, C_{16}^{-1}. The same arguments as above show that $a=0$.

From the above we get the following result.

Proposition 5. *The principal Stokes matrix associated with the WKB bases* $\left(\widetilde{G}_1^\sigma\right)$ *and* $\left(\widetilde{G}_2^\sigma\right)$, $\sigma=-1,\epsilon,\bar{\epsilon}$, *takes the form*

$$C_{21}=\begin{bmatrix}1&0&3\\0&1&0\\0&0&1\end{bmatrix}.$$

Moreover we have the following representations:

$$\begin{array}{rcl}\widetilde{\Phi}_1&=&\frac{1}{\pi\sqrt{3}}(\widetilde{G}_1^\epsilon+\widetilde{G}_1^{\bar{\epsilon}}-2\widetilde{G}_1^-),\\ \widetilde{\Phi}_2&=&\frac{i}{\sqrt{3}}(\widetilde{G}_1^{\bar{\epsilon}}-\widetilde{G}_1^\epsilon)\ \bmod\ \widetilde{\Phi}_1,\\ \widetilde{\Phi}_3&=&-\frac{4\pi}{\sqrt{3}}\widetilde{G}_1^-\ \bmod\ \left(\widetilde{\Phi}_1,\widetilde{\Phi}_2\right)\end{array}\tag{7.22}$$

(for $v\in\mathcal{R}_1$). Analogous representations hold in other rays of division:

$$\begin{array}{rcl}\widetilde{\Phi}_1&=&\frac{1}{\pi\sqrt{3}}(\widetilde{G}_j^-+\widetilde{G}_j^\epsilon+\widetilde{G}_j^{\bar{\epsilon}}),\ v\in\mathcal{R}_j,\ j=2,4,6,\\ &=&\frac{1}{\pi\sqrt{3}}(\widetilde{G}_j^-+\widetilde{G}_j^\epsilon+\widetilde{G}_j^{\bar{\epsilon}}-3\widetilde{G}_j^*),\ v\in\mathcal{R}_j,\ (j,*)=(1,-),(3,\epsilon),(5,\bar{\epsilon}).\end{array}\tag{7.23}$$

Note that in the ray $\mathcal{R}_1$ two dominating WKB solutions $\widetilde{G}^\epsilon$ and $\widetilde{G}^{\bar{\epsilon}}$ are of the the same order. So the coefficients between them in Eq. (7.22) are determined by the asymptotic of the oscillating integral (via the stationary phase formula). The coefficients before $\widetilde{G}^\epsilon$ and $\widetilde{G}^{\bar{\epsilon}}$ in Eq. (7.22) agree with Proposition 2, but the coefficient before $\widetilde{G}^-$ is different.

From the proof of Proposition 5 it is seen that using only the method from the Heading's book [He] we are not able to compute all the Stokes matrices, we obtain only one relation (7.19). On the other hand, only the knowledge of the Jordan decomposition of the composed Stokes matrices, like C_{31}, does not allow to obtain relation (7.19). Therefore the both methods should be used. Probably this fact is true in more general high order linear meromorphic ODE's.

Of course, the relative simplicity of the principal Stokes matrix can be explained by the fact that the domains of analyticity of the functions $\widetilde{G}_j^\sigma$ are larger than the sectors $\mathcal{S}_j$ (compare Section 5.2).

As we have mentioned, the Stokes matrices associated with the WKB solutions G^{σ}_{DM} from Remark 3 were calculated by A. Duval and C. Mitschi [DuMi]. Their calculations rely upon properties of the Mellin–Barnes integrals proved by C. Meijer [Me]. Anyway, their result completely agrees with ours.[8]

The analysis leading to Stokes operators associated with formal WKB solutions $\widetilde{H}^{\sigma}(w) \sim \sqrt{-\sigma w}e^{-3\sigma w/2}$ (see Eq. (5.11)) which are asymptotic series for analytic WKB solutions $\widetilde{H}^{\sigma}_j$ defined in sectors $\mathcal{S}_j$ about $w = \infty$ (see Eq. (5.17)) leads to the following result. Below the constants

$$L_1 = \sqrt{3}/2 \;\text{ and }\; L_3 = (-i/4)\sqrt{3/2\pi}$$

appear in the representation

$$H_4^- = L_1\widetilde{\Theta}_1 + L_2\widetilde{\Theta}_2 + L_3\widetilde{\Theta}_3, \quad w \in \mathcal{R}_4,$$

and are taken from Eq. (6.21).

Proposition 6. *The principal Stokes matrix associated with the WKB bases* $\left(\widetilde{H}^{\sigma}_4\right)$ *and* $\left(\widetilde{H}^{\sigma}_5\right)$, $\sigma = -1, \epsilon, \bar{\epsilon}$, *takes the form*

$$C_{54} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Moreover we have the following representation:

$$\begin{aligned} 4L_1\widetilde{\Theta}_1 &= 2\widetilde{H}_4^- - \widetilde{H}_4^{\epsilon} - \widetilde{H}_4^{\bar{\epsilon}}, \\ 4\pi i L_3\widetilde{\Theta}_2 &= -\widetilde{H}_4^{\epsilon} + \widetilde{H}_4^{\bar{\epsilon}}, \\ 4L_3\widetilde{\Theta}_3 &= 2(\widetilde{H}_4^- + \widetilde{H}_4^{\epsilon}) \qquad \bmod \quad \widetilde{\Theta}_2 \end{aligned} \tag{7.24}$$

for $w \in \mathcal{R}_4$ *and* $0 < t < 1$. *The representations in other rays* $\mathcal{R}_j$ *(and* $0 < t < 1$*) are presented in* [ZZ3, Prop. 5.5]. *This implies the following relations* mod $\left(\widetilde{\Theta}_1, \widetilde{\Theta}_2\right)$:

$$\begin{gathered} \widetilde{H}_1^- = 0, \;\; -\widetilde{H}_1^{\epsilon} = \widetilde{H}_1^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \;\; w \in \mathcal{R}_1 \\ \widetilde{H}_2^- = -\widetilde{H}_2^{\epsilon} = \widetilde{H}_2^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \;\; w \in \mathcal{R}_2 \\ \widetilde{H}_3^{\epsilon} = 0, \;\; \widetilde{H}_3^- = \widetilde{H}_3^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \;\; w \in \mathcal{R}_3 \\ \widetilde{H}_4^- = \widetilde{H}_4^{\epsilon} = \widetilde{H}_4^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \;\; w \in \mathcal{R}_4 \\ \widetilde{H}_5^{\bar{\epsilon}} = 0, \;\; \widetilde{H}_5^- = \widetilde{H}_5^{\epsilon} = L_3\widetilde{\Theta}_3, \;\; w \in \mathcal{R}_5 \\ \widetilde{H}_6^- = \widetilde{H}_6^{\epsilon} = -\widetilde{H}_6^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \;\; w \in \mathcal{R}_6 \\ \widetilde{H}_1^- = 0, \;\; \widetilde{H}_1^{\epsilon} = -\widetilde{H}_1^{\bar{\epsilon}} = L_3\widetilde{\Theta}_3, \;\; w \in \mathcal{R}_1. \end{gathered} \tag{7.25}$$

[8]In the sequent paper [Mit] Mitschi applied the results of [DuMi] to compute the differential Galois groups of some confluent hypergeometric equations. Previously these groups were calculated in algebro–geometrical way (which avoids calculation of the Stokes constants) by N. Katz [Ka1] and [Ka2]; the method of Katz was initiated in the paper [BBH].

Note that for $z > 0$, i.e. $w > 0$, the value of $\widetilde{\Theta}_3 \bmod \widetilde{\Theta}_2$ agrees with Eq. (6.21), which was obtained by calculation of corresponding mountain pass integrals.

Note also the difference between the data of the latter tables for the ray $\mathcal{R}_1$ (in the first and in the last row in Eq. (7.25)). It corresponds to the turning $w \longmapsto e^{2\pi i}w$. Here $\widetilde{\Theta}_1$ changes to $-\widetilde{\Theta}_1$, $\widetilde{\Theta}_2$ is unchanged, $\widetilde{\Theta}_3$ acquires a term proportional to $\widetilde{\Theta}_2$ and $\widetilde{H}^\sigma$ change to $-\widetilde{H}^\sigma$; all is OK.

7.2. **Stokes operators for the hypergeometric equation.** We deal with formal WKB solutions for the hypergeometric equation as well as for the corresponding Bessel type equations. By results of Section 5.2 the reductions to the normal (diagonal) form for associated with them systems are compatible. Recall that these formal solutions are of Gevrey type and in suitable domains are represented by analytic functions, but the above analytic constructions are not quite compatible. In the other hand, the analytic equivalences with corresponding Bessel type equations (using the matrices $\mathcal{H}_0$ and $\mathcal{H}_1$ in Section 5.3) imply compatibility of analytic and of formal solutions.

So, in order to avoid technicalities, we limit ourselves to the formal case. This is the way chosen in [ZZ3] for $d = 3$. In [ZZ1] the case $d = 2$ is done with complete details.

7.2.1. *The case* $d = 2$. Let $0 < t < 1$. Using Theorem 1 and Definition 3 we can replace in Proposition 4 $\widetilde{\Phi}_j$ and $\widetilde{\Theta_j}$ with φ_j and θ_j and the WKB solutions $\widetilde{G}_j^\pm$ and $\widetilde{H}_j^\pm$ with $g_{\text{princ}}^\pm$ and $h_{\text{princ}}^\pm$. Therefore, for $\arg x = 0$, we have

$$\begin{aligned}\varphi_1 &= \frac{1}{2\sqrt{\pi}}\left\{g_{\text{princ}}^+ + g_{\text{princ}}^-\right\}\\ &= \frac{1}{2\sqrt{\pi}}\left\{F^+(x)h_{\text{princ}}^+ + F^-(x)h_{\text{princ}}^-\right\}\\ &= \frac{1}{2\sqrt{\pi}}\cdot\frac{i}{\sqrt{\pi}}\left\{F^+ - F^-\right\}\cdot\theta_2 \mod \theta_1.\end{aligned}$$

For $\arg x = \pi$ we have

$$\begin{aligned}\varphi_1 &= \frac{1}{2\sqrt{\pi}}\left\{g_{\text{princ}}^- - g_{\text{princ}}^+\right\}\\ &= \frac{1}{2\sqrt{\pi}}\cdot\frac{-i}{\sqrt{\pi}}\left\{F^- - F^+\right\}\cdot\theta_2 \mod \theta_1.\end{aligned}$$

Here $F^\pm(x) = \frac{1}{x}e^{\pm ix\pi}\omega^\pm(1/x)$, $\omega^\pm = 1 + O(1/x)$ are defined in Definition 4 (compare also Propositions 2 and 3). The above pattern repeats as $\arg x$ increases by 2π.

We arrive at the following.

Proposition 7. *The connection coefficient $A_2(x)$ from Lemma 2 equals*

$$A_2(x) = \frac{i}{2\pi}\left\{F^+(x) - F^-(x)\right\}, \quad x \to \infty,$$

where the functions $F^{\pm}(x)$ are single valued.

Second proof of the formula (1.8). We note that the function $f_2(x) = -A_2(x)$ vanishes at the points $x = \pm 1, \pm 2, \ldots$. Since the function $\sin \pi x/x$ has simple zeroes at these points, we find that the function

$$f_2(x)/\left(\sin \pi x/x\right)$$

is entire on $\mathbb{C}$. By Proposition 7 it is bounded at infinity. Therefore it is a constant function equal $1/\pi$ (since $f(0) = 1$). ∎

7.2.2. *The case $d = 3$.* Here we follow the previous case with use of Propositions 5 and 6. For $0 < t < 1$, we have

$$\pi\sqrt{3}\varphi_1(t;x) = \begin{array}{ll} g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} - 2g^{-}_{\text{princ}}, & x \in \mathcal{R}_1, \\ g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} + g^{-}_{\text{princ}}, & x \in \mathcal{R}_2, \\ g^{-}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} - 2g^{\epsilon}_{\text{princ}}, & x \in \mathcal{R}_3, \\ g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} + g^{-}_{\text{princ}}, & x \in \mathcal{R}_4, \\ g^{-}_{\text{princ}} + g^{\epsilon}_{\text{princ}} - 2g^{\bar\epsilon}_{\text{princ}}, & x \in \mathcal{R}_5, \\ g^{\epsilon}_{\text{princ}} + g^{\bar\epsilon}_{\text{princ}} + g^{-}_{\text{princ}}, & x \in \mathcal{R}_6, \end{array}$$

where $g^{\sigma}_{\text{princ}} = F^{\sigma} h^{\sigma}_{\text{princ}}$.

We have also the following relation modulo (θ_1, θ_2) :

$$h^{-}_{\text{princ}} = 0, \quad h^{\bar\epsilon}_{\text{princ}} = -h^{\epsilon}_{\text{princ}} = L_3\theta_3, \quad x \in \mathcal{R}_1,$$

and other relations like in Eqs. (7.25), where $L_3 = -\frac{i}{8}\sqrt{3/2\pi}$.

This implies the following representations of the generating function $f_3(x) = -2A_3(x)$:

$$-i\,(2\pi)^{3/2} f_3(x) = \begin{array}{ll} F^{\bar\epsilon} - F^{\epsilon}, & x \in \mathcal{R}_1, \\ F^{\bar\epsilon} - F^{\epsilon} - F^{-}, & x \in \mathcal{R}_2, \\ F^{-} + F^{\bar\epsilon}, & x \in \mathcal{R}_3, \\ F^{\epsilon} + F^{\bar\epsilon} + F^{-}, & x \in \mathcal{R}_4, \\ F^{-} + F^{\epsilon}, & x \in \mathcal{R}_5, \\ F^{-} + F^{\epsilon} - F^{\bar\epsilon}, & x \in \mathcal{R}_6, \end{array} \tag{7.26}$$

where $F^{\sigma} = \frac{\pm 1}{x^{3/2}} e^{2\pi\sigma x/\sqrt{3}} \omega^{\sigma}(x^{-1/2})$ are the WKB type functions from Definition 4.

Since $F^{\sigma}(x) = F^{\sigma}_{\pm}(x^{1/2})$ depend on $x^{1/2}$ (see Eq. 5.22)), table (7.26) should be continued in order to turn twice around $x = \infty$. The corresponding formulas are related with compositions of the changes from Eqs. (7.26) with the monodromy of the functions $F_{\pm}^{\sigma}$:

$$\mathcal{M}_{\infty} : F_{\pm}^{\sigma} \longmapsto -F_{\mp}^{\sigma}. \tag{7.27}$$

We also see that the functions $F_{\pm}^{\sigma}$ are subject to Stokes phenomenon with the principal Stokes matrix relating solutions at the rays $\mathcal{R}_1$ and $\mathcal{R}_2$ of the form

$$C_{21} = \begin{bmatrix} 1 & p & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad p - q = 1. \tag{7.28}$$

We can state the fundamental result of the whole paper.

Theorem 2. *The collection $\left\{F_{\pm}^{\sigma}\right\}$ of WKB type functions is subject to the monodromy (7.27) around $x = \infty$ and the Stokes phenomenon with the constant principal matrix (7.28) (other Stokes matrices are obtained from this by applying the conjugation and rotation symmetries). The generating function $f_3(x)$, which is entire function of x, in each sector $\mathcal{S}_j$ near infinity is a linear combination with constant coefficients of the functions $F_{\pm}^{\sigma}$.*

Moreover, the functions $F_{\pm}^{\sigma}$ are WKB solutions to a sixth order differential equation near $x = \infty$ of the form

$$\partial_x^6 f + a_1 \partial_x^5 f + a_2 \partial_x^4 f + a_3 \partial_x^3 f + a_4 \partial_x^2 f + a_1 \partial_x f + a_6 f = 0 \tag{7.29}$$

with analytic coefficients

$$a_j(x) = \sum_{k \geq 0} a_{j,k} x^{-j} \tag{7.30}$$

such that

$$a_{3,0} = 2S_3(1)^3, \quad a_{6,0} = S_3(1)^6, \quad a_{1,1} = a_{4,1}, \quad a_{2,1} = a_{5,1}, \quad a_{3,1} = a_{6,1}. \tag{7.31}$$

Also the generating function $f_3(x)$ satisfies Eq. (7.29).

Proof. The first statement of the theorem (about the monodromy and the Stokes matrices) is already proved. From this it follows that the space generated by the functions $F_{\pm}^{\sigma}(x)$ near $x = \infty$ (or their analytic representatives) is invariant with respect to monodromy around $x = \infty$ and with respect to passing from one sector to an adjacent sector. Since the monodromy matrix $\mathcal{M}_{\infty}$ and the Stokes matrices have constant coefficients, also the spaces generated by the successive derivatives $\partial_x^i F_{\pm}^{\sigma}$ are invariant. As in other similar situations (see [Zo3]), we arrive to the determinant equation

$$\det \begin{bmatrix} f & \partial_x f & \dots & \partial_x^6 f \\ F_1 & \partial_x F_1 & \dots & \partial^6 F_1 \\ \dots & \dots & \dots & \dots \\ F_6 & \partial_x F_6 & \dots & \partial^6 F_6 \end{bmatrix} = 0$$

which is satisfied by the functions F_j (where we have ordered the functions $F_{\pm}^{\sigma} = F_j$). This equation is equivalent to Eq. (7.29), where the coefficients $a_j(x)$ are ratios of some minors of sixth dimension and are holomorphic and single valued functions of x.

The form (7.30) of the coefficients $a_j(x)$ and the relations (7.31) follow from the fact that the WKB solutions have the form $\sim e^{\sigma x S_3(1)} x^{-3/2}$. When we assume a solution $f \sim e^{\kappa x} x^{\gamma}$, then we should get the 'Hamilton–Jacobi equation' $\sum_j a_{j,0}\kappa^{6-j} = \left(\kappa^3 + S_3(1)\right)^2 = 0$ and the value $\gamma = -3/2$ implies the equation

$$6 \cdot (\sigma S_3(1))^5 \cdot \left(\frac{-3}{2}\right) + a_{3,0} \cdot 3 \cdot (\sigma S_3(1))^2 \cdot \left(\frac{-3}{2}\right) + \sum_j a_{j,1} \cdot (\sigma S_3(1))^j = 0,$$

which is satisfied for any $\sigma = -1, \epsilon, \bar{\epsilon}$. ∎

Remark 6. *It is highly interesting whether Eq. (7.29) can be prolonged to the whole $x-$plane with the other singularity at $x = 0$. Indeed, the function $f_3(x)$ is its solution and has very regular behavior at $x = 0$. So, maybe Eq. (7.29) has regular singularity at $x = 0$.*

But then each its coefficient $a_j(x)$ should be rational with pole at $x = 0$ of order $\leq j$. Moreover, since f_3 depends on x^3, our equation should be of the form

$$\begin{gathered} f^{(VI)} + c_1 x^{-1} f^{(V)} + c_2 x^{-2} f^{(IV)} + \left(c_3 + c_4 x^{-3}\right) f^{(III)} \\ + \left(c_5 x^{-1} + c_6 x^{-4}\right) f^{(II)} + \left(c_7 x^{-2} + c_8 x^{-5}\right) f^{(I)} \\ + \left(c_9 + c_{10} x^{-3} + c_{11} x^{6}\right) f = 0. \end{gathered} \tag{7.32}$$

Then we get the following recurrence for the coefficients in $f_3 = \sum b_k x^{3k}$:

$$\begin{gathered} \{c_{11} + 3kc_8 + 3k(3k-1)c_6 + 3k(3k-1)(3k-2)c_4 + 3k\ldots(3k-3)c_2 \\ +3k\ldots(3k-4)c_1 + 3k\ldots(3k-5)\}b_k + \\ \{c_{10} + (3k-3)c_7 + (3k-3)(3k-4)c_5 + (3k-3)\ldots(3k-5)c_3\}b_{k-1} \\ +c_9 b_{k-2} = 0. \end{gathered}$$

In a particular, for $k = 2$ we get an equation relating $b_0 = 1$, $b_1 = -\zeta(3)$ and $b_2 = \zeta(3,3) = \frac{1}{2}\left(\zeta(3)^2 - \zeta(6)\right)$ (where $\zeta(6) = \pi^6/945$). Since the coefficients c_j are potentially calculable, we could arrive at a quadratic equation for $\zeta(3)$ with coefficients which most probably belong to the field $\mathbb{Q}(\pi, \sqrt{3})$.

Recall that R. Apéry [Ap] *was the first who proved the irrationality of $\zeta(3)$. If our speculations turned out correct it would be quite spectacular achievement.*

Another question is about the values of the constants p, q in the principal Stokes matrix in Eq. (7.28). Probably $p = 0$ and $q = -1$.

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